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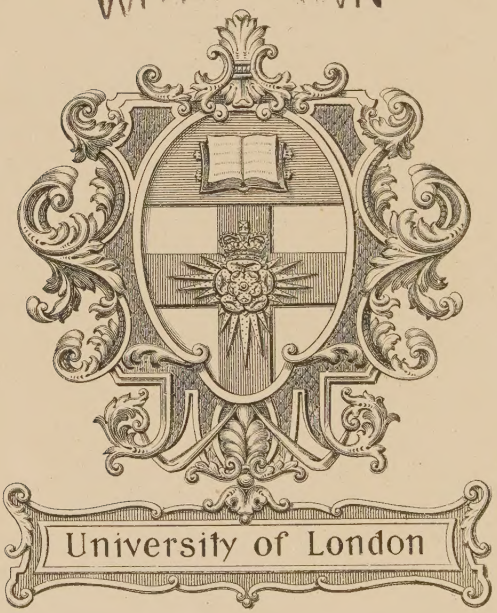

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. IV.

FROM NOVEMBER, 1871, TO NOVEMBER, 1873.

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LONDON MATHEMATICAL SOCIETY.

VOL. IV.

EIGHTH SESSION, 1871–2.

November 9th, 1871.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street.

Dr. SPOTTISWOODE, F.R.S., President, in the Chair.

The Reports of the Treasurer and Secretaries were read and adopted. From the Report of the Secretaries, it appeared that the number of Members since the last General Meeting, held November 10th, 1870, had increased from 110 to 114.

The Obituary of the Society, during the Session, comprised the name of one of its most distinguished members, and its first President. [An account of the tributes paid by the President and Treasurer to the memory of the late Prof. De Morgan has been given in the Society's Proceedings for April 13th, 1871, (No. 34).]

The communications made to the Society during the past Session had been as follows:—

"Sketch of recent researches upon Quartic and Quintic Surfaces:" by Prof. Cayley, V.P.

"On Elliptic Integrals:" by Prof. H. J. S. Smith, V.P.

"Note on the Theory of the Rational Transformation between Two Planes, and on Special Systems of Points;" and a "Second Memoir on Quartic Surfaces:" by Prof. Cayley, V.P.

"On the Polar Correlation of Two Planes, and its connexion with their Quadric Correspondence:" by Dr. Hirst, Treasurer.

"On Systems of Tangents to Plane Cubic and Quartic Curves:" by Mr. J. J. Walker.

"On the Order and Singularities of the Parallel of an Algebraical Curve:" by Mr. S. Roberts.

"Methods made use of in his 'Linear Associative Algebra':" by Prof. Peirce, of Harvard University.

"On an Analytical Theorem from a new point of view;" and "On a Problem in the Calculus of Variations:" by Prof. Cayley, V.P.

"A Property of Conical and Cylindrical Surfaces:" by Mr. C. W. Merrifield.

"On Skew Cubics:" by Prof. H. J. S. Smith, V.P.

"Remarks on the Mathematical Classification of Physical Quantities:" by Prof. J. Clerk-Maxwell.

"Historical Note upon Certain Formulæ (Gauss's Analogies) in Spherical Trigonometry:" by Mr. I. Todhunter.

"Stresses in Warren and Lattice Girders:" by Prof. M. W. Crofton.

"A Third Memoir on Quartic Surfaces:" by Prof. Cayley, V.P.

"On the Singularities of the Envelopes of a Non-Unicursal Series of Curves:" by Prof. Henrici, V.P.

"Note on the Explanation of Coronas, as given in Verdet's 'Leçons d'Optique Physique,' and other Works:" by Hon. J. W. Strutt.

"On the Problem of finding the Circle which cuts three given Circles at three given Angles:" by Mr. J. Griffiths.

"On Plücker's Models of Certain Quartic Surfaces:" by Prof. Cayley, V.P.

"On the Motion of a Plane under Certain Conditions:" by Mr. S. Roberts.

Several lesser communications were made, and descriptions given of models exhibited.

The same Mathematical Journals had been subscribed for as in the preceding Session. In future, however, the "Bulletin des Sciences Mathématiques et Astronomiques" will not be subscribed for, as it will be given to the Society in exchange for its own Proceedings.

The Proceedings of the Society had been sent to the Royal Societies of London and of Edinburgh; the Royal Irish Academy; the Cambridge Philosophical Society; the Philosophical Society of Manchester; the Institute of Actuaries; the British Museum; l'Académie des Sciences, Paris; l'École Polytechnique; la Société Philomathique; l'Académie Royale de Belgique; Akademie der Wissenschaften, Berlin; Royal Society of Göttingen; il Istituto Lombardo, Milan; the Smithsonian Institution, Washington, U.S.; United States Naval Observatory, Washington; Connecticut Academy, New Haven; Prof. Luigi Cremona, an Editor of the "Annali di Matematica"; Herr Borchardt, Berlin, Editor of "Crelle's Journal"; the Editor of "Nature."

In return for these, the Society had received the Proceedings of the Royal Society; the "Journal of the Institute of Actuaries"; the "Annali di Matematica"; the "Monatsberichte" for 1871, from the Akademie der Wissenschaften; "Crelle's Journal," from Herr Borchardt; "Nature" weekly from the Editor; Annuaire et Bulletins de l'Académie Royale des Sciences, des Lettres, et des Beaux Arts de Belgique; Journal de l'École Polytechnique; Report and Proceedings of the Smithsonian Institution, Washington; Transactions of the Connecticut Academy of Arts and Sciences (3 parts); Jahrbuch über die gesammten Fortschritte der Mathematik.

The following presents had been made to the Society:—

Royal Society's Catalogue, Vols. III., IV., V.

"On the Determination of the Orbit of a Planet from Three Observations," by A. Cayley, F.R.S.: from the Author.

"Géométrie Supérieure," by M. Chasles: from the President.

"Linear Associative Algebra," by Prof. Benjamin Peirce: from the Author.

"Nautical Almanac" for 1874: from the Nautical Almanac Office.

"Journal of the London Institution," Nos. 1—7.

"On Hills and Dales," and an Address to the Mathematical and Physical Section of the British Association, September 15th, 1870, by J. Clerk-Maxwell, F.R.S.: from the Author.

"The Annual of the Royal School of Naval Architecture and Marine Engineering:" from C. W. Merrifield, F.R.S., the Principal.

Ten Memoirs by M. Chasles (the titles of which are given p. 232, No. 34): from the Treasurer.

"Theorie und Anwendung des sogenannten Variationscalculs," von Dr. G. W. Strauch: from C. R. Hodgson, B.A.

Several early numbers of the "Cambridge Mathematical Journal" (10 numbers), and "Quarterly Journal of Pure and Applied Mathematics," March, 1855: from the President.

"Sulle Trasformazioni razionali nello Spazio," Nota 1ª del Prof. L. Cremona: from the Author.

Model of the Surface called the "Cylindroid": from Prof. R. Stawell Ball.

Appendix to Benjamin Anderson's Journey to Musadu: from the Connecticut Academy.

"Rapport sur les Progrès de Géométrie," par M. Chasles: from the President.

"Verzeichniss der Abhandlungen der Königlich-Preussischen Akademie der Wissenschaften von 1710—1870, in Alphabetischer Folge der Verfasser," Berlin, 1871.

The following Models had been added to the Society's collection:—

Of the tubular surface which is the envelope of a sphere of constant radius which moves with its centre on a given parabola.

Of the cubic surface whose equation is

$$xyz-(\tfrac{3}{7})^3(x+y+z-1)^3=0.$$

No Cartes de Visite had been received during the Session.

After the Reports had been read, and Mr. R. Wormell, M.A., appointed Auditor for the ensuing Session, the Meeting proceeded to the election of the new Council.

The following gentlemen were declared by the Scrutators to be duly elected:—

President, Dr. Spottiswoode, F.R.S.; Vice-Presidents, Profs. Cayley, Henrici, H. J. S. Smith, and *Mr. S. Roberts; Treasurer, Dr. Hirst; Hon. Secs., Messrs. M. Jenkins and R. Tucker; other Members of the Council, Profs. Clifford and Crofton, Messrs. Cotterill, Merrifield, Stirling, Sylvester, and Walker, and the *Hon. J. W. Strutt. (The gentlemen marked * were not on the preceding Council list.)

A vote of thanks was carried to the retiring Members of the Council.

Mr. A. Freeman, M.A., F.R.A.S., Fellow of St. John's College, Cambridge, was proposed for election.

The President entered into an explanation of the reasons which had led the Council to think the time had arrived for increasing the number of Honorary Foreign Members, and mentioned that at this date it had been resolved to recommend that the number be raised to six. It being unanimously agreed that it was desirable so to enlarge this class of Members, the Chairman proceeded to nominate, for the Council, the following five Foreign Mathematicians:—Prof. Betti, Dr. Clebsch, Prof. Cremona, M. Hermite, and Dr. Hesse. [Though this list had not been drawn up with reference to geographical divisions, yet it now gave two representatives for France, two for Germany, and two for Italy, M. Chasles being the other representative for the first-named country.]

Mr. Sylvester made a communication "On the partition of an even number into two primes."

The following presents were received:—

"Monatsbericht" for August, 1871.

"Nautical Almanac" for 1875.

"Annali di Matematica," April, 1871.

"Crelle's Journal," 74 Band, erstes Heft.

The following is an abstract of Mr. Sylvester's communication:—

In one of his minor papers, Euler has enunciated, as a theorem resting entirely on intuition from a comparatively small number of instances, that every even number may be decomposed into a sum of two primes. The object of Mr. Sylvester's communication was to obtain some measure of the probable number of ways in which such decomposition can be effected for any given number; if it can be shown to be probably greater than the square root of the number itself, it will follow, from

generally admitted principles of the theory of chances, that the probability of the theorem being universally true above any assigned limit, if *proved* to be true up to that limit, may be represented by an infinite product of terms which will approach as near as we please to unity the higher the limit is taken.

The mere fact of the theorem, as Euler gave it, being proved up to 100,000,000 or any other number, however great, would leave the probability of its being universally true absolutely zero, just as the fact of the sun having risen 100,000,000 times would not contribute an atom of probability to the supposition that it would continue to rise for all time to come. In the case before us, on the contrary, the probability of the theorem being universally true by a sufficiently copious induction may be made to approach as near as we please to absolute certitude. The Author considers that he has established beyond the reach of reasonable doubt that the order of the magnitude which represents the mean probable value of the number of modes of effecting the resolution of a very large even number into two prime numbers is that of the square of the number of primes inferior to the given number divided by the number itself, or, which (thanks to the discoveries of Legendre and Tchebicheff) we know to be the same thing, the number of the decompositions in question bears a finite ratio (assignable within limits) to the number to be decomposed divided by the square of its Napierian logarithm. If we agree provisionally to call preter-primes in respect to n those numbers which are prime themselves and also when subtracted from n leave prime remainders, the Author shows that the probable number of such preter-primes (*i. e.* the most probable value attainable under our present conditions of knowledge) may be found approximately by multiplying the number of ordinary primes inferior to n by the product of a set of fractions depending in part on the order of the magnitude and in part on the constitution of the number n. If n is the double of a prime, the product in question is got by multiplying together all the quantities $\frac{p-2}{p-1}$, where p is every odd prime between unity and the square root of n; but if n itself contains any such primes among its factors, then the corresponding factors are to be omitted out of the product. We thus see that if two even numbers of considerable magnitude lie adjacent and tolerably near to each other, one of which is the double of a prime, but the other six times a prime, the number of preter-primes relative to the latter will be about twice as many as those relative to the former. For the purpose of greater simplicity of explanation, the formula of approximation has been stated above with less accuracy than it admits of being stated with; instead of the total number of odd primes being multiplied by the product of factors last described, those only should have been taken which are not intermediate between 2 and $\sqrt{n}$, and

the result so modified should have been stated to be the probable value, not of the total number of preter-primes, but only of such of them (by far the larger number) as are not of the excluded class above described, nor, subtracted from n, give rise to remainders belonging to such class. The Author has found, by actual trial on an extensive scale, that the estimated values of the number of decompositions never differ by more than a moderate, and in some cases exceedingly slight, percentage from their actual values, determined by the use of Borchardt's tables.

The same methods enable him also to assign a probable value to the number of modes of resolving an odd number into the sum of one prime and the double of another, and in general leads to an approximate representation of the number of solutions in prime numbers of any system of linear equations of which the total number of solutions is limited, and even to resolve approximately such questions as that of determining how many prime numbers there are inferior to a given limit which are followed by prime numbers differing from them by any assigned interval.

Since the communication made to the Mathematical Society, the Secretaries have been informed by Mr. Sylvester that he has verified his results by quite a different method. The exact number of the solutions of the equation $x+y=n$ in prime numbers may be expressed algebraically by means of the method of Generating Functions in terms of the inferior primes to n. The expression will be found to consist of two parts—one a constant multiple of n, the other a function of the roots of unity corresponding to the several inferior primes and their combinations. The former non-periodic part may obviously be regarded as the mean value of the expression, and Mr. Sylvester has found that it is identical with the value obtained by the method of averages previously employed.

December 14*th*, 1871.

Dr. SPOTTISWOODE, F.R.S., President, in the Chair.

Mr. A. Freeman, M.A., F.R.A.S., was elected an Ordinary Member, and the following gentlemen Foreign Members of the Society:—Prof. Betti, Dr. Clebsch, Prof. Cremona, M. Hermite, and Dr. Hesse.

Mr. Sylvester made a communication "On the Theorem that an Arithmetical Progression which contains more than one, contains an infinite number of prime numbers."

Profs. Cayley and Smith took part in a discussion on the subject.

Prof. Clifford gave an account of a paper he is preparing for the Society.

Prof. Cayley drew attention to the question of the determination of the surfaces capable of division into infinitesimal squares by means of their curves of curvature.

Mr. S. Roberts exhibited a thread model of a homographic transformation of the developable surface which circumscribes a system of confocal quadrics. The surface is generated by planes touching an ellipse at a constant inclination, and its equation is obtained by writing p^2z^2 for r^2 in $\varphi(x^2, y^2, r^2) = 0$, representing the plane parallel of the ellipse.

The following presents were received:—

"Annali di Matematica," April, 1871.

Extrait du "Bulletin des Sciences Mathématiques et Astronomiques": "Sur une surface de cinquième ordre et sur représentation sur le plan," par M. G. Darboux; "Sur une série de lignes analogues aux lignes géodésiques," par M. G. Darboux; "Sur les équations aux dérivées partielles du second ordre," par M. G. Darboux: from the Author.

"Memorie del Reale Istituto Lombardo," and "Rendiconti," 17 parts.

"Rapporti sui progressi delle Scienze i sopra alcuni recenti studj di Chimica Organica," del Dottor L. Gabba.

The following is an abstract of Mr. Sylvester's communication:—

This celebrated theorem is one of those which no one would think of doubting, but which are of extreme difficulty to prove. A pretended proof had been given by Legendre a good part of a century ago, and occupies a whole chapter in the "Théorie des Nombres"; but the first real demonstration was accomplished by Lejeune Dirichlet in his great memoir published in the Berlin "Transactions" for the year 1837.

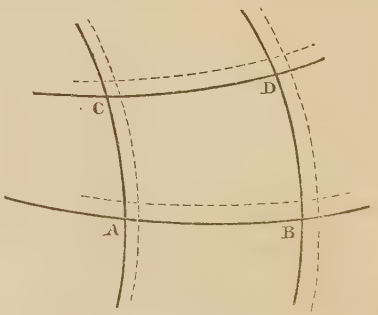
The present communication is limited to the case of Arithmetical Progressions, proceeding according to the common difference 4 or 6. The fundamental theorem employed is an identical equation, on the one side of which are algebraical fractions of the form $\frac{x^p}{1-x^{2p}}$, where p represents any combination of the simple powers of any system of primes taken with the positive or negative sign, according as p contains an even or an odd number of factors, and on the other side simple powers of x, whose indices are all the odd numbers not containing any one of the given system of primes as a factor. In the case of progressions with the common difference 4, all the primes of the form $4q+3$ and their primary combinations figure as indices on the first side of the

equation, and consequently the powers of x on the other side have for their indices combinations of factors of the form $4q+1$. By writing in place of x the square root of negative unity into x, it is shown instantaneously that if the number of primes of the form $4q+3$ were finite, a finite series of fractions converging to an infinite value as x approaches to unity would be equal to another such series which would remain finite, which is of course absurd. The proof applicable to the case of progressions of the form $4q+1$ is not quite so simple: it depends on showing that, if their number were only i in number, it would be possible to have a rational integer function of the degree $2i$ in the logarithm of n greater than a finite multiple of n for a value of n unlimitedly great, which is known to be absurd.

A process precisely similar applies, *mutatis mutandis*, to the case of progressions of the form $6q+1$, $6q+5$, the sole difference being that, instead of substituting for x, x multiplied by the square root of negative unity, we must now substitute for it x multiplied successively by the two prime sixth-roots of unity, and subtract the results from one another. The method here successfully employed in the treatment of these elementary cases appears to differ fundamentally from Dirichlet's method, in regard of the circumstance that it deals with an infinitesimal variation in the value of the variable, whereas in Dirichlet's method the infinitesimal variation takes place in the index of the power of the variable.

The substance of Prof. Cayley's remarks is contained in the following abstract:—

Geometrically, the question is as follows:—Consider any two curves of curvature AB, CD of one set, and any two AC, BD of the other set, as shown by the continuous lines of the figure: drawing the consecutive curves as shown by dotted lines, the curve consecutive to AB at an arbitrary (infinitesimal) distance from AB, the other three curves may be drawn at such distances that the elements at A, B, and C shall be each of them a square; but this being so, the element at D will not be in general a square, and it is only for certain surfaces that it is so. But if (whatever the curves of curvature AB, CD, AD, BC may be) the element at D is a square, then it is clear that the whole surface can be, by means of its curves of curvature, divided into infinitesimal squares.



Analytically, if for a given surface the equations of its curves of curvature are expressed in the form $h = f(x, y, z)$, $k = \phi(x, y, z)$; then the coordinates x, y, z can be expressed each of them as a function of the parameters h, k, and we have for the element of distance between two consecutive points on the surface

$$dx^2 + dy^2 + dz^2 = A dh^2 + C dk^2,$$

where A, C are in general each of them a function of h and k. The condition for the divisibility into squares is that the quotient $A \div C$ shall be of the form function $h \div$ function k.

It was shown by M. Bertrand that, in a triple system of orthotomic isothermal surfaces, each surface possesses the property in question of divisibility into squares by means of its curves of curvature. But in such a triple system, each surface of the system is necessarily a quadric; so that the theorem comes to this, that a quadric surface is, by means of its curves of curvature, divisible into squares. The analytical verification is at once effected: taking the equation of the surface to be

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

then the expressions for the coordinates in terms of the parameters h, k of a curve of curvature are

$$x^2 = \frac{a(a+h)(a+k)}{(a-b)(a-c)},$$

$$y^2 = \frac{b(b+h)(b+k)}{(b-c)(b-a)},$$

$$z^2 = \frac{c(c+h)(c+k)}{(c-a)(c-b)},$$

and we have $4(dx^2 + dy^2 + dz^2)$

$$= (h-k)\left\{ \frac{h\,dh^2}{(a+h)(b+h)(c+h)} - \frac{k\,dk^2}{(a+k)(b+k)(c+k)} \right\};$$

so that $A \div C$ is of the required form.

But there is nothing to show that the property is confined to quadric surfaces; and the question of the determination of the surfaces possessing the property appears to be one of considerable difficulty, and which has not hitherto been examined.

January 11*th*, 1872.

Dr. SPOTTISWOODE, F.R.S., President, and subsequently Prof. CAYLEY, V.P., F.R.S., in the Chair.

Mr. J. W. L. Glaisher, B.A., F.R.A.S., Fellow of Trinity College, Cambridge, was proposed for election; and Major F. Close, R.A., was admitted into the Society by the President.

The papers read were—Prof. Cayley "On the Surfaces the Loci of the Vertices of Cones which satisfy Six Conditions;" Mr. J. W. L. Glaisher "On the Constants that occur in certain Summations by Bernoulli's Series;" Mr. W. Barrett Davis "On Methods used in the Construction of Tables of Divisors;" Mr. S. Roberts "On the Parallel Surfaces of Conicoids and Conics."

The following presents were made to the Society:—

"Annali di Matematica," September, 1871.

"Theory of Heat," by Prof. Maxwell, F.R.S.: from the Author.

"Nogle Bemærkninger om de Mathematiske Satzer $\frac{0}{a}=0$, $\frac{a}{0}=\infty$, og $\frac{0}{0}=x$," af S. A. Sexe, 1869.

"Yderligere Bemærkninger om Imaginære Størrelser," by the same, 1870.

"Om nogle Punkter i den Elementære Arithmetik," by the same, no date.

"Nogle Bemærkninger om Imaginære Størrelser," by the same, 1868.

"Om Cirklers Berøring," af C. M. Guldberg, 1861.

"Om Ligningen af 5^{te} Grad," by the same, 1869.

"Om den Algebraiske Ligning af n^{te} Grad, hvis Rodder repræsenteres ved Formelen $x=R_1^{\frac{1}{n}}+R_2^{\frac{1}{n}}$," by the same, 1870.

"Om Vædskers Udvidelse ved Konstant Tryk," by the same, 1869.

"Repräsentation der Imaginären der Plangeometrie," von M. S. Lie, 1869.

"Traité Élémentaire des Fonctions Elliptiques," par Dr. O. J. Broch, two Parts 1866 and 1867: the last ten works from l'Université Royale de Norvège à Christiania.

On the Surfaces each the Locus of the Vertex of a Cone which passes through m given Points and touches $6-m$ given Lines. By Prof. CAYLEY.

I consider the surfaces, each of them the locus of the vertex of a (quadri-)cone which passes through m given points and touches $6-m$ given lines; viz., calling the given points $a, b, c \ldots$ and the given lines $\alpha, \beta, \gamma \ldots$, the surfaces in question are:

	Order
$abcdef$	4
$abcdea$	8
$abcda\beta$	16
$abca\beta\gamma$	24
$aba\beta\gamma\delta$	24
$aa\beta\gamma\delta\epsilon$	14
$a\beta\gamma\delta\epsilon\zeta$	8

I remark that the orders of these several surfaces are in effect determined by the investigations of M. Chasles in regard to the conics in space which satisfy seven conditions. The surface $abcdef$ was long ago considered by M. Chasles, and it is treated of in my "Memoir on Quartic Surfaces," and in the same Memoir the surface $a\beta\gamma\delta\epsilon\zeta$ is also referred to: these two surfaces, and also the surfaces $aa\beta\gamma\delta\epsilon$ and $aba\beta\gamma\delta$ are considered by Dr. Hierholzer* in his excellent paper "Ueber Kegelschnitte im Raume," Clebsch and Neumann, t. II. (1870), pp. 563—586, and to him are due the equations given in the sequel for the surfaces $abcdef$ and $a\beta\gamma\delta\epsilon\zeta$: the researches of the present Memoir are, in fact, a continuation and development of those in the Memoir last referred to.

Table of Singularities, and Explanations in regard thereto.

1. We have on the before-mentioned surfaces respectively certain simple or multiple points, right lines, and curves, as shown in the following Table:

---

* I was grieved to hear of Dr. Hierholzer's death last autumn, at Carlsruhe, at the early age of 30.

	$abcdef$, 4	$abcde\alpha$, 8	$abcda\beta$, 16	$abca\beta\gamma$, 24	$aba\beta\gamma\delta$, 24	$aa\beta\gamma\delta\epsilon$, 14	$a\beta\gamma\delta\epsilon\zeta$, 8	
Points a	$6\times(2)$	$5\times(4)$	$4\times(8)$	$3\times(8)$	$2\times(4)$	$1\times(2)$	0	(0)
Lines ab	$15\times(1)$, C	$10\times(2)$, C	$6\times(4)$, C	$3\times(4)$, C	$1\times(2)$, C	0	0	(1)
α	0	$1\times(2)$, C	$2\times(4)$, C	$3\times(8)$, C	$4\times(8)$, C	$5\times(4)$, C	$6\times(2)$, C	(2)
$[ab, \alpha, \beta, \gamma]$	0	0	0	$6\times(4)$, P	†$8\times(2+2)$, L	0	0	(3)
$[\alpha, \beta, \gamma, \delta]$	0	0	0	0	$2\times(8)$, P	$10\times(2)$, L	$30\times(1)$, L	(4)
$[ab, cd, \alpha, \beta]$	0	0	$6\times(2)$, P	0	0	0	0	(5)
abc, def	$10\times(1)$, P	0	0	0	0	0	0	(6)
abc, de, α	0	*$10\times(1)$, P	0	0	0	0	0	(7)
abc, α, β	0	0	‡$4\times(2+2)$, P	$3\times(4)$, P	0	0	0	(8)
Cubic $abcdef$	$1\times(1)$, C	0	0	0	0	0	0	(9)
Quadriquadric $a\beta\gamma, \delta\epsilon\zeta$	0	0	0	0	0	0	$(10)\times(1)$, L	(10)
Excuboquartic $a\beta\gamma, \delta\epsilon, \alpha$	0	0	0	0	0	§$10\times(1)$, L	0	(11)

* Tangent plane along line is plane abc.
† Tacnodal line, each sheet of surface touched along line by hyperboloid $a\beta\gamma$.
‡ Tacnodal line, each sheet touched along line by plane abc.
§ Surface touched along line by hyperboloid $a\beta\gamma$.

2. In the Table, the upper margin refers to the surfaces, and the left-hand margin to the points, lines, and curves situate on these surfaces respectively; the body of the Table showing the number, and in ( ) the multiplicity, of these points, lines, and curves in regard to the several surfaces respectively. Thus, points a; for the surface $abcdef$, $6\times(2)$, there are 6 such points, each of them a 2-conical (ordinary conical) point on the surface: so $abcdea$, $5\times(4)$, there are 5 such points, each a 4-conical point on the surface (viz., instead of the tangent plane there is a quartic cone); and so on. Similarly, lines ab (viz., these are the lines joining two points a, b); for the surface $abcdef$, $15\times(1)$, there are 15 such lines, each a simple line on the surface; surface $abcdea$, $10\times(2)$, there are 10 such lines, each a double (ordinary nodal) line on the surface; and so on. We have in two places the multiplicity $(2+2)$, which refers to a tacnodal line, as presently explained. The corner letters C, P, L denote respectively proper cone, plane-pair, and line-pair, as afterwards explained.

3. The lines and curves referred to in the left-hand margin are:

(1) ab, line joining the points a and b.

(2) a, line a.

(3) $[ab, a, \beta, \gamma]$, pair of lines meeting each of the four lines, or say the tractors of the four lines ab, a, β, γ. As regards the surface $aba\beta\gamma\delta$, the multiplicity is given as $(2+2)$, viz. the line is (not an ordinary nodal, but) a tacnodal line, each sheet touching along the whole line the hyperboloid $a\beta\gamma$.

(4) $[a, \beta, \gamma, \delta]$, tractors of the four lines a, β, γ, δ.

(5) $[ab, cd, a, \beta]$, tractors of the four lines ab, cd, a, β.

(6) abc, def, line of intersection of the planes abc and def.

(7) abc, de, a, line in the plane abc joining the intersections of this plane by the lines de and a respectively.

(8) abc, a, β, line in the plane abc joining the intersections of this plane by the lines a and β respectively. As regards the surface $abcda\beta$, the multiplicity is given as $(2+2)$, viz. each line is (not an ordinary nodal, but) a tacnodal line, each sheet touching along the whole line the plane abc.

(9) Cubic $abcdef$, cubic curve through the six points a, b, c, d, e, f, common intersection of the cones each having its vertex at one of the points and passing through the other five.

(10) Quadriquadric $a\beta\gamma$, $\delta\epsilon\zeta$, intersection of the quadric surfaces $a\beta\gamma$ and $\delta\epsilon\zeta$, that is, the quadric surfaces through the lines a, β, γ and δ, ϵ, ζ respectively.

(11) Excuboquartic $a\beta\gamma$, $\delta\epsilon$, a, quartic curve generated as follows: viz., taking any line whatever which meets the lines a, β, γ (or say any generating line of the quadric $a\beta\gamma$), the plane

through this line and the point a meets the lines δ, ϵ in two points respectively; and the line joining these meets the generating line in a point having for its locus the excubo-quartic curve in question (theory further considered in the sequel).

Special forms of (Quadri-)Cones.

4. We have to consider the special forms of (quadri-)cones; these are: 1°. The sharp-cone, or plane-pair; that is, a pair of two planes, intersecting in a line called the axis, the vertex being in this case an indeterminate point on the axis. Observe that a plane-pair passes through a given point when either of its planes passes through such point; it touches a given line when its axis meets the given line. 2°. The flat-cone, or line-pair; viz., this is a pair of intersecting lines, their point of intersection being the vertex of the line-pair, and the plane of the two lines being the *diametral* of the line-pair. Observe that the line-pair passes through a given point when its diametral passes through such point; it touches a given line when either of its lines meets the given line. 3°. There is a third kind, the line-pair-plane; viz., the two planes of the plane-pair may come to coincide, retaining, however, a definite line of intersection, or axis: or again, the two lines of a line-pair may come to coincide, retaining a definite plane or diametral; that is, in either case we have a plane passing through a line; and which is to be considered indifferently as two coincident planes intersecting in the line, or as two coincident lines lying in the plane. But there is not, in the present Memoir, any occasion to consider this third kind of special cone.

The letters C, P, L in the Table denote that the cone is a (proper) cone, plane-pair, or line-pair, as the case may be.

Singular Lines and Curves on the Surfaces.

5. We may establish *à priori* the existence, and even to some extent the multiplicity, of the several lines and curves on the surfaces $abcdef$, ... $\alpha\beta\gamma\delta\epsilon\zeta$. Thus:

1°. Lines ab: take for the vertex of the cone a point at pleasure on the line ab; the cone passing through b will *ipso facto* pass through a; and the conditions are thus that the cone shall pass through b and satisfy four other conditions—in all, five conditions: and there is thus a cone with the point in question as vertex; that is, the line ab is situate on the surface. Moreover, for the surfaces $abcdef$, $abcdea$, $abcda\beta$, $abca\beta\gamma$, $aba\beta\gamma\delta$ respectively, for a given position of the vertex on the line ab, the number of cones is 1, 2, 4, 4, 2 respectively: and these are the multiplicities of the line ab on the several surfaces respectively.

2°. Lines α: take for the vertex of the cone a point at pleasure on the line α; then the cone *ipso facto* touches the line α, and there are only five other conditions to be satisfied; that is, we have a cone with the vertex in question; or the line α is situate on the surface. Moreover, for the surfaces $abcde\alpha$, $abcd\alpha\beta$, $abc\alpha\beta\gamma$, $ab\alpha\beta\gamma\delta$, $aa\beta\gamma\delta\epsilon$, $\alpha\beta\gamma\delta\epsilon\zeta$ respectively, the number of cones is 1, 2, 4, 4, 2, 1 respectively: and it may be seen that the multiplicities of the line α are the doubles of these numbers, or are $= 2, 4, 8, 8, 4, 2$ for the several surfaces respectively.

3°. Lines $[ab, \alpha, \beta, \gamma]$: taking the vertex in one of these tractors, the cone cannot be a proper cone, but (if it exist) it must be either a line-pair having the tractor for one of its lines, or else a plane-pair having the tractor for its axis. The two cases are:

Surface $abc\alpha\beta\gamma$. Cone is a plane-pair, the two planes intersecting in the tractor, and passing, the one of them through the points a, b, the other through the point c. The vertex being an indeterminate point on the tractor, the tractor is situate on the surface.

Surface $ab\alpha\beta\gamma\delta$. Cone is a line-pair, one line being the tractor, the other a line drawn in the plane of the tractor and ab to meet δ, and which meets the tractor in an arbitrary point thereof: the tractor is thus a line on the surface.

4°. Lines $[\alpha, \beta, \gamma, \delta]$: taking the vertex in one of these tractors, then, as in the last case, the cone is either a line-pair having the tractor for one of its lines or a plane-pair having the tractor for its axis. The three cases are:

Surface $ab\alpha\beta\gamma\delta$. Cone is a plane-pair, the two planes intersecting in the tractor and passing through the points a, b respectively.

Surface $aa\beta\gamma\delta\epsilon$. Cone is a line-pair, one line being the tractor, the other a line in the plane of the tractor and a, meeting the line ϵ and meeting the tractor in an indeterminate point.

Surface $\alpha\beta\gamma\delta\epsilon\zeta$. Cone is a line-pair, one line being the tractor, the other a line drawn from an indeterminate point of the tractor to meet the lines ϵ and ζ.

5°. Lines $[ab, cd, \alpha, \beta]$. Cone is a plane-pair, the two planes intersecting in the tractor, and passing through the points a, b and the points c, d respectively.

6°. Line abc, def. Cone is a plane-pair, consisting of the two planes abc and def.

7°. Line abc, de, α. Cone is a plane-pair, the two planes intersecting in the line; one plane being abc, the other a plane through the line de.

8°. Line abc, α, β. There are two cases:

Surface $abcd\alpha\beta$. Cone is a plane-pair, the two planes intersecting

in the line; the one being *abc*, and the other passing through the point *d*.

Surface $abc\alpha\beta\gamma$. Cone is a line-pair; one line being *abc*, α, β, the other a line in the plane *abc* meeting the line δ, and meeting the line *abc*, α, β in an indeterminate point.

9°. Cubic *abcdef*. Each point of the cubic is the vertex of a proper cone passing through the cubic, and therefore through the six points; that is, the cubic is a line on the surface *abcdef*.

10°. Quadriquadric $\alpha\beta\gamma$, $\delta\epsilon\zeta$. Cone is a line-pair; viz., it is composed of the lines drawn from any point of the curve, one of them to meet the lines α, β, γ, and the other to meet the lines δ, ϵ, ζ.

11°. Excuboquartic $\alpha\beta\gamma$, $\delta\epsilon$, a. Cone is a line-pair; the two lines being, one of them a line at pleasure meeting α, β, γ, the other the line which, in the plane of the other line and the point a, meets the lines δ, ϵ.

Mode of obtaining the several Equations: Notations and Formulæ.

6. The equations of the several surfaces are obtained by taking as centre of projection an assumed position of the vertex, and projecting everything upon an arbitrary plane; the projections of the given points and lines are points and lines in the arbitrary plane, and the section of the cone by this plane is a conic; the equation of the surface is thus obtained as the condition that there shall be a conic passing through m given points and touching $6-m$ given lines.

7. We take as current coordinates (X, Y, Z, W), or when plane-coordinates are employed $(\xi, \eta, \zeta, \omega)$: the coordinates of the vertex are throughout represented by (x, y, z, w); but in explanations, &c., these are also used as current coordinates. The plane of projection is taken to be $W=0$. The coordinates of the given points a, &c., are taken to be (x_a, y_a, z_a, w_a), &c. There is no confusion occasioned by so doing, and I retain the ordinary letters (a, b, c, f, g, h) for the six coordinates of a line, it being understood that these letters so used have no reference whatever to the given points a, b, &c.; viz., the coordinates of the given lines α, &c., are $(a_\alpha, b_\alpha, c_\alpha, f_\alpha, g_\alpha, h_\alpha)$, &c.; there is sometimes occasion to consider the coordinates of other lines ab, &c., but the notation will always be explained.

8. I write l, m, n, p, q, r for the coordinates of the line joining the vertex (x, y, z, w) with a point (x', y', z', w'); viz.,

$$\begin{aligned} l &= yz'-y'z, & p &= xw'-x'w, \\ m &= zx'-z'x, & q &= yw'-y'w, \\ n &= xy'-x'y, & r &= zw'-z'w, \end{aligned}$$

($l_a = yz_a - y_a z$, &c., this being explained when necessary); and also

$$\begin{aligned} \mathrm{P} &= \quad . \quad hy-gz+aw, \\ \mathrm{Q} &= -hx \quad . \quad +fz+bw, \\ \mathrm{R} &= \quad gx-fy \quad . \quad +cw, \\ \mathrm{S} &= -ax-by-cz \quad . \end{aligned},$$

($\mathrm{P}_a = h_a y - g_a z + a_a w$, &c., this being explained when necessary).

This being so, then projecting from the vertex (x, y, z, w), say on the plane $\mathrm{W}=0$, the x, y, z coordinates of the projection of a point a are as $p_a : q_a : r_a$ ($p_a = xw_a - x_a w$, &c.); and the equation of the projection of a line α is

$$\mathrm{P}_\alpha \mathrm{X} + \mathrm{Q}_\alpha \mathrm{Y} + \mathrm{R}_\alpha \mathrm{Z} = 0,$$

($\mathrm{P}_\alpha = h_\alpha y - g_\alpha z + a_\alpha w$, &c.). We thus have, in the projection on the plane $\mathrm{W}=0$, the m points and $6-m$ lines situate in and touched by the conic.

The following notations and formulæ are convenient:

9. $pabc = 0$ is the equation of the plane through the points a, b, c; viz.,

$$pabc = \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ x_b, & y_b, & z_b, & w_b \\ x_c, & y_c, & z_c, & w_c \end{vmatrix}.$$

Of course $pbac = -pabc$, &c. Observe that here, and in the notations which follow, the letter p is used as referring to the coordinates (x, y, z, w), and that the index of p (=1 when no index is expressed) shows the degree in these coordinates.

10. $paa = 0$ is the equation of the plane through the point a and the line α; viz., paa is the foregoing determinant, if for a moment b, c are any two points on the line α; or, what is the same thing,

$$paa = \mathrm{P}_a x + \mathrm{Q}_a y + \mathrm{R}_a z + \mathrm{S}_a w,$$

where

$$\begin{aligned} \mathrm{P}_a &= \quad . \quad hy_a - gz_a + aw_a, \\ \mathrm{Q}_a &= -hx_a \quad . \quad +fz_a + bw_a, \\ \mathrm{R}_a &= \quad gx_a - fy_a \quad . \quad +cw_a, \\ \mathrm{S}_a &= -ax_a - by_a - cz_a \quad . \end{aligned};$$

and (a, b, c, f, g, h) are the coordinates of the line α: observe that $paa = paa$.

11. $p^2\alpha\beta\gamma = 0$ is the equation of the quadric surface through the lines α, β, γ; viz., we have

$$\begin{aligned} p^2\alpha\beta\gamma = {} & (agh)\,x^2 + (bhf)\,y^2 + (cfg)\,z^2 + (abc)\,w^2 \\ & + [(abg) - (cah)]\,xw \\ & + [(bch) - (abf)]\,yw \\ & + [(caf) - (bcg)]\,zw \\ & + [(bfg) + (chf)]\,yz \\ & + [(cgh) + (afg)]\,zx \\ & + [(ahf) + (bgh)]\,xy, \end{aligned}$$

where $$agh = \begin{vmatrix} a_\alpha, & g_\alpha, & h_\alpha \\ a_\beta, & g_\beta, & h_\beta \\ a_\gamma, & g_\gamma, & h_\gamma \end{vmatrix} \quad \&c.,$$

$(a_\alpha, b_\alpha, c_\alpha, f_\alpha, g_\alpha, h_\alpha)$, $(a_\beta, \ldots)$, $(a_\gamma, \ldots)$ being the coordinates of the given lines α, β, γ. Observe that $p^2\beta a\gamma = -p^2 a\beta\gamma$, &c.

12. It is to be noticed that, writing

$$\begin{aligned} \mathrm{P} &= \quad . \quad hy - gz + aw, \\ \mathrm{Q} &= -hx \quad . \quad + fz + bw, \\ \mathrm{R} &= \quad gx - fy \quad . \quad + cw, \\ \mathrm{S} &= -ax - by - cz \quad . \quad , \end{aligned}$$

viz., $\mathrm{P}_\alpha = h_\alpha y - g_\alpha z + a_\alpha w$, &c., then that we have identically

$$\begin{vmatrix} \lambda, & \mu, & \nu, & \rho \\ \mathrm{P}_\alpha, & \mathrm{Q}_\alpha, & \mathrm{R}_\alpha, & \mathrm{S}_\alpha \\ \mathrm{P}_\beta, & \mathrm{Q}_\beta, & \mathrm{R}_\beta, & \mathrm{S}_\beta \\ \mathrm{P}_\gamma, & \mathrm{Q}_\gamma, & \mathrm{R}_\gamma, & \mathrm{S}_\gamma \end{vmatrix} = -(\lambda x + \mu y + \nu z + \rho w) \, . \, p^2 a\beta\gamma,$$

and further that we have identically

$$-p^2 a\beta\gamma = \mathrm{L}_{a\beta}\mathrm{P}_\gamma + \mathrm{M}_{a\beta}\mathrm{Q}_\gamma + \mathrm{N}_{a\beta}\mathrm{R}_\gamma + \Omega_{a\beta}\mathrm{S}_\gamma,$$

where $$\begin{aligned} \mathrm{L} &= (af' - a'f)\,x + (bf' - b'f)\,y + (cf' - c'f)\,z - (bc' - b'c)\,w, \\ \mathrm{M} &= (ag' - a'g)\,x + (bg' - b'g)\,y + (cg' - c'g)\,z - (ca' - c'a)\,w, \\ \mathrm{N} &= (ah' - a'h)\,x + (bh' - b'h)\,y + (ch' - c'h)\,z - (ab' - a'b)\,w, \\ \Omega &= (gh' - g'h)\,x + (hf' - h'f)\,y + (fg' - f'g)\,z \\ &\qquad + (af' - a'f + bg' - b'g + ch' - c'h)\,w; \end{aligned}$$

and $\mathrm{L}_{a\beta}$, &c. are the values of L, &c. on substituting therein $(a_\alpha, \ldots)$ and $(a_\beta, \ldots)$ for the unaccented and accented letters respectively.

13. Observe that we have

$$\begin{aligned} \mathrm{L} + (a'f + b'g + c'h)\,x &= \quad . \quad - c'\mathrm{Q} + b'\mathrm{R} - f'\mathrm{S}, \\ \mathrm{M} + (\quad\ \ ,,\quad\ \ )\,y &= \quad c'\mathrm{P} \quad . \quad - a'\mathrm{R} - g'\mathrm{S}, \\ \mathrm{N} + (\quad\ \ ,,\quad\ \ )\,z &= -b'\mathrm{P} + a'\mathrm{Q} \quad . \quad - h'\mathrm{S}, \\ \Omega + (\quad\ \ ,,\quad\ \ )\,w &= \quad f'\mathrm{P} + g'\mathrm{Q} + h'\mathrm{R} \quad . \quad ; \end{aligned}$$

and similarly

$$\begin{aligned} -\mathrm{L} + (af' + bg' + ch')\,x &= \quad . \quad - c\mathrm{Q}' + b\mathrm{R}' - f\mathrm{S}', \\ -\mathrm{M} + (\quad\ \ ,,\quad\ \ )\,y &= \quad c\mathrm{P}' \quad . \quad - a\mathrm{R}' - g\mathrm{S}', \\ -\mathrm{N} + (\quad\ \ ,,\quad\ \ )\,z &= -b\mathrm{P}' + a\mathrm{Q}' \quad . \quad - h\mathrm{S}', \\ -\Omega + (\quad\ \ ,,\quad\ \ )\,w &= \quad f\mathrm{P} + g\mathrm{Q} + h\mathrm{R} \quad . \quad ; \end{aligned}$$

whence also

$$\begin{aligned} \quad . \quad h'\mathrm{M} - g'\mathrm{N} + a'\Omega &= -(a'f + b'g + c'h)\,\mathrm{P}', \\ -h'\mathrm{L} \quad . \quad + f'\mathrm{N} + b'\Omega &= -(\quad\ \ ,,\quad\ \ )\,\mathrm{Q}', \\ g'\mathrm{L} - f'\mathrm{M} \quad . \quad + c'\Omega &= -(\quad\ \ ,,\quad\ \ )\,\mathrm{R}', \\ -a'\mathrm{L} - b'\mathrm{M} - c'\mathrm{N} \quad . \quad &= -(\quad\ \ ,,\quad\ \ )\,\mathrm{S}'; \end{aligned}$$

and

$$\begin{aligned} . \quad hM - gN + a\Omega &= (af' + bg' + ch')\ \mathrm{P}, \\ -hL \quad . \quad + fN + b\Omega &= (\quad\ \ ,,\quad\ \ )\ \mathrm{Q}, \\ gL - fM \quad . \quad + c\Omega &= (\quad\ \ ,,\quad\ \ )\ \mathrm{R}, \\ -aL - bM - cN \quad . \quad &= (\quad\ \ ,,\quad\ \ )\ \mathrm{S}. \end{aligned}$$

14. $p^3a\,.\,a\beta\,.\,\gamma\delta = 0$ is the equation of the cubic surface through the lines $\alpha, \beta, \gamma, \delta$ and $a\alpha, a\gamma\delta$ (viz., $a\alpha\beta$ is the line from a to meet α, β, and so $a\gamma\delta$ is the line from a to meet γ, δ). Observe that the conditions which determine this cubic surface thus are that the cubic shall pass through

a; the points of $a\alpha\beta$ on α and β respectively, 3 other points on α, 3 on β, and 1 on $a\alpha\beta$;

also the points of $a\gamma\delta$ on γ and δ respectively, 3 other points on γ, 3 on δ, and 1 on $a\gamma\delta$; in all, $1+9+9 = 19$ points;

viz., the conditions completely determine the surface.

15. We have

$$p^3a\,.\,a\beta\,.\,\gamma\delta = \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ \mathrm{L}_{a\beta}, & \mathrm{M}_{a\beta}, & \mathrm{N}_{a\beta}, & \Omega_{a\beta} \\ \mathrm{L}_{\gamma\delta}, & \mathrm{M}_{\gamma\delta}, & \mathrm{N}_{\gamma\delta}, & \Omega_{\gamma\delta} \end{vmatrix},$$

viz., this determinant, equated to zero, gives the equation of the surface.

To prove this, take as before the unaccented letters (a, b, c, f, g, h) to refer to the line α, and the letters with one, two, and three accents to refer to the lines β, γ, δ respectively; write also L, M, N, Ω and L′, M′, N′, Ω' for $\mathrm{L}_{a\beta}$, &c., and $\mathrm{L}_{\gamma\delta}$, &c., respectively. Referring to the foregoing expressions for L, M, N, Ω, and observing that for a point on the line α, the values of P, Q, R, S are each $= 0$, then for such a point we have $\mathrm{L} + (a'f + b'g + c'h)\,x = 0$, &c., that is, $\mathrm{L} : \mathrm{M} : \mathrm{N} : \Omega = x : y : z : w$, and these values satisfy the equation of the surface, which is thus a surface passing through the line α; and similarly it passes through the lines β, γ, δ.

To show that the surface passes through the line $a\alpha\beta$, take the coordinates of the point a to be $0, 0, 0, 1$; then the line $a\alpha\beta$ is given as the intersection of the planes $ax + by + cz = 0$ and $a'x + b'y + c'z = 0$, that is, $\mathrm{S} = 0$ and $\mathrm{S}' = 0$. And the equation of the surface, writing therein $x_a, y_a, z_a, w_a = 0, 0, 0, 1$, becomes

$$\begin{vmatrix} x, & y, & z \\ \mathrm{L}, & \mathrm{M}, & \mathrm{N} \\ \mathrm{L}', & \mathrm{M}', & \mathrm{N}' \end{vmatrix} = 0,$$

or, as this may be written,

$$\begin{vmatrix} \mathrm{S}, & \mathrm{S}', & z \\ a\mathrm{L} + b\mathrm{M} + c\mathrm{N}, & a'\mathrm{L} + b'\mathrm{M} + c'\mathrm{N}, & \mathrm{N} \\ a\mathrm{L}' + b\mathrm{M}' + c\mathrm{N}', & a'\mathrm{L}' + b'\mathrm{M}' + c'\mathrm{N}', & \mathrm{N}' \end{vmatrix} = 0;$$

and, for a point on the line $a\alpha\beta$, this is

$$\begin{vmatrix} aL+bM+cN, & a'L+b'M+c'N \\ aL'+bM'+cN', & a'L'+b'M'+c'N' \end{vmatrix} = 0.$$

But in the equations $-a'L-b'M-c'N = -(a'f+b'g+c'h)\,S'$, and $-aL-bM-cN = (af'+bg'+ch')\,S$, writing $S=0$ and $S'=0$, we have $aL+bM+cN=0$ and $a'L+b'M+c'N=0$, and the equation is satisfied; that is, the surface passes through the line $a\alpha\beta$, and similarly it passes through the line $a\gamma\delta$.

Surface abcdef.

16. The equation may be written

$$pabe\,.\,pcde\,.\,pacf\,.\,pdbf - pabf\,.\,pcdf\,.\,pace\,.\,pdbe = 0,$$

where $pabe=0$ is the equation of the plane through the points a, b, e; and the like for the other symbols. The form is one out of 45 like forms, depending on the partitionment

$$\left\{\begin{matrix} ab\,.\,cd \\ ac\,.\,db \\ ad\,.\,bc \end{matrix}\right\}(ef),$$

of the six letters.

17. *Investigation.*—In the projection, the six points (p_a, q_a, r_a) are situate on a conic; the condition for this is

$$(p, q, r)^2 = 0,$$

where the left-hand side represents the determinant obtained by writing successively (p_a, q_a, r_a), &c., for (p, q, r). The equation in question may be written $abe\,.\,cde\,.\,acf\,.\,dbf - abf\,.\,cdf\,.\,ace\,.\,dbe = 0$;

where $$abe = \begin{vmatrix} p_a, & q_a, & r_a \\ p_b, & q_b, & r_b \\ p_e, & q_e, & r_e \end{vmatrix}$$ &c.;

and substituting for $p_a, \ldots$, their values, we have $abe = w^2\,.\,pabe$, whence the foregoing result.

18. *Singularities.*—The form of the equation shows at once that*

(0)† The point a is a 2-conical point; in fact, for this point we have $pabe=0$, $pacf=0$, $pabf=0$, $pace=0$.

(1) The line ab a simple line; in fact, for any point of this line we have $pabe=0$, $pabf=0$.

(2) The line $abe.cdf$ a simple line; in fact, for any point of this line we have $pabe=0$, $pcdf=0$.

(9) To show analytically that the cubic curve $abcdef$ is a line on

* Of course, as regards the present surface and the other surfaces for which the equation is given in an unsymmetrical form, the conclusion obtained in regard to any point or line of the surface applies to every point or line of the same kind. Thus $a5$ being a simple line, we have also ad a simple line, although the equation, as written down, does not put this in evidence.

† The bracketed numbers refer to the lines of the Table.

the surface, observe that the equation of the surface is satisfied if we have simultaneously (λ being arbitrary)

$$pabe \,.\, pacf - \lambda \,.\, pabf \,.\, pace = 0,$$
$$\lambda \,.\, pcde \,.\, pdbf - \quad pcdf \,.\, pdbe = 0.$$

The first of these equations is a cone, vertex a, which passes through the points b, e, c, f, and which, if λ is properly determined, will pass through the point d; the second is a cone, vertex d, which passes through the points b, e, c, f, and which, if λ is properly determined, will pass through the point a; the two determinations of λ are

$$dabe \,.\, dacf - \lambda \,.\, dabf \,.\, dace = 0,$$
$$\lambda \,.\, acde \,.\, adbf - \quad acdf \,.\, adbe = 0;$$

giving the same value of λ; and the equations then represent cones, the first having a for its vertex, and passing through d, b, e, c, f; the second having d for its vertex, and passing through a, b, e, c, f; the two intersect in the line ad, and in the cubic curve $abcdef$, which is thus a curve on the surface.

Surface abcdea.

19. The equation may be written

$$(pabe \,.\, pcde \,.\, p^2aac \,.\, db - pace \,.\, pdbe \,.\, p^2aab \,.\, cd)^2$$
$$+ 4pabe \,.\, pcde \,.\, pace \,.\, pdbe \,.\, pabc \,.\, pdbc \,.\, paa \,.\, pda = 0,$$

or what is the same thing,

$$(pabe \,.\, pcde \,.\, p^2aac \,.\, db + pace \,.\, pdbe \,.\, p^2aab \,.\, cd)^2$$
$$+ 4pabe \,.\, pcde \,.\, pace \,.\, pdbe \,.\, pbad \,.\, pcad \,.\, pba \,.\, pca = 0,$$

(the equivalence of the two depending on the identity

$$-p^2aab \,.\, cd \,.\, p^2aac \,.\, db$$
$$+ pabc \,.\, pdbc \,.\, paa \,.\, pda - pbad \,.\, pcad \;\; pba \,.\, pca = 0)$$

where, as before, $pabe = 0$ is the equation of the plane through the points a, b, e; $p^2aacdb = 0$ the equation of the quadric surface through the lines a, ac, db; and $paa = 0$ the equation of the plane through the point a and the line a.

The above forms are 2 out of 30 like forms, as appears by the partitionment

$$\left\{\begin{matrix} ab, & cd \\ ac, & db \\ ad, & bc \end{matrix}\right\} ea.$$

20. *Investigation.* — In the projection, the equation of the conic through the five points may be written

$$\left| \begin{matrix} (X, Y, Z)^2 \\ (p, \;\; q, \;\; r)^2 \end{matrix} \right| = 0,$$

where the symbol denotes a determinant the last five lines of which are obtained by giving to (p, q, r) the suffixes a, b, c, d, e respectively. This is at once transformed into

$$abe \,.\, cde \,.\, ac\Delta \,.\, db\Delta - ace \,.\, dbe \,.\, ab\Delta \,.\, cd\Delta = 0,$$

[Surface *abcdea*.]

or, what is the same thing,

$$pabe\,.\,pcde\,.\,ac\Delta\,.\,db\Delta - pace\,.\,pdbe\,.\,ab\Delta\,.\,cd\Delta = 0,$$

or say, $pabe\,.\,pcde\,(A''X+B''Y+C''Z)(A'''X+B'''Y+C'''Z)$

$$-pace\,.\,pdbe\,(AX+BY+CZ)(A'X+B'Y+C'Z),$$

where *pabe*, &c. signify as before, and

$$AX+BY+CZ = \begin{vmatrix} X, & Y, & Z \\ p_a, & q_a, & r_a \\ p_b, & q_b, & r_b \end{vmatrix}$$

and so for $A'X+B'Y+C'Z$, &c., the suffixes for A′, B′, C′ being (c, d), and those for A″, B″, C″ and A‴, B‴, C‴ being (a, c) and (d, b) respectively.

21. Passing to the reciprocal equation, and making the conic touch the line a, we obtain the equation of the surface in the form

$$\left\{ pabe\,.\,pcde \begin{vmatrix} P_a, & Q_a, & R_a \\ A'', & B'', & C'' \\ A''', & B''', & C''' \end{vmatrix} - pace\,.\,pdbe \begin{vmatrix} P_a, & Q_a, & R_a \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} \right\}^2$$

$$+4pace\,.\,pdbe\,.\,pabe\,.\,pcde \begin{vmatrix} P_a, & Q_a, & R_a \\ A, & B, & C \\ A'', & B'', & C'' \end{vmatrix} \begin{vmatrix} P_a, & Q_a, & R_a \\ A', & B', & C' \\ A''', & B''', & C''' \end{vmatrix} = 0,$$

(where $P_a = h_a y - g_a z + a_a w = 0$) or in the equivalent form wherein we have in the first term + instead of −, and in the second term the determinants

$$\begin{vmatrix} P_a, & Q_a, & R_a \\ A, & B, & C \\ A''', & B''', & C''' \end{vmatrix}, \begin{vmatrix} P_a, & Q_a, & R_a \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix}$$

22. [The question, in fact, is to find the reciprocal of the form

$$\lambda(ax+by+cz)(a'x+b'y+c'z) - \mu(a''x+b''y+c''z)(a'''x+b'''y+c'''z) = 0;$$

taking ξ, η, ζ for the reciprocal variables, the coefficient of ξ^2 is

$$\{\lambda(bc'+b'c) - \mu(b''c'''+b'''c'')\}^2 - (2\lambda bb' - 2\mu b''b''')(2\lambda cc' - 2\mu c''c'''),$$

viz., this is

$$\lambda^2(bc'-b'c)^2 + \mu^2(b''c'''-b'''c'')^2$$

$$+2\lambda\mu\{2bb'c''c''' + 2b''b'''cc' - (bc'+b'c)(b''c'''+b'''c'')\},$$

or, as it may be written,

$$\{\lambda(bc'-b'c) \pm \mu(b''c'''-b'''c'')\}^2$$

$$+2\lambda\mu \left\{ \begin{matrix} 2bb'c''c''' + 2b''b'''cc' \\ \mp(bc-b'c)(b''c'''-b'''c'') \\ -(bc'+b'c)(b''c'''+b'''c'') \end{matrix} \right\}.$$

Taking the upper signs, this is

$$\{\lambda(bc'-b'c) + \mu(b''c'''-b'''c'')\}^2$$

$$+4\lambda\mu \left(\begin{matrix} bb'c''c''' + b''b'''cc' \\ -bc'b''c''' - b'cb'''c'' \end{matrix} \right);$$

viz., the term in $\lambda\mu$ is $= +4\lambda\mu\,(bc'''-b'''c)(b'c''-b''c')$.

[Surface *abcdea*.]

Taking the lower signs, it is

$$\{\lambda(bc'-b'c)-\mu(b''c'''-b'''c'')\}^2 + 4\lambda\mu\begin{pmatrix} bb'c''c''' + b''b'''cc' \\ -bc'b'''c''-b'cb''c''' \end{pmatrix};$$

viz., the term in $\lambda\mu$ is $4\lambda\mu(bc''-b''c)(b'c'''-b'''c')$;

and it is thence easy to infer the forms of the other coefficients and to obtain the reciprocal equation in the two equivalent forms

$$\{\lambda\begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} + \mu\begin{vmatrix} \xi, & \eta, & \zeta \\ a'', & b'', & c'' \\ a''', & b''', & c''' \end{vmatrix}\}^2 + 4\lambda\mu\begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a''', & b''', & c''' \end{vmatrix}\begin{vmatrix} \xi, & \eta, & \zeta \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0,$$

$$\{\lambda\begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} - \mu\begin{vmatrix} \xi, & \eta, & \zeta \\ a'', & b'', & c'' \\ a''', & b''', & c''' \end{vmatrix}\}^2 + 4\lambda\mu\begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a'', & b'', & c'' \end{vmatrix}\begin{vmatrix} \xi, & \eta, & \zeta \\ a', & b', & c' \\ a''', & b''', & c''' \end{vmatrix} = 0,$$

which are the required auxiliary formulæ.]

23. To reduce the foregoing result, we have

$$\mathrm{A, B, C} = \left\| \begin{matrix} xw_a - wx_a, & yw_a - wy_a, & zw_a - wz_a \\ xw_b - wx_b, & yw_b - wy_b, & zw_b - wz_b \end{matrix} \right\|$$

proportional to the three determinants which contain w, of the set

$$\left\| \begin{matrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ x_b, & y_b, & z_b, & w_b \end{matrix} \right\|, \quad \text{viz., } \mathrm{A} = w\begin{vmatrix} y, & z, & w \\ y_a, & z_a, & w_a \\ y_b, & z_b, & w_b \end{vmatrix}, \text{ &c.};$$

and similarly A′, B′, C′ are proportional to the three determinants which contain w, of the set

$$\left\| \begin{matrix} x, & y, & z, & w \\ x_c, & y_c, & z_c, & w_c \\ x_d, & y_d, & z_d, & w_d \end{matrix} \right\|, \quad \text{viz., } \mathrm{A}' = w\begin{vmatrix} y, & z, & w \\ y_c, & z_c, & w_c \\ y_d, & z_d, & w_d \end{vmatrix}, \text{ &c.}$$

Hence, omitting the factor w, and writing (a, b, c, f, g, h) and (a′, b′, c′, f′, g′, h′) for the coordinates of the lines ab and cd respectively, we have

$$\begin{aligned} \mathrm{A} &= \mathrm{h}y - \mathrm{g}z + \mathrm{a}w, & \mathrm{A}' &= \mathrm{h}'y - \mathrm{g}'z + \mathrm{a}'w, \\ \mathrm{B} &= -\mathrm{h}x + \mathrm{f}z + \mathrm{b}w, & \mathrm{B}' &= -\mathrm{h}'x + \mathrm{f}'z + \mathrm{b}'w, \\ \mathrm{C} &= \mathrm{g}x - \mathrm{f}y + \mathrm{c}w, & \mathrm{C}' &= \mathrm{g}'x - \mathrm{f}'y + \mathrm{c}'w; \end{aligned}$$

and thence

$$\begin{aligned} \mathrm{BC}' - \mathrm{B}'\mathrm{C} &= \Omega x - \mathrm{L}w, \\ \mathrm{CA}' - \mathrm{C}'\mathrm{A} &= \Omega y - \mathrm{M}w, \\ \mathrm{AB}' - \mathrm{A}'\mathrm{B} &= \Omega z - \mathrm{N}w, \end{aligned}$$

where

$$\begin{aligned} \mathrm{L} &= (\mathrm{af}' - \mathrm{a}'\mathrm{f})\,x + (\mathrm{bf}' - \mathrm{b}'\mathrm{f})\,y + (\mathrm{cf}' - \mathrm{c}'\mathrm{f})\,z - (\mathrm{bc}' - \mathrm{b}'\mathrm{c})\,w, \\ \mathrm{M} &= (\mathrm{ag}' - \mathrm{a}'\mathrm{g})\,x + (\mathrm{bg}' - \mathrm{b}'\mathrm{g})\,y + (\mathrm{cg}' - \mathrm{c}'\mathrm{g})\,z - (\mathrm{ca}' - \mathrm{c}'\mathrm{a})\,w, \\ \mathrm{N} &= (\mathrm{ah}' - \mathrm{a}'\mathrm{h})\,x + (\mathrm{bh}' - \mathrm{b}'\mathrm{h})\,y + (\mathrm{ch}' - \mathrm{c}'\mathrm{h})\,z - (\mathrm{ab}' - \mathrm{a}'\mathrm{b})\,w, \\ \Omega &= (\mathrm{gh}' - \mathrm{g}'\mathrm{h})\,x + (\mathrm{hf}' - \mathrm{h}'\mathrm{f})\,y + (\mathrm{fg}' - \mathrm{f}'\mathrm{g})\,z \\ &\qquad - (\mathrm{af}' - \mathrm{a}'\mathrm{f} + \mathrm{bg}' - \mathrm{b}'\mathrm{g} + \mathrm{ch}' - \mathrm{c}'\mathrm{h})\,w; \end{aligned}$$

[Surface $a'bcde\,a$.]

and consequently

$$\begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} = \Omega\,(xP_\alpha + yQ_\alpha + zR_\alpha) - w\,(LP_\alpha + MQ_\alpha + NR_\alpha)$$

$$= -w\,(LP_\alpha + MQ_\alpha + NR_\alpha + \Omega S_\alpha);$$

or omitting the factor $-w$, say it is $= LP_\alpha + MQ_\alpha + NR_\alpha + \Omega S_\alpha$, viz., this is $= p^2\alpha\, ab\,.\,cd$.

We have similarly $\begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A'', & B'', & C'' \\ A''', & B''', & C''' \end{vmatrix}$ taken to be $= p^2\alpha\, ac\,.\,db$.

24. We have in like manner the other two determinants

$\begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A, & B, & C \\ A'', & B'', & C'' \end{vmatrix}$ and $\begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A', & B', & C' \\ A''', & B''', & C''' \end{vmatrix}$ to be $= p^2\alpha\, ab\,.\,ac$ and $p^2\alpha\, cd\,.\,db$ respectively.

But we have $p^2\alpha\, ab\,.\,ac = pa\alpha\,.\,pabc$,

(viz., geometrically the hyperboloid through the lines α, ab, ac breaks up into the plane $pa\alpha$ through the line α and point a, and the plane $pabc$ through the points a, b, c).

And similarly

$$p^2\alpha\, cd\,.\,db = -p^2\alpha\, dc\,.\,db = +p^2\alpha\, db\,.\,dc = pad\,.\,pdbc;$$

whence, substituting for the several determinants, we have the foregoing equation of the surface.

25. *Singularities.*—The form of the equation shows that—

(0) The point a is a 4-conical point: in fact, for this point we have $pabe = 0$, $p^2\alpha\, ac\,.\,db = 0$, $pace = 0$, $p^2\alpha\, ab\,.\,cd = 0$.

(1) The line ab a double line: in fact, for any point of the line we have $pabe = 0$, $p^2\alpha\, ab\,.\,cd = 0$, $pabc = 0$.

(2) The line α a double line: in fact, for any point of the line we have $p^2\alpha\, ac\,.\,db = 0$, $p^2\alpha\, ab\,.\,cd = 0$, $pa\alpha = 0$, $pd\alpha = 0$.

(7) The line $abe\,.\,cd\,.\,\alpha$ a simple line: in fact, for any point of the line we have $pabe = 0$, $p^2\alpha\, ab\,.\,cd = 0$. Observe that, on writing in the equation $pabe = 0$, the equation becomes $(p^2\alpha\, ab\,.\,cd)^2 = 0$; so that the surface along the line in question touches the plane $pabe$.

Surface abcdaβ.

26. The equation of the surface is

$$\text{Norm}\,\{\sqrt{pa\alpha\,.\,pa\beta}\,.\,pbcd - \sqrt{pb\alpha\,.\,pb\beta}\,.\,pcda + \sqrt{pc\alpha\,.\,pc\beta}\,.\,pdab - \sqrt{pd\alpha\,.\,pd\beta}\,.\,pabc\} = 0,$$

where the norm is the product of 8 factors.

As before, $paa=0$ is the equation of the plane through the point a and the line a; and $pbcd=0$ the equation of the plane through the points b, c, d. The form is unique.

27. *Investigation.*—In the projection, the equation of the conic touching the projections of the lines α, β is

$$\sqrt{(P_\alpha X+Q_\alpha Y+R_\alpha Z)(P_\beta X+Q_\beta Y+R_\alpha Z)}+AX+BY+CZ=0,$$

where A, B, C are arbitrary coefficients. To make this pass through the projection of the point a, we must write $X:Y:Z=p_a:q_a:r_a$; viz., we thus have

$$\begin{aligned} P_\alpha X+Q_\alpha Y+R_\alpha Z = &\ w_a(xP_\alpha+yQ_\alpha+zR_\alpha) \\ &-w\ (x_aP_\alpha+y_aQ_\alpha+z_aR_\alpha) \\ =&-w\ (x_aP_\alpha+y_aQ_\alpha+z_aR_\alpha+w_aS_\alpha) \\ =&-w\,.\,paa\,; \end{aligned}$$

and similarly

$$P_\beta X+Q_\beta Y+R_\alpha Z=-w\,.\,pa\beta.$$

We thus have $w\sqrt{paa\,.\,pa\beta}+Ap_a+Bq_a+Cr_a=0.$

Or forming the like equations for the points b, c, d respectively, and eliminating, the equation is

$$\begin{vmatrix} \sqrt{paa\,.\,pa\beta}, & p_a, & q_a, & r_a \\ \sqrt{pba\,.\,pb\beta}, & p_b, & q_b, & r_b \\ \sqrt{pca\,.\,pc\beta}, & p_c, & q_c, & r_c \\ \sqrt{pda\,.\,pd\beta}, & p_d, & q_d, & r_d \end{vmatrix}=0;$$

which, substituting for (p_a, q_a, r_a), &c., their values, viz., $p_a=xw_a-x_aw$, &c., is readily converted into

$$\begin{vmatrix} & x, & y, & z, & w \\ \sqrt{paa\,.\,pa\beta}, & x_a, & y_a, & z_a, & w_a \\ \sqrt{pba\,.\,pb\beta}, & x_b, & y_b, & z_b, & w_b \\ \sqrt{pca\,.\,pc\beta}, & x_c, & y_c, & z_c, & w_c \\ \sqrt{pda\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix}=0.$$

or, what is the same thing,

$$\begin{aligned} &\sqrt{paa\,.\,pa\beta}\,.\,pbcd-\sqrt{pba\,.\,pb\beta}\,.\,pcda \\ &\qquad+\sqrt{pca\,.\,pc\beta}\,.\,pdab-\sqrt{pda\,.\,pd\beta}\,.\,pabc=0\,; \end{aligned}$$

or, taking the norm, we have the form mentioned above.

28. *Singularities.*—The equation shows that—

(0) The point a is an 8-conical point; in fact, for the point in question $paa=0$, $pa\beta=0$, $pcda=0$, $pdab=0$, $pabc=0$; each factor is of the form 0^1, and the norm is 0^8.

(1) The line ab is a 4-tuple line. To show this, observe in the first

instance, that we may obtain the 8 factors of the norm by giving to the radical $\sqrt{paa\,.\,pa\beta}$ the sign +, and to the other three radicals the signs +, −, at pleasure. For a point on the line in question, we have $pdab = 0$, $pabc = 0$; hence the norm is the product of the four equal factors

$$\sqrt{paa\,.\,pa\beta}\,.\,pbcd - \sqrt{pba\,.\,pb\beta}\,.\,pcda,$$

and the other four equal factors obtained by writing herein + instead of −.

Now for a point on the line ab, we may write for x, y, z, w the values ux_a+vx_b, uy_a+vy_b, uz_a+vz_b, uw_a+vw_b, where u, v are arbitrary coefficients. We have

$$\begin{aligned}
paa &= u\,.\,aaa + v\,.\,baa &&= v\,.\,baa &&= -v\,.\,aba,\\
pa\beta & &&= v\,.\,ba\beta &&= -v\,.\,ab\beta,\\
pba &= u\,.\,aba + v\,.\,bba &&= u\,.\,aba, & &\\
pb\beta & &&= u\,.\,ab\beta, & &\\
pbcd &= u\,.\,abcd + v\,.\,bbcd &&= u\,.\,abcd, & &\\
pcda &= u\,.\,acda + v\,.\,bcda &&= v\,.\,bcda &&= -v\,.\,abcd,
\end{aligned}$$

where $aba = 0$ is the condition that the points a, b and the line α may be in the same plane (or, what is the same thing, that the lines ab and α may intersect), viz., baa is $= P_a x_b + Q_a y_b + R_a z_b + S_a w_b$. And similarly $abcd = 0$ is the condition that the four points a, b, c, d may be in a plane; viz., we have

$$abcd = \begin{vmatrix} x_a, & y_a, & z_a, & w_a \\ x_b, & y_b, & z_b, & w_b \\ x_c, & y_c, & z_c, & w_c \\ x_d, & y_d, & z_d, & w_d \end{vmatrix}.$$

Substituting, we have $\sqrt{paa\,.\,pa\beta}\,.\,pbcd$ and $\sqrt{pba\,.\,pb\beta}\,.\,pcda$, each equal (save as to sign) to $uv\sqrt{aba\,.\,ab\beta}\,.\,abcd$; that is, the four equal factors of one set will vanish. The vanishing factors are of the form 0^1, and the norm is 0^4, that is, the line in question, ab, is a 4-tuple line.

(2) The line α is a 4-tuple line; in fact, for any point of the line we have $paa = 0$, $pba = 0$, $pca = 0$, $pda = 0$; each factor of the norm is therefore evanescent, of the form 0^1, and the norm itself is thus $= 0^4$.

29. (5) The line (ab, cd, α, β) is a double line. To show this, take $z = 0$, $w = 0$ as the equations of the line in question; then we have $h_\alpha = 0$, $h_\beta = 0$, $z_a w_b - z_b w_a = 0$; or say $w_a = \lambda z_a$, $w_b = \lambda z_b$: and $z_c w_d - z_d w_c = 0$; or say $w_c = \mu z_c$, $w_d = \mu z_d$ (λ and μ arbitrary coefficients). Putting for shortness

$$\mathrm{I} = (g-\lambda a)\,x - (f+\lambda b)\,y, \quad \mathrm{J} = (g-\mu a)\,x - (f+\mu b)\,y;$$

viz., $\mathrm{I}_a = (g_a - \lambda a_a)\,x - (f_a + \lambda b_a)\,y$, &c.; and writing $z = 0$, $w = 0$, we have $paa\,.\,pa\beta = z_a^2 \mathrm{I}_a \mathrm{I}_\beta$, $pba\,.\,pb\beta = z_b^2 \mathrm{I}_a \mathrm{I}_\beta$, $pca\,.\,pc\beta = z_c^2 \mathrm{J}_a \mathrm{J}_\beta$, $pda\,.\,pd\beta = z_d^2 \mathrm{J}_a \mathrm{J}_\beta$;

[Surface $abcd\,a\beta$.]

and the factor of the norm (reverting to the expression thereof as a determinant) is

$$\begin{vmatrix} & x, & y & & \\ z_a\sqrt{I_aI_\beta}, & x_a, & y_a, & z_a, & \lambda z_a \\ z_b\sqrt{I_aI_\beta}, & x_b, & y_b, & z_b, & \lambda z_b \\ z_c\sqrt{J_aJ_\beta}, & x_c, & y_c, & z_c, & \mu z_c \\ z_d\sqrt{J_aJ_\beta}, & x_d, & y_d, & z_d, & \mu z_d \end{vmatrix}$$

which vanishes. In fact, resolving the determinant into a set of products of the form $\pm 2.13.45$, where the single symbol denotes a term of the top line, and the binary symbols refer to the second and third lines, and the fourth and fifth lines respectively (denoting minors composed with the terms in these pairs of lines respectively); then each product will contain a term 14, 15, or 45, and the minor so designated (to whichever of the two pairs of lines it belongs) is $=0$. The factor is thus evanescent, being, as it is easy to see, $=0^1$. There are two factors which vanish; viz., taking the first radical to be $+$, the second radical must be also $+$, but the third and fourth radicals may be either both $+$ or both $-$; the norm is thus $=0^2$, viz., the line (ab, cd, α, β) is a double line.

30. (8) The line abc, α, β is a double line. To prove this, take $w=0$ for the equation of the plane abc, and $(z=0, w=0)$ for those of the line in question; we have $h_\alpha=0$, $h_\beta=0$, $w_a=0$, $w_b=0$, $w_c=0$; and writing $I_\alpha=-g_\alpha x+f_\alpha y$, $I_\beta=-g_\beta x+f_\alpha y$, then for $z=0$, $w=0$, the factor expressed as a determinant is

$$\begin{vmatrix} & x, & y, & . & . \\ z_a\sqrt{I_aI_\beta}, & x_a, & y_a, & z_a, & . \\ z_b\sqrt{I_aI_\beta}, & x_b, & y_b, & z_b, & . \\ z_c\sqrt{I_aI_\beta}, & x_c, & y_c, & z_c, & . \\ \sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix}$$

which is

$$= w_d\sqrt{I_aI_\beta}\begin{vmatrix} . & x, & y, & . \\ z_a, & x_a, & y_a, & z_a \\ z_b, & x_b, & y_b, & z_b \\ z_c, & x_c, & y_c, & z_c \end{vmatrix}$$

and consequently vanishes, the form being 0^1. There are two such factors, viz., the radical $\sqrt{pd\alpha\,.\,pd\beta}$ may be either $+$ or $-$, hence the norm is $=0^2$.

31. But it is to be further shown that the line is tacnodal, each sheet of the surface being touched along the line by the plane $w=0$: we have to show that the factor operated upon by $\Delta = X\delta_x+Y\delta_y+Z\delta_z+W\delta_w$,

[Surface $abcd\alpha\beta$.]

'educes itself for $z=0$, $w=0$ to a multiple of W. Considering the 'actor in the form of a determinant, the result of the operation is

$$\begin{vmatrix} & X, & Y, & Z, & W \\ \sqrt{p\alpha\alpha \,.\, p\alpha\beta}, & x_a, & y_a, & z_a, & . \\ \sqrt{pb\alpha \,.\, pb\beta}, & x_b, & y_b, & z_b, & . \\ \sqrt{pc\alpha \,.\, pc\beta}, & x_c, & y_c, & z_c, & . \\ \sqrt{pd\alpha \,.\, pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix} + \begin{vmatrix} & x, & y, & . & . \\ \Delta\sqrt{p\alpha\alpha \,.\, p\alpha\beta}, & x_a, & y_a, & z_a, & . \\ \Delta\sqrt{pb\alpha \,.\, pb\beta}, & x_b, & y_b, & z_b, & . \\ \Delta\sqrt{pc\alpha \,.\, pc\beta}, & x_c, & y_c, & z_c, & . \\ \Delta\sqrt{pd\alpha \,.\, pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix};$$

the first term is

$$\begin{vmatrix} & X, & Y, & Z, & W \\ z_a\sqrt{I_\alpha I_\beta}, & x_a, & y_a, & z_a, & . \\ z_b\sqrt{I_\alpha I_\beta}, & x_b, & y_b, & z_b, & . \\ z_c\sqrt{I_\alpha I_\beta}, & x_c, & y_c, & z_c, & . \\ \sqrt{pd\alpha \,.\, pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix},$$

where the first column may be replaced by

$$\begin{matrix} -Z\sqrt{I_\alpha I_\beta} \\ \vdots \\ \sqrt{pd\alpha \,.\, pd\beta} - z_d\sqrt{I_\alpha I_\beta}, \end{matrix}$$

and the term in question thus becomes

$$\{w_d Z\sqrt{I_\alpha I_\beta} + W(-z_d\sqrt{I_\alpha I_\beta} + \sqrt{pd\alpha \,.\, pd\beta})\} \,.\, abc,$$

if for shortness

$$\begin{vmatrix} x_a, & y_a, & z_a \\ x_b, & y_b, & z_b \\ x_c, & y_c, & z_c \end{vmatrix} = abc.$$

As regards the second term, we have

$$\Delta\sqrt{p\alpha\alpha \,.\, p\alpha\beta} = \frac{p\alpha\alpha \,.\, \Delta p\alpha\beta + p\alpha\beta \,.\, \Delta p\alpha\alpha}{2\sqrt{p\alpha\alpha \,.\, p\alpha\beta}},$$

which is

$$= \frac{I_\alpha \Delta p\alpha\beta + I_\beta \Delta p\alpha\alpha}{2\sqrt{I_\alpha I_\beta}}.$$

But $$p\alpha\alpha = x(-g_a z_a) + y(f_a z_a) + z(g_a x_a - f_a y_a) + w(a_a x_a - b_a y_a - c_a z_a)$$
$$= x_a(g_a z - a_a w) + y_a(-f_a y - b_a w) + z_a(-g_a x + f_a y - c_a w);$$

and thence

$$\Delta p\alpha\alpha = x_a(g_a Z - a_a W) + y_a(-f_a Z - b_a W) + z_a(-g_a X + f_a Y - c_a W),$$

with the like formula for $\Delta p\alpha\beta$; hence

$$\frac{I_\alpha \Delta p\alpha\beta + I_\beta \Delta p\alpha\alpha}{2\sqrt{I_\alpha I_\beta}} = Ax_a + By_a + Cz_a.$$

[Surface *abcd* $\alpha\beta$.]

where

$$A = \frac{1}{2\sqrt{I_\alpha I_\beta}} \{I_\beta(g_\alpha Z - a_\alpha W) + I_\alpha(g_\beta Z - a_\beta W)\},$$

$$B = \frac{1}{2\sqrt{I_\alpha I_\beta}} \{I_\beta(-f_\alpha Z - b_\alpha W) + I_\alpha(-f_\beta Z - b_\beta W)\},$$

$$C = \frac{I}{2\sqrt{I_\alpha I_\beta}} \{I_\beta(-g_\alpha X + f_\alpha Y - c_\alpha W) + I_\alpha(-g_\beta X + f_\beta Y - c_\delta W)\}.$$

The term in question is thus

$$\begin{vmatrix} & x, & y, & . & . \\ Ax_a + By_a + Cz_a, & x_a, & y_a, & z_a, & . \\ Ax_b + By_b + Cz_b, & x_b, & y_b, & z_b, & . \\ Ax_c + By_c + Cz_a, & x_c, & y_c, & z_c, & . \\ \Delta\sqrt{pd\alpha \,.\, pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix}$$

viz., replacing the first column by

$$\begin{matrix} -Ax - By \\ \vdots \\ \Delta\sqrt{pd\alpha \,.\, pd\beta} - Ax_d - By_d - Cx_d; \end{matrix}$$

this is $= (Ax + By)\, w_d \,.\, abc;$

and we have

$$Ax + By = \frac{1}{2\sqrt{I_\alpha I_\beta}} \begin{matrix} [\ I_\beta(\ g_\alpha x - f_\alpha y) + I_\alpha(\ g_\beta x - f_\beta y)]\, Z \\ [+I_\beta(-a_\alpha x - b_\alpha y) + I_\alpha(-a_\beta x - b_\beta y)]\, W \end{matrix}$$

$$= \frac{1}{2\sqrt{I_\alpha I_\beta}} (-2I_\alpha I_\beta Z - MW),$$

if for shortness

$$M = (-g_\beta x + f_\beta y)(a_\alpha x + b_\alpha y) + (-g_\alpha x + f_\alpha y)(a_\beta x + b_\beta y);$$

viz., the whole term is

$$w_d \left\{ -\sqrt{I_\alpha I_\beta}\, Z - \frac{\frac{1}{2}M}{\sqrt{I_\alpha I_\beta}} W \right\} abc.$$

Hence the first and second terms together are

$$= W \left\{ -z_d \sqrt{I_\alpha I_\beta} + \sqrt{pd\alpha \,.\, pd\beta} - \frac{\frac{1}{2}M}{\sqrt{I_\alpha I_\beta}} w_d \right\} abc;$$

viz., this is a multiple of W, which was the theorem to be proved.

Surface $abc\alpha\beta\gamma$.

32. The equation is

$$\text{Norm} \begin{vmatrix} \sqrt{pa\alpha}, & \sqrt{pb\alpha}, & \sqrt{pc\alpha} \\ \sqrt{pa\beta}, & \sqrt{pb\beta}, & \sqrt{pc\beta} \\ \sqrt{pa\gamma}, & \sqrt{pb\gamma}, & \sqrt{pc\gamma} \end{vmatrix} = 0,$$

[Surface $abc\alpha\beta\gamma$.]

where the norm is a product of 16 factors, each of the order $\frac{3}{2}$. As before, $paa = 0$ is the equation of the plane through the point a and the line α; viz., paa has the value already mentioned.

33. *Investigation.* — In the projection, the equation of the conic touching the projections of the lines α, β, γ is

$$A\sqrt{P_\alpha X + Q_\alpha Y + R_\alpha Z} + B\sqrt{P_\beta X + Q_\beta Y + R_\beta Z} + C\sqrt{P_\gamma X + Q_\gamma Y + R_\gamma Z} = 0;$$

and to make this pass through the projection of the point a, we must write herein $X : Y : Z = p_a : q_a : r_a$. As before, we have

$$\begin{aligned} P_\alpha X + Q_\alpha Y + R_\alpha Z = {} & w_a (x P_\alpha + y Q_\alpha + z R_\alpha) \\ & - w (x_a P_\alpha + y_a Q_\alpha + z_a R_\alpha) \\ = {} & - w (x_a P_\alpha + y_a Q_\alpha + z_a R_\alpha + w_a S_\alpha) \\ = {} & - w \,.\, paa ; \end{aligned}$$

and so for the other terms; the equation thus is

$$A\sqrt{paa} + B\sqrt{pa\beta} + C\sqrt{pa\gamma} = 0;$$

or forming the like equations in regard to the points b, c respectively, and eliminating, we have a determinant $= 0$, and then, taking the norm, we obtain the above-written equation of the surface.

34. *Singularities.*—The equation of the surface shows that—

(0) The point a is 8-conical: in fact, for the point in question we have $paa = 0$, $pa\beta = 0$, $pa\gamma = 0$; each factor is $0^{\frac{1}{2}}$, and the norm is 0^8.

(1) The line ab is 4-tuple. To prove this, observe that the sixteen factors are obtained by attributing at pleasure the signs $+, -$ to the radicals $\sqrt{pb\beta}$, $\sqrt{pc\beta}$, $\sqrt{pb\gamma}$, $\sqrt{pc\gamma}$; hence there are four factors in which $\sqrt{pb\beta}$, $\sqrt{pb\gamma}$ have determinate signs, but in which we attribute to the radicals $\sqrt{pc\beta}$, $\sqrt{pc\gamma}$ the signs $+$ or $-$ at pleasure. It is to be shown that the four factors each vanish for a point on the line ab; that is, on writing therein for x, y, z, w that values $ux_a + vx_b$, $uy_a + vy$, &c. But we thus have, as before, $paa = -v \,.\, aba$ and $pba = u \,.\, aba$, with the like formulæ with β and γ in place of α. The factor thus becomes

$$\sqrt{-uv} \left| \begin{matrix} \sqrt{aba}, & \sqrt{aba}, & \sqrt{pca} \\ \sqrt{ab\beta}, & \sqrt{ab\beta}, & \sqrt{pc\beta} \\ \sqrt{ab\gamma}, & \sqrt{ab\gamma}, & \sqrt{pc\gamma} \end{matrix} \right|,$$

which vanishes, being $= 0^1$; and the norm is thus $= 0^4$, viz., the line is 4-tuple.

[Surface $abc\alpha\beta\gamma$.]

(2) The line a is 8-tuple: in fact, for a point on the line we have $paa=0$, $pba=0$, $pca=0$, whence each factor vanishes being $=0^{\frac{1}{2}}$, and the norm is therefore 0^8.

(3) The line $(ab, \alpha, \beta, \gamma)$ is 4-tuple: in fact, writing $z=0$, $w=0$ for the equations of the line, we have $h_a=0$, $h_\beta=0$, $h_\gamma=0$, and $z_a w_b - z_b w_a = 0$, or say $w_a=\lambda z_a$, $w_b=\lambda z_b$. Hence, writing

$$\mathrm{I} = (g-\lambda a)\,x-(f+\lambda b)\,y,$$

viz., $\mathrm{I}_a = (g_a-\lambda a_a)\,x-(f_a+\lambda b_a)\,y$, &c., for $z=0$, $w=0$, we have $paa = z_a \mathrm{I}_a$, $pba = z_b \mathrm{I}_a$; and similarly $pa\beta = z_a \mathrm{I}_\beta$, $pb\beta = z_b \mathrm{I}_\beta$, and $pa\gamma = z_a \mathrm{I}_\gamma$, $pb\gamma = z_b \mathrm{I}_\gamma$. The factor thus is

$$\sqrt{z_a z_b} \left| \begin{matrix} \sqrt{\mathrm{I}_a}, & \sqrt{\mathrm{I}_a}, & \sqrt{pca} \\ \sqrt{\mathrm{I}_\beta}, & \sqrt{\mathrm{I}_\beta}, & \sqrt{pc\beta} \\ \sqrt{\mathrm{I}_\gamma}, & \sqrt{\mathrm{I}_\gamma}, & \sqrt{pc\gamma} \end{matrix} \right|$$

which vanishes, being $=0^1$; there are four such factors, or the norm is 0^4; whence the line is 4-tuple.

(8) The line $abc\,.\,\alpha\,.\,\beta$ is a 4-tuple line. To prove it, take as before $w=0$ for the equation of the plane abc, and $(z=0, w=0)$ for the equations of the line in question. We have $h_a=0$, $h_\beta=0$, $w_a=0$, $w_b=0$, $w_c=0$; whence (if $z=0$, $w=0$), writing for shortness $\mathrm{I} = gx-fy$ (viz., $\mathrm{I}_a = g_a x - f_a y$, $\mathrm{I}_\beta = g_\beta x - f_\beta y$), we have paa, pba, $pca = \mathrm{I}_a z_a$, $\mathrm{I}_a z_b$, $\mathrm{I}_a z_c$, and similarly $pa\beta$, $pb\beta$, $pc\beta = \mathrm{I}_\beta z_a$, $\mathrm{I}_\beta z_b$, $\mathrm{I}_\beta z_c$: the factor thus is

$$\left| \begin{matrix} \sqrt{\mathrm{I}_a z_a}, & \sqrt{\mathrm{I}_a z_b}, & \sqrt{\mathrm{I}_a z_c} \\ \sqrt{\mathrm{I}_\beta z_a}, & \sqrt{\mathrm{I}_\beta z_b}, & \sqrt{\mathrm{I}_\beta z_c} \\ \sqrt{pa\gamma}, & \sqrt{pb\gamma}, & \sqrt{pc\gamma} \end{matrix} \right|$$

which vanishes, being $=0^1$: and there are four such factors, obtained by giving to the radicals the signs $+$, $-$ at pleasure: hence the norm is $=0^4$.

Surface $aba\beta\gamma\delta$.

35. The equation is

$$\mathrm{Norm}\,\{\sqrt{paa\,.\,pba}\,.\,p^2\beta\gamma\delta - \sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta a + \sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta a\beta - \sqrt{pa\delta\,.\,pb\delta}\,.\,p^2 a\beta\gamma\} = 0,$$

where the norm is the product of 8 factors each of the order 3. As before, $paa=0$ is the equation of the plane through the point a and the line a; viz., paa has the value previously mentioned: and $p^2\beta\gamma\delta = 0$ is the equation of the quadric surface through the lines β, γ, δ.

36. *Investigation.*—In the projection, taking ξ, η, ζ as current line-

coordinates, the equation of the conic passing through the projections of the points a, b is

$$\sqrt{(p_a\xi+q_a\eta+r_a\zeta)(p_b\xi+q_b\eta+r_b\zeta)}+A\xi+B\eta+C\zeta=0,$$

where A, B, C are arbitrary coefficients. To make this touch the projection of the line a, we must write $\xi:\eta:\zeta=P_a:Q_a:R_a$; and then

$$\begin{aligned} p_a\xi+q_a\eta+r_a\zeta = &\quad p_aP_a+q_aQ_a+r_aR_a \\ = &\quad w_a(xP_a+yQ_a+zR_a) \\ &- w(x_aP_a+y_aQ_a+z_aR_a) \\ = &- w(x_aP_a+y_aQ_a+z_aR_a+w_aS_a) \\ = &- w\,.\,paa, \end{aligned}$$

and similarly $\qquad p_b\xi+q_b\eta+r_b\zeta=-w\,.\,pba.$

Whence the equation is

$$w\sqrt{paa\,.\,pba}+AP_a+BQ_a+CR_a=0;$$

and forming the like equations for the lines β, γ, δ respectively, and eliminating, we have

$$\begin{vmatrix} \sqrt{paa\,.\,pba}, & P_a, & Q_a, & R_a \\ \sqrt{pa\beta\,.\,pb\beta}, & P_\beta, & Q_\beta, & R_\beta \\ \sqrt{pa\gamma\,.\,pb\gamma}, & P_\gamma, & Q_\gamma, & R_\gamma \\ \sqrt{pa\delta\,.\,pb\delta}, & P_\delta, & Q_\delta, & R_\delta \end{vmatrix} = 0;$$

which, throwing out a factor w, becomes

$$\begin{aligned} &\sqrt{paa\,.\,pba}\,.\,p^2\beta\gamma\delta-\sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta a \\ &\qquad+\sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta a\beta-\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2a\beta\gamma=0; \end{aligned}$$

or, taking the norm, we have the above written equation.

37. *Singularities.*—The equation shows that

(0) The point a is a 4-conical point; in fact, for the point in question we have $paa=0$, $pa\beta=0$, $pa=0$, $pa\delta=0$; each factor is $=0^{\frac{1}{2}}$, and the norm is $=0^4$.

(1) The line ab is a 2-tuple line. To prove this, we have for the coordinates of a point on the line in question ux_a+vx_b, uy_a+vy_b, &c.; the values of paa, pba become as before $-v\,.\,aba$, $+u\,.\,aba$, and similarly for $pa\beta$, $pb\beta$, &c.; so that, omitting the constant factor $\sqrt{-uv}$, the value of the factor is

$$aba\,.\,p^2\beta\gamma\delta-ab\beta\,.\,p^2\gamma\delta a+ab\gamma\,.\,p^2\delta a\beta-ab\delta\,.\,p^2a\beta\gamma.$$

Taking (a, b, c, f, g, h) for the coordinates of the line ab, we have

$$aba=af_a+bg_a+ch_a+fa_a+gb_a+hc_a,$$

with the like expressions for $a\beta b$, &c.; and substituting for $p\beta\gamma\delta$, &c., their values, the factor is

[Surface $ab\alpha\beta\gamma\delta$.]

	x^2	y^2	z^2	w^2	xw	yw	zw	yz	zx	xy
a	$fagh$			$fabc$	$fabg - fach$	$fbch$	$-fbcg$		$fcgh$	$fbgh$
b		$gbhf$		$gabc$	$-gach$	$gbch - gabf$	$gcaf$	$gchf$	$hafg$	$gahf$
c			$hcfg$	$habc$	$habg$	$-habf$	$hcaf - hbcg$	$hbfg$		
f		$abhf$	$acfg$			$abch$	$-abcg$	$abfg + achf$	$acgh$	$abgh$
g	$bagh$		$bcfg$		$-bach$		$bcaf$	$bchf$	$bcgh + bafg$	$bahf$
h	$cagh$	$cbhf$			$cabg$	$-cabf$		$cbfg$	$cafg$	$cahf + cbgh$

viz., the value of the factor is $\{a(fagh) + g(bagh) + h(cagh)\}\, x^2 + \&c.$,
where $fagh = f_\alpha a_\beta g_\gamma h_\delta$ is the determinant

$$\left| \begin{matrix} f, & a, & g, & h \\ \vdots & & & \end{matrix} \right|,$$

the suffixes in the four lines being α, β, γ, δ respectively.

Collecting, this is

$$\begin{array}{l}
(\quad . \quad bchfy - bcfgz + fabcw)(\quad . \quad hy - gz + aw) \\
(-cagbx \quad . \quad + acfgz + gabcw)(-hx \quad . \quad + fz + bw) \\
(+bagbx - abhfy \quad . \quad + habcw)(\quad gx - fy \quad . \quad + cw) \\
(-afghx - bfghy - cfghz \quad . \quad)(\quad ax + by + cz \quad . \quad) \\
\quad + bcgh\,[w\,(ax + by + cz) - x\,(\quad . \quad hy - gz + aw)] \\
\quad + cahf\,[w\,(ax + by + cz) - y\,(-hx \quad . \quad + fz + bw)] \\
\quad + abfg\,[w\,(ax + by + cz) - z\,(\quad gx - fy \quad . \quad + cw)] = 0\,;
\end{array}$$

or what is the same thing, $AP + BQ + CR + DS = 0$, where

$$\begin{array}{l}
A = \{-bcghx + cbhfy - bcfgz + fabcw\}, \\
B = \{-cagbx + cahfy + acfgz + gabcw\}, \\
C = \{+bagbx - abhfy + abfgz + habcw\}, \\
D = \{-afghx - bfghy - cfghz + (bcgh + cahf + abfg)w\},
\end{array}$$

$$\begin{array}{l}
P = (\quad . \quad hy - gz + aw), \\
Q = (-hx \quad . \quad + fz + bw), \\
R = (\quad gx - fy \quad . \quad + cw), \\
S = (\quad ax + by + cz \quad . \quad) = 0,
\end{array}$$

where the right-hand factors vanish for the values $ux_a + vx_b$ of the coordinates.

38. It thus appears so far that the factor is $= 0^1$; it is, in fact, $= 0^2$, viz., we can show that, operating upon it with

$$\Delta = X d_x + Y d_y + Z d_z + W d_w,$$

the value (for any point of the line ab) is $= 0$. We have

$$\Delta \sqrt{paa \,.\, pba} \,.\, p^2\beta\gamma\delta$$

$$= \frac{paa \,.\, lba + pba \,.\, laa}{2\sqrt{paa \,.\, pba}}\, p^2\beta\gamma\delta + \sqrt{paa \,.\, pba} \,.\, \Delta \,.\, p^2\beta\gamma\delta,$$

where $lba\,(= \Delta pba)$ is what pba becomes on writing therein (X, Y, Z, W) in place of (x, y, z, w). Writing, as before, for x, y, z, w the values $ux_a + vx_b$, &c., we have $paa = -v \,.\, aba$, $pba = u \,.\, aba$; and putting for shortness

$$-v \,.\, lba + u \,.\, laa = lka, \text{ \&c.},$$

the expression in question, divided by $\sqrt{-uv}$, is

$$\begin{array}{l}
= -2vu\,\{aba \,.\, \Delta p^2\beta\gamma\delta - \text{\&c.}\} \\
\qquad + \{lka \,.\, \quad p^2\beta\gamma\delta - \text{\&c.}\},
\end{array}$$

where, denoting the determinants

$$\left| \begin{array}{cccc} X & Y & Z & W \\ ux_a - vx_b, & uy_a - vy_b, & uz_a - vz_b, & uw_a - vw_b \end{array} \right|$$

by (a', b', c', f', g', h'), we have

$$lka = a' f_a + b' f_a + c' g_a + f' a_a + g' b_a + h' c_a.$$

[Surface $ab\alpha\beta\gamma\delta$.]

But $aba \,.\, \Delta p^2\beta\gamma\delta = \Delta aba \,.\, p^2\beta\gamma\delta$, since aba is independent of (x, y, z, w); and the expression is

$$= -2vu\,\Delta\,(\mathrm{AP}+\mathrm{BQ}+\mathrm{CR}+\mathrm{DS}) + \mathrm{AP}'+\mathrm{BQ}'+\mathrm{CR}'+\mathrm{DS}',$$

where P′, Q′, R′, S′ denote $\mathrm{h}'y-\mathrm{g}'z+\mathrm{a}'w$, &c., and where, finally, x, y, z, w are to be replaced by ux_a+vx_b, &c. Since for these values P, Q, R, S vanish, the expression becomes

$$= -2vu\,(\Delta\Delta\mathrm{P}+\mathrm{B}\Delta\mathrm{Q}+\mathrm{C}\Delta\mathrm{R}+\mathrm{D}\Delta\mathrm{S}) + \mathrm{AP}'+\mathrm{BQ}'+\mathrm{CR}'+\mathrm{DS}';$$

that is,

$$= \mathrm{A}\,(\mathrm{P}'-2uv\Delta\mathrm{P})+\mathrm{B}\,(\mathrm{Q}'-2uv\Delta\mathrm{Q})+\mathrm{C}\,(\mathrm{R}'-2uv\Delta\mathrm{R})+\mathrm{D}\,(\mathrm{S}'-2uv\Delta\mathrm{S})$$

and we have, in fact, $\mathrm{P}'-2uv\Delta\mathrm{P}=0$, &c. For, writing for a moment

$$x,\ y,\ z,\ w = ux_a+vx_b,\ uy_a+vy_b,\ uz_a+vz_b,\ uw_a+vw_b,$$
$$x',\ y',\ z',\ w' = ux_a-vx_b,\ uy_a-vy_b,\ uz_a-vz_b,\ uw_a-vw_b;$$

then, for instance, $\mathrm{S}' = \mathrm{a}'x+\mathrm{b}'y+\mathrm{c}'z,$

where $\mathrm{a}', \mathrm{b}', \mathrm{c}' = \mathrm{Y}z'-\mathrm{Z}y',\ \mathrm{Z}x'-\mathrm{X}z',\ \mathrm{X}y'-\mathrm{Y}x';$

and thence

$$\mathrm{S}' = -\begin{vmatrix} \mathrm{X}, & \mathrm{Y}, & \mathrm{Z} \\ x, & y, & z \\ x', & y', & z' \end{vmatrix}$$
$$= 2uv\,(\mathrm{aX}+\mathrm{bY}+\mathrm{cZ})$$
$$= 2uv\,\Delta\mathrm{S};$$

and similarly for the other equations. The factor is thus $=0^2$; there is only one such factor, and the line ab is double.

(2.) The line a is an 8-tuple line: in fact, for a point on the line we have $pa\alpha = 0$, $pb\alpha = 0$, $p^2\gamma\delta\alpha = 0$, $p^2\delta\alpha\beta = 0$, $p^2\alpha\beta\gamma = 0$; and the factor vanishes, being $=0^1$. Each of the factors is 0^1, and the norm is $=0^8$.

39. (3.) The line $[ab, \alpha, \beta, \gamma]$ is a double line. To prove this, observe first that for a point on this line we have $p^2\alpha\beta\gamma = 0$.

Taking as before $z=0$, $w=0$ for the equation of the line ab, α, β, γ, we have $h_a=0$, $h_\beta=0$, $h_\gamma=0$, and $z_aw_b-z_bw_a=0$; or say $w_a=\lambda z_a$, $w_b=\lambda z_b$; whence, writing for shortness $\mathrm{I}=-(g-\lambda a)\,x+(f+\lambda b)\,y$, viz., $\mathrm{I}_a=-(g_a-\lambda a_a)\,x+(f_a+\lambda b_a)\,y$, we have (when $z=0$, $w=0$) $pa\alpha = z_a\mathrm{I}_a$, $pb\alpha = z_b\mathrm{I}_a$, or omitting the factor $\sqrt{z_az_b}$, $\sqrt{pa\alpha \,.\, pb\alpha} = \mathrm{I}_a$; and so for $\sqrt{pa\beta \,.\, pb\beta}$ and $\sqrt{pa\gamma \,.\, pb\gamma}$. The factor thus is

$$\mathrm{I}_a \,.\, p^2\beta\gamma\delta - \mathrm{I}_\beta \,.\, p^2\gamma\delta\alpha + \mathrm{I}_\gamma \,.\, p^2\delta\alpha\beta;$$

viz., writing $z=0$, $w=0$ in the expressions of $p^2\beta\gamma\delta$, &c., this may be written

$$\Sigma\,[(g-\lambda a)\,x-(f+\lambda b)\,y]\,\{(agh)\,x^2+[(ahf)+(bgh)]\,xy+(bhf)\,y^2\}.$$

Observe that Σ denotes a sum of three terms of the form

$$\alpha\,.\,\beta\gamma\delta - \beta\,.\,\gamma\delta\alpha + \gamma\,.\,\delta\alpha\beta.$$

Adding thereto a fourth term $-\delta\,.\,\alpha\beta\gamma$, the value of the sum would be $= \alpha\beta\gamma\delta$, or the sum of the three terms is $= \alpha\beta\gamma\delta + \delta\,.\,\alpha\beta\gamma$, where the symbols represent determinants. But in each case the determinant $\alpha\beta\gamma$ is $= 0$, as containing the column $h_\alpha, h_\beta, h_\gamma$, the terms of which are each $= 0$: thus $\Sigma g\,.\,agh$ is $= gagh - g_\delta\,.\,agh$, where in $gagh$ the suffixes are $\alpha, \beta, \gamma, \delta$, and in agh they are α, β, γ: that is, we have $\Sigma g\,.\,agh = gagh$. And the whole expression thus is

$$\begin{aligned} = \quad & x^3\,(gagh - \lambda aagh) \\ & + x^2y\,(gahf - \lambda aahf + gbgh - \lambda abgh - fagh - \lambda bagh) \\ & + xy^2\,(gbhf - \lambda abhf \qquad\qquad - fahf - \lambda bahf - fbgh - \lambda bbgh) \\ & + y^3\,(\qquad\qquad\qquad\qquad\qquad - fhbf - \lambda bbhf), \end{aligned}$$

where $gahf$ denotes the determinant $\begin{vmatrix} g, & a, & h, & f \\ \vdots & & & \end{vmatrix}$, with the suffixes $\alpha, \beta, \gamma, \delta$, in the four lines respectively, and so in other cases: the terms, such as $gagh$, which contain a twice-repeated letter, vanish of themselves; and in the coefficients of x^2y and xy^2, the terms which do not separately vanish destroy each other in pairs $gahf - fagh = 0$, &c.; whence the factor vanishes, being $= 0^1$; there are two such factors (viz., the zero term $\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma$ may be taken with the sign + or − at pleasure), and the norm is thus $= 0^2$.

40. But the line is tacnodal, each sheet of the surface touching along the line in question the hyberboloid $p^2\alpha\beta\gamma$. To prove this, write

$$\Delta = \mathrm{X}\delta_x + \mathrm{Y}\delta_y + \mathrm{Z}\delta_z + \mathrm{W}\delta_w;$$

we have for the hyperboloid, writing $z=0$, $w=0$,

$$\Delta p^2\alpha\beta\gamma = (afg\,.\,x + bfg\,.\,y)\,\mathrm{Z} + (abg\,.\,x - abf\,.\,y)\,\mathrm{W}.$$

And it is to be shown that

$$\begin{aligned} \Delta\,(\sqrt{paa\,.\,pba}\,.\,p^2\beta\gamma\delta - \sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta\alpha \\ + \sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta\alpha\beta \mp \sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma) \end{aligned}$$

each contain the factor $\Delta p^2\alpha\beta\gamma$; or what is the same thing, that

$$\Delta\Sigma\sqrt{paa\,.\,pba}\,.\,p^2\beta\gamma\delta$$

contains the factor in question, Σ denoting the sum of the first three terms of the original expression. The value is

$$= \Sigma\left(\frac{paa\,.\,\mathrm{P}ba + pba\,.\,\mathrm{P}aa}{2\sqrt{paa\,.\,pba}}\,p^2\beta\gamma\delta + \sqrt{paa\,.\,pba}\,.\,\Delta p^2\beta\gamma\delta\right);$$

where $\mathrm{P}aa, = \Delta paa$, denotes what paa becomes on writing therein X, Y, Z, W for x, y, z, w; and the like as to $\mathrm{P}ba$. Substituting for paa and pba their values $z_a\mathrm{I}_a$ and $z_b\mathrm{I}_a$, and multiplying by $\sqrt{z_az_b}$, the expression is $= \Sigma\,\{(z_a\,\mathrm{P}ba + z_b\,\mathrm{P}aa)\,p^2\beta\gamma\delta + 2z_az_b\mathrm{I}_a\,\Delta p^2\beta\gamma\delta\}$,

[Surface $ab\alpha\beta\gamma\delta$.]

where we have

$$\begin{aligned}
&z_a\,\mathrm{P}b\alpha + z_b\,\mathrm{P}a\alpha \\
&\quad = z_b\{x_a(\mathrm{Z}g_a - \mathrm{W}a_a) + y_a(-\mathrm{Z}f_a - \mathrm{W}b_a) \\
&\qquad\qquad + z_a[\mathrm{X}(-g_a+\lambda a_a) + \mathrm{Y}(f_a+\lambda b_a) + (\lambda\mathrm{Z}-\mathrm{W})c_a]\} \\
&\qquad + z_a\{x_b(\mathrm{Z}g_a - \mathrm{W}a_a) + y_b(-\mathrm{Z}f_a - \mathrm{W}b_a) \\
&\qquad\qquad + z_b[\mathrm{X}(-g_a+\lambda a_a) + \mathrm{Y}(f_a+\lambda b_a) + (\lambda\mathrm{Z}-\mathrm{W})c_a]\} \\
&\quad = (z_b x_a + z_a x_b)(\mathrm{Z}g_a - \mathrm{W}a_a) \\
&\qquad + (z_b y_a + z_a y_b)(-\mathrm{Z}f_a - \mathrm{W}b_a) \\
&\qquad + 2z_a z_b\{\mathrm{X}(-g_a+\lambda a_a) + \mathrm{Y}(f_a+\lambda b_a) + (\lambda\mathrm{Z}-\mathrm{W})c_a\}.
\end{aligned}$$

Also
$$\begin{aligned}
z_a z_b \mathrm{I}_a &= z_a z_b\{(-g_a+\lambda a_a)x + (f_a+\lambda b_a)y\}, \\
p^2\beta\gamma\delta &= x^2 . agh + xy(ahf+bgh) + y^2 . hbf, \\
\Delta p^2\beta\gamma\delta &= \mathrm{X} . 2x . agh + y(ahf+bgh) \\
&\quad + \mathrm{Y} . x(ahf+bgh) + 2y . hbf \\
&\quad + \mathrm{Z} . x(cgh+afg) + y(bfg+chf) \\
&\quad + \mathrm{W} . x(abg-cah) + y(bch-abf).
\end{aligned}$$

41. The whole expression is a linear function of X, Y, Z, W, and it is easy to see *à priori*, or to verify, that the coefficients of X, Y each of them vanish. The coefficient of Z is

$$\begin{aligned}
&= \Sigma\{(z_b x_a + z_a x_b)g_a - (z_b y_a + z_a y_b)f_a + 2\lambda z_a z_b c_a\}\, p^2\beta\gamma\delta \\
&\quad + \Sigma z_a z_b[(-g_a+\lambda a_a)x + (f_a+\lambda b_a)y][x(cgh+afg) + y(bfg+chf)],
\end{aligned}$$

with a like expression for the coefficient of W.

The foregoing expression may be written

$$\begin{aligned}
&(z_b x_a + z_a x_b)\,\Sigma\, g[agh . x^2 + (ahf+bgh)xy + bhf . y^2] \\
&-(z_b y_a + z_a y_b)\,\Sigma f[agh . x^2 + (ahf+bgh)xy + bhf . y^2] \\
&+2\lambda z_a z_b\,\Sigma\{c[agh . x^2 + (ahf+bgh)xy + bhf . y^2] \\
&\qquad\qquad + (ax+by)[(cgh+afg)x + (bfg+chf)y]\} \\
&+2z_a z_b\,\Sigma(-gx+fy)[(cgh+afg)x + (bfg+chf)y].
\end{aligned}$$

The first sum is
$$\begin{aligned}
&x^2 . gagh + xy(gahf+gbgh) + y^2 . gbhf \\
&\quad = -xy . afgh - y^2 . bfgh \\
&\quad = -h_\gamma y(afg . x + bfg . y);
\end{aligned}$$

where afg, bfg denote determinants with the suffixes α, β, γ. Similarly the second sum is $= -h_\beta x(afg . x + bfg . y)$;

the third sum is $(a_\delta x + b_\delta y)(afg . x + bfg . y)$,

and the fourth sum is $(-g_\delta x + f_\delta y)(afg . x + bfg . y)$.

The whole coefficient of Z thus contains the factor $(afg . x + bfg . y)$; and similarly it would appear that the whole coefficient of W contains the factor $(abg . x - abf . y)$, the other factor being the same in each case; viz., the two terms together are

[Surface $ab\alpha\beta\gamma\delta$.]

$$\left\{\begin{array}{l} -(z_b x_a + z_a x_b)\, h_y y \\ +(z_b y_a + z_a y_b)\, h_y x \\ +2\lambda z_a z_b (a_y x + b_y y) \\ +2 z_a z_b (-g_y x + f_y y) \end{array}\right\} \{ Z(ajg\,.\,x + bfy\,.\,y) + W(abg\;\; x - abf\,.\,y) \};$$

where the second factor is $\Delta p^2 \alpha\beta\gamma$, which is the required result. See *post*, Nos. 59 *et seq.*

42. (4.) The line $[\alpha, \beta, \gamma, \delta]$ is an 8-tuple line; in fact, for any point of the line in question we have $p^2\beta\gamma\delta = 0$, $p^2\gamma\delta\alpha = 0$, $p^2\delta\alpha\beta = 0$, $p^2\alpha\beta\gamma = 0$; whence each factor is 0^1, or the norm is 0^8.

I notice that the surface meets the quadric $p^2\alpha\beta\gamma$ in

lines	α, β, γ	each 8 times,		24
„	$(\alpha, \beta, \gamma, \delta)$	„	„	16
„	$(ab, \alpha, \beta, \gamma)$	„ 4	„	8
				$24 \times 2 = 48$

Surface $aa\beta\gamma\delta\epsilon$.

43. The equation is

$$(p^2 a\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^3 aa\gamma\,.\,\delta\beta + p^2 a\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^3 aa\beta\,.\,\gamma\delta)^2 - 4p^2 a\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^2 a\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^2 a\beta\gamma\,.\,p^2\delta\beta\gamma\,.\,paa\,.\,p\delta a = 0;$$

or what is the same thing,

$$(p^2 a\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^3 aa\gamma\,.\,\delta\beta - p^2 a\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^3 aa\beta\,.\,\gamma\delta)^2 - 4p^2 a\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^2 a\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^2\beta a\delta\,.\,p^2\gamma a\delta\,.\,p\beta a\,.\,p\gamma a = 0;$$

the equivalence of the two depending on the identity

$$p^3 aa\beta\,.\,\gamma\delta\,.\,p^3 aa\gamma\,.\,\delta\beta - p^2 a\beta\gamma\,.\,p^2\delta\beta\gamma\,.\,paa\,.\,p\delta a + p^2\beta a\delta\,.\,p^2\gamma a\delta\,.\,p\beta a\,.\,p\gamma a = 0;$$

where, as before, $p^2\alpha\beta\epsilon = 0$ is the equation of the quadric through the lines α, β, ϵ, and $paa = 0$ is the equation of the plane through the line α and the point a; viz., $p^2 a\beta\epsilon$, &c., and paa, &c., have the values already mentioned: $p^3 aa\beta\,.\,\gamma\delta = 0$ as already mentioned is the cubic surface through the lines α, β, γ, δ and $aa\beta$, $a\gamma\delta$.

44. *Investigation.* — In the projection, using line-coordinates, the equation of the conic touching the five lines may be written

$$\left|\begin{array}{c} (\xi,\ \eta,\ \zeta)^2 \\ (\mathrm{P},\ \mathrm{Q},\ \mathrm{R})^2 \end{array}\right| = 0;$$

where the symbol denotes a determinant the last five lines of which are obtained by giving to (P, Q, R) the suffixes $\alpha, \beta, \gamma, \delta, \epsilon$ respectively. This is at once transformed into

$$a\beta\epsilon\,.\,\gamma\delta\epsilon\,.\,a\gamma\Delta\,.\,\delta\beta\Delta - a\gamma\epsilon\,.\,\delta\beta\epsilon\,.\,a\beta\Delta\,.\,\gamma\delta\Delta = 0,$$

or what is the same thing,

$$p^2\alpha\beta\epsilon \,.\, p^2\gamma\delta\epsilon \,.\, \alpha\gamma\Delta \,.\, \delta\beta\Delta - p^2\alpha\gamma\varepsilon \,.\, p^2\delta\beta\epsilon \,.\, \alpha\beta\Delta \,.\, \gamma\delta\Delta = 0;$$

or say

$$p^2\alpha\beta\epsilon \,.\, p^2\gamma\delta\epsilon\,(A''\xi+B''\eta+C''\zeta)(A'''\xi+B'''\eta+C'''\zeta) - p^2\alpha\gamma\epsilon \,.\, p^2\delta\beta\epsilon\,(A\xi+B\eta+C\zeta)(A'\xi+B'\eta+C'\zeta) = 0;$$

where $p^2\alpha\beta\epsilon$, &c., signify as before; and

$$A\xi+B\eta+C\zeta = \begin{vmatrix} \xi, & \eta, & \zeta \\ P_\alpha, & Q_\alpha, & R_\alpha \\ P_\beta, & Q_\beta, & R_\beta \end{vmatrix}$$

and so for $A'\xi+B'\eta+C'\zeta$, &c., the suffixes for A′, B′, C′ being (γ, δ); and those for $A''\xi+B''\eta+C''\zeta$ and $A'''\xi+B'''\eta+C'''\zeta$ being (α, γ) and (δ, β) respectively.

45. Passing to the reciprocal equation, and making the conic pass through the point α, we obtain the equation of the surface in the form

$$\left\{ p^2\alpha\gamma\epsilon \,.\, p^2\delta\beta\epsilon \begin{vmatrix} p_\alpha, & q_\alpha, & r_\alpha \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} - p^2\alpha\beta\epsilon \,.\, p^2\gamma\delta\epsilon \begin{vmatrix} p_\alpha, & q_\alpha, & r_\alpha \\ A'', & B'', & C'' \\ A''', & B''', & C''' \end{vmatrix} \right\}^2$$

$$+ 4p^2\alpha\gamma\epsilon \,.\, p^2\delta\beta\epsilon \,.\, p^2\alpha\beta\epsilon \,.\, p^2\gamma\delta\epsilon \begin{vmatrix} p_\alpha, & q_\alpha, & r_\alpha \\ A, & B, & C \\ A'', & B'', & C'' \end{vmatrix} \begin{vmatrix} p_\alpha, & q_\alpha, & r_\alpha \\ A', & B', & C' \\ A''', & B''', & C''' \end{vmatrix} = 0.$$

Or in the equivalent form, where in the first term we have + instead of −, and in the second term the determinants are

$$\begin{vmatrix} p_\alpha, & q_\alpha, & r_\alpha \\ A, & B, & C \\ A''', & B''', & C'' \end{vmatrix}, \quad \begin{vmatrix} p_\alpha, & q_\alpha, & r_\alpha \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix}.$$

46. To reduce this result, observe that we have

$$A, B, C = \left\| \begin{matrix} hy-gz+aw, & -hx+fz+bw, & gx-fy+cw \\ h'y-g'z+a'w, & -h'x+f'z+b'w, & g'x-f'y+c'w \end{matrix} \right\|$$

where, for convenience, I retain the unaccented and accented letters $(a, \ldots)$, $(a', \ldots)$ instead of these letters with the suffixes α and β respectively. Writing as before

$$\begin{aligned} L &= (af'-a'f)\,x+\ldots \\ M &= (ag'-a'g)\,x+\ldots \\ N &= (ah'-a'h)\,x+\ldots \\ \Omega &= (gh'-g'h)\,x+\ldots \end{aligned}$$

then

$$\begin{aligned} A &= \Omega x - Lw, \quad \text{and similarly} \quad A' = \Omega' x - L' w, \\ B &= \Omega y - Mw, \qquad\qquad\quad B' = \Omega' y - M' w, \\ C &= \Omega z - Nw, \qquad\qquad\quad C' = \Omega' z - N' w; \end{aligned}$$

where for L′, M′, N′, Ω′ we have $(a'', \ldots)$ and $(a''', \ldots)$. Hence

$$BC' - B'C = w \begin{vmatrix} y, & z, & w \\ M, & N, & \Omega \\ M', & N', & \Omega' \end{vmatrix}$$

[Surface $\alpha\alpha\beta\gamma\delta\epsilon$.]

with like expressions for $CA' - C'A$ and $AB' - A'B$; and substituting, we have

$$\begin{vmatrix} p_a, & q_a, & r_a \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} = w \begin{vmatrix} p_a, & q_a, & r_a, \\ x, & y, & z, & w \\ L, & M, & N, & \Omega \\ L', & M', & N', & \Omega' \end{vmatrix}$$

or substituting for p_a, q_a, r_a their values $xw_a - wx_a$, $yw_a - wy_a$, $zw_a - wz_a$, this is

$$\begin{vmatrix} p_a, & q_a, & r_a \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} = -w^2 \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L, & M, & N, & \Omega \\ L', & M', & N', & \Omega' \end{vmatrix}$$

whence, omitting the factors w^2, the equation is

$$\left\{ p^2a\gamma\epsilon \,.\, p^2\delta\beta\epsilon \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L, & M, & N, & \Omega \\ L', & M', & N', & \Omega' \end{vmatrix} - p^2a\beta\epsilon \,.\, p^2\gamma\delta\epsilon \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L'', & M'', & N'', & \Omega'' \\ L''', & M''', & N''', & \Omega''' \end{vmatrix} \right\}^2$$

$$+4p^2a\gamma\epsilon \,.\, p^2\delta\beta\epsilon \,.\, p^2a\beta\epsilon \,.\, p^2\gamma\delta\epsilon \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L, & M, & N, & \Omega \\ L'', & M'', & N'', & \Omega'' \end{vmatrix} \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L', & M', & N', & \Omega' \\ L''', & M''', & N''', & \Omega''' \end{vmatrix} = 0,$$

where I recall that for (L, ...), (L', ...), (L'', ...), (L''', ...) the suffixes are (a, β), (γ, δ), (a, γ), and (δ, β) respectively. The values of the first two determinants thus are $p^3aa\beta \,.\, \gamma\delta$ and $p^3aa\gamma \,.\, \delta\beta$ respectively: that of the third is $p^3aa\beta \,.\, a\gamma$; viz., this is $= p^2a\beta\gamma \,.\, paa$. Similarly, that of the fourth is $p^3a\gamma\delta \,.\, \delta\beta$, which is $= -p^3a\delta\gamma \,.\, \delta\beta = +p^3a\delta\beta \,.\, \delta\gamma$; or finally this is $= p^2\delta\beta\gamma \,.\, pa\delta$. And we have thus the before-mentioned equation of the surface.

47. *Singularities.*—The equation of the surface shows that—

(0) The point a is a 2-conical point: in fact, we have for this point $p^3aa\beta \,.\, \gamma\delta = 0$, $p^3aa\gamma \,.\, \delta\beta = 0$, $paa = 0$, $pa\delta = 0$.

(2) The line a is a 4-tuple line: in fact, for any point on this line $p^2a\beta\epsilon = 0$, $p^3aa\beta \,.\, \gamma\delta = 0$, $p^2a\gamma\epsilon = 0$, $p^3aa\gamma \,.\, \delta\beta = 0$, $p^2a\beta\gamma = 0$, $p^2aa = 0$.

(4) The line $(\alpha, \beta, \gamma, \epsilon)$ is a 2-tuple line: in fact, for any point on the line we have $p^2a\beta\epsilon = 0$, $p^2a\gamma\epsilon = 0$.

(10) The excuboquartic $a\beta\epsilon \,.\, \gamma\delta \,.\, a$ is a simple curve: in fact, for any point of this curve we have $p^2a\beta\epsilon = 0$, $p^3aa\beta \,.\, \gamma\delta = 0$, these two surfaces intersecting in the lines a, β and the curve. It is, moreover, obvious that the surface is touched along the curve by the hyperboloid $p^2a\beta\epsilon$.

[Surface $aa\beta\gamma\delta\epsilon$.]

I notice that the surface meets the quadric $p^2\alpha\beta\gamma$ in

lines	α, β, γ	each 4 times,	12
„	$(\alpha, \beta, \gamma, \delta)$	„ twice,	4
„	$(\alpha, \beta, \gamma, \epsilon)$	„ „	4
curve	$a\alpha\beta\gamma.\delta\epsilon$	„ „	8
			$14\times 2 = 28$

Surface $\alpha\beta\gamma\delta\epsilon\zeta$.

48. The equation of the surface may be written

$$p^2\alpha\beta\epsilon \,.\, p^2\gamma\delta\epsilon \,.\, p^2\alpha\gamma\zeta \,.\, p^2\delta\beta\zeta - p^2\alpha\beta\zeta \,.\, p^2\gamma\delta\zeta \,.\, p^2\alpha\gamma\epsilon \,.\, p^2\delta\beta\epsilon = 0,$$

where $p^2\alpha\beta\epsilon = 0$ is the equation of the quadric through the lines α, β, ϵ; viz., $p^2\alpha\beta\epsilon$ has the value already mentioned.

The form is one of 45 like forms depending on the partitionment

$$\left\{\begin{matrix} \alpha\beta \,.\, \gamma\delta \\ \alpha\gamma \,.\, \delta\beta \\ \alpha\delta \,.\, \beta\gamma \end{matrix}\right\}(\epsilon, \zeta)$$

of the six letters.

49. *Investigation.*—The projections of the six lines are tangents to a conic: the condition for this is $(P, Q, R)^2 = 0$, where the left-hand side represents the determinant obtained by writing successively $(P_\alpha, Q_\alpha, R_\alpha)$, &c. for (P, Q, R). The equation may be written

$$\alpha\beta\epsilon \,.\, \gamma\delta\epsilon \,.\, \alpha\gamma\zeta \,.\, \delta\beta\zeta - \alpha\beta\zeta \,.\, \gamma\delta\zeta \,.\, \alpha\gamma\epsilon \,.\, \delta\beta\gamma = 0,$$

where

$$\alpha\beta\epsilon = \begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ P_\beta, & Q_\beta, & R_\beta \\ P_\epsilon, & Q_\epsilon, & R_\epsilon \end{vmatrix}$$

and substituting for P_α, &c., their values, we have $\alpha\beta\epsilon = w \,.\, p^2\alpha\beta\epsilon$; whence the foregoing result.

50. *Singularities.*—The equation shows that—

(2) The line α is a 2-tuple line: in fact, for each point of the line we have $p^2\alpha\beta\epsilon = 0$, $p^2\alpha\gamma\zeta = 0$, $p^2\alpha\beta\zeta = 0$, $p^2\alpha\gamma\epsilon = 0$.

(4) The line $(\alpha, \beta, \epsilon, \zeta)$ is a simple line: in fact, for each point of the line we have $p^2\alpha\beta\epsilon = 0$, $p^2\alpha\beta\zeta = 0$.

(9) The quadriquadric $\alpha\beta\epsilon \,.\, \gamma\delta\zeta = 0$ is a simple curve on the surface: in fact, for each point of the curve we have $p^2\alpha\beta\epsilon = 0$, $p^2\gamma\delta\zeta = 0$.

It may be remarked that the surface meets the hyperboloid $p^2\alpha\beta\epsilon$ in

lines	α, β, ϵ	each twice,	6
„	$[\alpha, \beta, \epsilon, \gamma]$	„ once,	2
„	$[\alpha, \beta, \epsilon, \delta]$	„ „	2
„	$[\alpha, \beta, \epsilon, \zeta]$	„ „	2
curve	$\alpha\beta\epsilon \,.\, \gamma\delta\zeta$	„ „	4
			$2\times 8 = 16$

51. It might be thought that there should be on the surface some curve $\alpha\beta\gamma\delta\epsilon\zeta$, such as the cubic *abcdef* on the surface *abcdef*; but I cannot find that this is so. The equation of the surface is satisfied if we have simultaneously (λ being arbitrary)

$$p^2\alpha\beta\epsilon \,.\, p^2\alpha\gamma\zeta - \lambda p^2\alpha\beta\zeta \,.\, p^2\alpha\gamma\epsilon = 0,$$
$$\lambda p^2\gamma\delta\epsilon \,.\, p^2\delta\beta\zeta - p^2\gamma\delta\zeta \,.\, p^2\delta\beta\epsilon = 0;$$

which equations represent quartic surfaces, the first of them having α for a double line, and passing through the lines β, γ, ϵ, ζ ($13+4\times5 = 33$ conditions, so that the equation of such a surface contains only an arbitrary parameter λ); and the second having δ for a double line, and passing through the lines β, γ, ϵ, ζ. But I see no condition by which λ can be determined so as to have the same value in the two equations respectively. Of course, leaving it arbitrary, the two quartic surfaces intersect in the lines β, γ, ϵ, ζ and in a curve of the order 12 depending on the arbitrary value of λ, which curve lies on the surface $\alpha\beta\gamma\delta\epsilon\zeta$.

The Excuboquartic $\alpha\beta\gamma$, $\delta\epsilon$, a.

52. The notion is, that we have a fixed point a, two fixed lines δ, ϵ, and a singly infinite series of lines, or say the generating lines of a skew surface: each generating line determines, with the point a, a plane; and if in this plane we draw, meeting the lines δ, ϵ, a line to meet the generating line in a point P, then the locus of this point P is the curve about to be considered.

53. In the case in question, the singly infinite series of lines are the lines which meet each of the lines α, β, γ, or say they are the generatrices of the hyperboloid $\alpha\beta\gamma$: the locus, or curve $\alpha\beta\gamma$, $\delta\epsilon$, a, is, as mentioned above, an excuboquartic. It is not necessary for the purpose of the memoir, but it is interesting to consider in conjunction therewith the excuboquartic arising in like manner from the directrices of the hyperboloid; it will appear that the two curves are the complete intersection of the quadric $\alpha\beta\gamma$ by a quartic surface. Observe that the two curves are given as follows: viz., considering for the quadric $\alpha\beta\gamma$ any tangent-plane through the point a, and drawing in this plane, to meet the lines δ and ϵ, a line, this meets the section of the quadric surface by the tangent-plane in two points, the locus of which is the aggregate of the two curves: viz., the section being a line-pair, the two points belong, one of them to a generatrix and the other to a directrix of the quadric surface.

54. It is convenient to take $x=0$, $y=0$ for the equations of the line δ; $z=0$, $w=0$ for those of the line ϵ: for then, for any plane $Ax+By+Cz+Dw=0$, the line in this plane and meeting the lines δ and ϵ, has for its equations $Ax+By=0$, $Cz+Dw=0$; or, what is

the same thing, for the plane $P=0$ the equations of the line are $P_{xy}=0$, $P_{zw}=0$, where P_{xy}, P_{zw} denote the terms in x, y and in z, w respectively.

I take also x_0, y_0, z_0, w_0 for the coordinates of the point a, and $PS-QR=0$ for the equation of the quadric surface, P, Q, R, S being given linear functions of (x, y, z, w): we have then say $P-\theta R=0$, $Q-\theta S=0$ for the equations of any generatrix, and $P-\phi Q=0$, $R-\phi S=0$ for the equations of any directrix of the hyperboloid.

The equation of the plane through the point a and the generatrix $P-\theta R=0$, $Q-\theta S=0$, is clearly

$$(Q_0-\theta S_0)(P-\theta R)-(P_0-\theta R_0)(Q-\theta S)=0;$$

so that for the line in this plane, meeting the lines δ and ϵ, we have

$$(Q_0-\theta S_0)(P_{xy}-\theta R_{xy})-(P_0-\theta R_0)(Q_{xy}-\theta S_{xy})=0,$$
$$(Q_0-\theta S_0)(P_{zw}-\theta R_{zw})-(P_0-\theta R_0)(Q_{zw}-\theta S_{zw})=0;$$

and joining thereto the equations

$$\theta=\frac{P}{R}=\frac{Q}{S}=\frac{P_{xy}+P_{zw}}{R_{xy}+R_{zw}}=\frac{Q_{xy}+Q_{zw}}{R_{xy}+R_{zw}},$$

(equivalent in all to three equations,) the elimination of θ gives the required curve: the equations thus are

$$PS-QR=0,$$
$$(Q_0S-QS_0)(P_{xy}R-PR_{xy})-(P_0R-PR_0)(Q_{xy}S-QS_{xy})=0,$$

or, as the second equation may also be written,

$$(Q_0S-QS_0)(P_{xy}R_{zw}-P_{zw}R_{xy})-(P_0R-PR_0)(Q_{xy}S_{zw}-Q_{zw}S_{xy})=0;$$

viz., the second equation represents a cubic surface having upon it the lines $(P=0, R=0)$ and $(Q=0, S=0)$: it therefore intersects the quadric $PS-QR=0$ in these two lines, and besides in an excuboquartic curve, which is the required locus.

55. Representing the determinants

$$\begin{vmatrix} P, & Q, & R, & S \\ P_0, & Q_0, & R_0, & S_0 \end{vmatrix} \quad \text{by } a', b', c', f', g', h', \quad \text{viz., } \begin{aligned} a' &= QR_0-Q_0R, \ \dots \\ f' &= PS_0-P_0S, \ \dots; \end{aligned}$$

$$\begin{vmatrix} P_{xy}, & Q_{xy}, & R_{xy}, & S_{xy} \\ P_{zw}, & Q_{zw}, & R_{zw}, & S_{zw} \end{vmatrix} \quad \text{by } a, b, c, f, g, h, \quad \text{viz., } a=Q_{xy}R_{zw}-Q_{zw}R_{xy}, \ \dots;$$

so that $(a', \dots)$ are linear functions, $(a, \dots)$ quadric functions of the coordinates; the equation of the cubic surface is $gb'-bg'=0$, viz., the excuboquartic arising from the generatrices is the partial intersection of the quadric $PS-QR=0$ and the cubic $gb'-g'b=0$; the two surfaces besides intersecting in the lines $(P=0, R=0)$ and $(Q=0, S=0)$.

It appears, in the same manner, that the excuboquartic arising from the directrices is the partial intersection of the quadric $PS-QR=0$

and the cubic $hc' - ch' = 0$; the two surfaces besides intersecting in the lines $(P=0, Q=0)$ and $(R=0, S=0)$.

56. But the elimination may be performed in a different manner, as follows: from the first two equations in θ, multiplying by P_{zw}, $-P_{xy}$ and adding, and so with Q_{zw}, $-Q_{xy}$, &c., we obtain

$$\begin{aligned}
&(Q_0-\theta S_0)(\quad -\theta b)-(P_0-\theta R_0)(-c+\theta f) = 0,\\
&(Q_0-\theta S_0)(\quad c+\theta a)-(P_0-\theta R_0)(\quad \theta g) = 0,\\
&(Q_0-\theta S_0)(-b\quad)-(P_0-\theta R_0)(\quad a+\theta h) = 0,\\
&(Q_0-\theta S_0)(\quad f-\theta h)-(P_0-\theta R_0)(\quad g\quad) = 0.
\end{aligned}$$

We then have $$\theta = \frac{-c+\theta f}{a+\theta h} = \frac{c+\theta a}{f-\theta h},$$

or, what is the same thing,

$$h\theta^2 + (a-f)\,\theta + c = 0.$$

Using this equation, written in the form $(a+\theta h)\,\theta = -c+\theta f$, to transform the first or third of the four equations in θ, we obtain

$$-aP_0 - bQ_0 - cR_0 + \theta\,(-hP_0 \;.\; + fR_0 + bS_0) = 0;$$

and using the same equation, written in the form $(f-\theta h)\,\theta = c+\theta a$, to transform the second or fourth equation, we obtain

$$gP_0 - fQ_0 + cS_0 + \theta\,(hQ_0 - gR_0 + aS_0) = 0;$$

and hence, eliminating θ, we obtain

$$\begin{aligned}
&(hQ_0 - gR_0 + aS_0)(-aP_0 - bQ_0 - cR_0)\\
&\qquad -(-hP_0 \;.\; + fR_0 + bS_0)(gP_0 - fQ_0 \;.\; + cS_0) = 0,
\end{aligned}$$

which, as being of the second order in $(a, \ldots)$, represents a quartic surface. The equation remains unaltered by the interchange of Q, R, and the consequent interchanges among (a, b, c, f, g, h): hence the quartic surface contains not only the excuboquartic arising from the generatrices, but also that arising from the directrices; and these two curves are the complete intersection of the quartic by the quadric $PS - QR = 0$.

57. I obtain this same result also as follows. Consider a point (P_1, Q_1, R_1, S_1) on the quadric surface; $P_1S_1 - Q_1R_1 = 0$; the tangent plane at the point is $PS_1 - QR_1 - RQ_1 + SP_1 = 0$;
and if this passes through the point α, then

$$P_0S_1 - Q_0R_1 - R_0Q_1 + S_0P_1 = 0.$$

The line which in the tangent-plane meets the lines δ, ϵ is given, as before, by the equations

$$\begin{aligned}
P_{xy}S_1 - Q_{xy}R_1 - R_{xy}Q_1 + S_{xy}P_1 &= 0,\\
P_{zw}S_1 - Q_{zw}R_1 - R_{zw}Q_1 + S_{zw}P_1 &= 0.
\end{aligned}$$

Remembering the significations of $(a, \ldots)$, the last three equations give

$$\begin{aligned} S_1 : R_1 : -Q_1 : -P_1 = \quad & . \quad hQ_0 - gR_0 + aS_0 \\ : & -hP_0 \quad . \quad +fR_0 + bS_0 \\ : & \ gP_0 - fR_0 \quad . \quad +cS_0 \\ : & -aP_0 - bQ_0 - cR_0 \quad . \ ; \end{aligned}$$

and substituting these values in $S_1P_1 - Q_1R_1 = 0$, we have the above equation of the quadric surface.

58. Or again, changing the notation, I take the equation of the quadric surface to be

$$(a, b, c, d, f, g, h, l, m, n \mathbin{\text{\textbigtimes}} x, y, z, w)^2 = 0.$$

A tangent-plane hereof is $\xi x + \eta y + \zeta z + \omega w = 0$,

where ξ, η, ζ, ω are any quantities satisfying the relation

$$(A, B, C, D, F, G, H, L, M, N \mathbin{\text{\textbigtimes}} x, y, z, w)^2,$$

the capitals denoting the inverse coefficients.

Supposing that the tangent-plane passes through a fixed point α, coordinates $(\alpha, \beta, \gamma, \delta)$, we have

$$\alpha\xi + \beta\eta + \gamma\zeta + \delta\omega = 0;$$

and if the equations of the lines δ, ϵ are as before $(x=0, y=0)$ and $(z=0, w=0)$; then for the line in the tangent-plane meeting the lines δ, ϵ, we have $\xi x + \eta y = 0, \quad \zeta z + \omega w = 0.$

These last equations may be represented by

$$\xi = ly, \quad \eta = -lx, \quad \zeta = mw, \quad \omega = -mz.$$

And substituting these values

$$(A, \ldots \mathbin{\text{\textbigtimes}} ly, -lx, mw, -mz)^2 = 0,$$
$$(\alpha, \ldots \mathbin{\text{\textbigtimes}} ly, -lx, mw, -mz)^1 = 0,$$

that is $(Ay^2 - 2Hxy + Bx^2, -Fxw + Gyw - Lyz + Mxz,$
$Cw^2 - 2Nwz + Dz^2 \mathbin{\text{\textbigtimes}} l, m)^2 = 0,$

and $(\alpha y - \beta x, \gamma w - \delta z \mathbin{\text{\textbigtimes}} l, m) = 0.$

Whence, eliminating l, m, we have the quartic equation

$$(Ay^2 - 2Hxy + Bx^2, -Fxw + Gyw - Lyz + Mxz,$$
$$Cw^2 - 2Nzw + Dz^2 \mathbin{\text{\textbigtimes}} \gamma w - \delta z, \beta x - \alpha y)^2 = 0.$$

Further Investigation as to the Surface $ab\alpha\beta\gamma\delta$.

59. The theorem that in the surface $ab\alpha\beta\gamma\delta$, the equation of which is

$$\text{Norm}\{\sqrt{p\alpha a \,.\, pba}\,.\,p^2\beta\gamma\delta - \sqrt{p\alpha\beta \,.\, pb\beta}\,.\,p^2\gamma\delta\alpha$$
$$+ \sqrt{pa\gamma \,.\, pb\gamma}\,.\,p^2\delta\alpha\beta - \sqrt{pa\delta \,.\, pb\delta}\,.\,p^2\alpha\beta\gamma\} = 0;$$

the lines $(ab, \alpha, \beta, \gamma)$ are tacnodal, each sheet touching along the line the quadric $p^2\alpha\beta\gamma$, may be proved in a different manner by investigating the intersection of the surface with the quadric $p^2\alpha\beta\gamma$.

For this purpose take the equation of the quadric to be $yz-xw=0$; the equations of the lines α, β, γ will be

$$\begin{pmatrix} z-\lambda_\alpha w=0 \\ x-\lambda_\alpha y=0 \end{pmatrix},\quad \begin{pmatrix} z-\lambda_\beta w=0 \\ x-\lambda_\beta y=0 \end{pmatrix},\quad \begin{pmatrix} z-\lambda_\gamma w=0 \\ x-\lambda_\gamma y=0 \end{pmatrix};$$

and we may write (a, b, c, f, g, h) for the coordinates of the line δ. The equation of the surface will be

$$\text{Norm}\left\{\Sigma\left[\pm\sqrt{p\alpha\alpha\,.\,pb\alpha}\,(\lambda_\beta-\lambda_\gamma)\left\{\begin{matrix}(a-f)\,xz-(\lambda_\beta+\lambda_\gamma)\,yz+\lambda_\beta\lambda_\gamma yw \\ +(b-g)\,\lambda_\beta\lambda_\gamma(yz-xw) \\ +c(z-\lambda_\beta w)(z-\lambda_\gamma w) \\ +h(x-\lambda_\beta y)(x-\lambda_\gamma z)\end{matrix}\right\}\right]\right.$$

$$\left.-\sqrt{p\alpha\delta\,.\,pb\delta}\,(\lambda_\beta-\lambda_\gamma)(\lambda_\gamma-\lambda_\alpha)(\lambda_\alpha-\lambda_\beta)(yz-xw)\right\};$$

where Σ denotes the sum of the three terms obtained by the cyclical interchange of α, β, γ; and

$$p\alpha\alpha=(z_a-\lambda w_a)(x-\lambda y)-(x_a-\lambda y_a)(z-\lambda w),$$
$$pb\alpha=(z_b-\lambda w_b)(x-\lambda y)-(x_b-\lambda y_b)(z-\lambda w);$$

λ here standing for λ_α; and similarly for $p\alpha\beta$, &c.

60. To obtain the intersection with $xw-yz=0$, writing $w=\frac{yz}{x}$,

then
$$p\alpha\alpha=\left[z_a-\lambda w_a-\frac{z}{x}(x_a-\lambda y_a)\right](x-\lambda y),\quad(\lambda=\lambda_\alpha),$$

$$pb\alpha=\left[z_b-\lambda w_b-\frac{z}{x}(x_b-\lambda y_b)\right](x-\lambda y);$$

or say
$$\sqrt{p\alpha\alpha\,.\,pb\alpha}=\sqrt{\mathrm{M}_\alpha}\,(x-\lambda_\alpha y);$$

also the expression in $\{\ \}$ becomes

$$=\left\{(a-f)\frac{z}{x}+c\frac{z^2}{x^2}+h\right\}(x-\lambda_\beta y)(x-\lambda_\gamma y);$$

so that the norm in question is

$$\text{Norm}\,\Sigma\sqrt{\mathrm{M}_\alpha}\,(\lambda_\beta-\lambda_\gamma)\left\{(a-f)\frac{z}{x}+c\frac{z^2}{x^2}+h\right\}(x-\lambda_\alpha y)(x-\lambda_\beta y)(x-\lambda_\gamma y);$$

or say

$$\text{Norm}\,\Sigma\sqrt{\mathrm{M}_\alpha}\,(\lambda_\beta-\lambda_\gamma)\left\{hx^2+(a-f)zx+cz^2\right\}(x-\lambda_\alpha y)(x-\lambda_\beta y)(x-\lambda_\gamma y);$$

where M_α is now considered to stand for

$$\left\{(z_ax-zx_a)-\lambda(w_ax-y_az)\right\}\left\{(z_bx-zx_b)-\lambda(w_bx-y_bz)\right\}.$$

Observing that the norm was originally the product of 8 factors, this breaks up into

$$\left\{hx^2+(a-f)zx+cz^2\right\}^8\left\{(x-\lambda_\alpha y)(x-\lambda_\beta y)(x-\lambda_\gamma y)\right\}^8=0,$$

and
$$\text{Norm}^2\sqrt{\mathrm{M}_\alpha}\,(\lambda_\beta-\lambda_\gamma)=0,$$

where the new norm is the product of 4 factors.

61. Writing for greater convenience λ, μ, ν in place of λ_α, λ_β, λ_γ, and

observing that M_α is a quadric function of λ_α, that is of λ, the last mentioned norm is $\quad \text{Norm}\,\sqrt{A+B\lambda+C\lambda^2}\,(\mu-\nu)$,

which is easily seen to be

$$= (4AC-B^2)(\mu-\nu)^2(\nu-\lambda)^2(\lambda-\mu)^2.$$

Or writing for a moment

$$(A+B\lambda+C\lambda^2) = (P-Q\lambda)(P'-Q'\lambda),$$

whence $\quad A = PP', \quad B = -(PQ'+P'Q), \quad C = QQ';$

then $\quad 4AC-B^2 = -(PQ'-P'Q)^2;$

and we have $\quad P, Q = z_a x - z x_a,\ w_a x - y_a z,$

$$P', Q' = z_b x - z x_b,\ w_b x - y_b z,$$

whence $\quad PQ'-P'Q = (z_a w_b - z_b w_a)\,x^2$

$$+[y_a z_b - y_b z_a - (x_a w_b - x_b w_a)]\,xz$$

$$+(x_a y_b - x_b y_a)\,z^2;$$

viz., if (a, b, c, f, g, h) are the coordinates of the line ab, this is

$$= hx^2 + (a-f)\,xz + cz^2.$$

Hence, omitting the constant factor $(\mu-\nu)^4(\nu-\lambda)^4(\lambda-\mu)^4$ [that is $(\lambda_\beta-\lambda_\gamma)^4(\lambda_\gamma-\lambda_\alpha)^4(\lambda_\alpha-\lambda_\beta)^4$], the foregoing equation norm$^2 = 0$ becomes

$$[hx^2 + (a-f)\,xz + cz^2]^4 = 0,$$

and the intersections of the quadric with the surface are obtained by combining the equation $xw-yz=0$ with the several equations

$$\{hx^2+(a-f)\,zx+cz^2\}^8 = 0,$$

$$\{(x-\lambda_\alpha y)(x-\lambda_\beta y)(x-\lambda_\gamma y)\}^8 = 0,$$

$$\{hx^2+(a-f)\,zx+cz^2\}^4 = 0;$$

viz., these are

lines $(\alpha, \beta, \gamma, \delta)$	each	8	times	16
line $(x=0, z=0)$		16	„	16
lines α, β, γ	each	8	„	24
line $(x=0, y=0)$		24	„	24
lines $[ab, \alpha, \beta, \gamma]$	each	4	„	8
line $(x=0, z=0)$		8	„	8

$$(16+24+8)\times 2 = 48+48.$$

But it is clear that the lines $(x=0, y=0)$ and $(x=0, z=0)$ are introduced by the process of elimination, and are no part of the intersection. The complete intersection consists of the lines $(\alpha, \beta, \gamma, \delta)$ each 8 times, the lines (α, β, γ) each 8 times, and the lines $[ab, \alpha, \beta, \gamma]$ each 4 times. But the last mentioned lines being only double lines on the surface, this means that the two sheets each touch the quadric surface, or that the lines are tacnodal.

On the Constants that occur in Certain Summations by Bernoulli's Series. *By* J. W. L. Glaisher, B A., F.R.A.S., Fellow of Trinity College, Cambridge.

(*Read January* 11*th*, 1872.)

In the applications of Bernoulli's series,

$$\Sigma u_x = C + \int u_x dx - \tfrac{1}{2}u_x + \frac{B_1}{1.2}\frac{du_x}{dx} - \frac{B_2}{1.2.3.4}\frac{d^3u_x}{dx^3} + \dots \quad \dots\dots(1),$$

the constant of integration C has to be determined separately for every different form of u_x; and in many cases [as, *e. g.*, when $u_x = \log x$ and $C = \log \sqrt{(2\pi)}$] the determination is very difficult. If u_x be of the form x^n, we may write the formula (1)

$$1^n + 2^n + \dots x^n = \Phi(n) + \frac{x^{n+1}}{n+1} + \tfrac{1}{2}x^n + \tfrac{1}{12}n x^{n-1} - \tfrac{1}{720}n(n-1)(n-2)x^{n-3} + \dots \quad \dots\dots(2),$$

in which $\Phi(n)$, the constant of integration with regard to x, is written for C.

If n be an even integer, it is well known that

$$\Phi(-n) = \tfrac{1}{2}\frac{(2\pi)^n B_{\frac{1}{2}n}}{1.2 \dots n} \quad \dots\dots(3),$$

$B_{\frac{1}{2}n}$ being the $\frac{1}{2}n^{\text{th}}$ Bernoulli's number; for in this case, since $n < -1$, the series on the left-hand side of (2), when continued to infinity, is convergent, and $\Phi(-n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \dots$.

If $n = -1$, the second term on the right-hand side of (2) is replaced by a logarithm, and the constant is well known as Euler's constant ·577215.... It was with the view of exhibiting the series of constants to which Euler's constant and the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \dots$, viz. $\frac{\pi^2}{6}$, belong, that Table I. was formed of the values of $\Phi\left(-\frac{1}{m}\right)$ and $\Phi\left(-1-\frac{1}{m}\right)$ for $m = 1, 2, \dots 20$.

The Table was calculated by means of (1); that is to say, that the functions were calculated from the formulæ

$$\Phi\left(-\frac{1}{m}\right) = 1^{-\frac{1}{m}} + 2^{-\frac{1}{m}} \dots + x^{-\frac{1}{m}} - \frac{m}{m-1}x^{1-\frac{1}{m}} - \tfrac{1}{2}x^{-\frac{1}{m}} + \frac{1}{12m}x^{-1-\frac{1}{m}} - \frac{1}{720m^3}(m+1)(2m+1)x^{-3-\frac{1}{m}} + \dots \quad \dots\dots(4),$$

$$\Phi\left(-1-\frac{1}{m}\right)=1^{-1-\frac{1}{m}}+2^{-1-\frac{1}{m}}\ldots+x^{-1-\frac{1}{m}}+mx^{-\frac{1}{m}}-\tfrac{1}{2}x^{-1-\frac{1}{m}}$$

$$+\frac{1}{12m}(m+1)\,x^{-2-\frac{1}{m}}-\frac{1}{720m^3}(m+1)(2m+1)(3m+1)x^{-4-\frac{1}{m}}+\ldots(5);$$

x was taken $=10$, so as to render the calculation of the descending series as easy as possible; and the terms written down in (4) and (5) were all that were included in the calculation.

The series $1^{-\frac{1}{m}}+2^{-\frac{1}{m}}+\ldots$ (m integral) is divergent, but $1^{-1-\frac{1}{m}}+2^{-1-\frac{1}{m}}+\ldots$ converges so that

$$\Phi\left(-1-\frac{1}{m}\right)=\frac{1}{1^{1+\frac{1}{m}}}+\frac{1}{2^{1+\frac{1}{m}}}+\frac{1}{3^{1+\frac{1}{m}}}+\ldots;$$

and the second column of Table I. (p. 55) may be regarded as a Table of the sums of series of this form.

It will be noticed in this table $\left[\text{viz., that of } \Phi\left(-1-\frac{1}{m}\right)\right]$ that

$$\Phi\left(-1-\frac{1}{m}\right)=m+\text{a decimal fraction},$$

and it is this fractional portion which tends to ·577215... as its limit; for, from (5),

$$1^{-1-\frac{1}{m}}+2^{-1-\frac{1}{m}}\ldots+x^{-1-\frac{1}{m}}=\Phi\left(-1-\frac{1}{m}\right)-m-\left[m\left(x^{-\frac{1}{m}}-1\right)\right]+\ldots,$$

and the limit of the term in square brackets, as m increases indefinitely, is $\log x$; so that Euler's constant

$$\gamma=\text{the limit of } \Phi\left(-1-\frac{1}{m}\right)-m \quad \ldots\ldots\ldots\ldots\ldots (6),$$

when m is infinite; $\Phi(-2)$ is, of course, $=\frac{1}{6}\pi^2$.

The descending series in (2) may be thrown into the form of a definite integral by means of the result

$$\mathrm{B}_n=4n\int_0^\infty\frac{t^{2n-1}dt}{e^{2\pi t}-1},$$

whence

$$1^n+2^n\ldots+x^n=\Phi(n)+\frac{x^{n+1}}{n+1}+\tfrac{1}{2}x^n$$

$$+2\int_0^\infty\frac{dt}{e^{2\pi t}-1}\left(nx^{n-1}t-\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}t^3+\ldots\right)$$

$$=\Phi(n)+\frac{x^{n+1}}{n+1}+\tfrac{1}{2}x^n-i\int_0^\infty\frac{(x+ti)^n-(x-ti)^n}{e^{2\pi t}-1}\,dt,$$

in which i is written for $\sqrt{-1}$. This expresses $\Phi(n)$ as a definite integral involving a quantity x, which may have any integral value. The simplest form is found by making $x=1$, when, after reduction,

we find $\Phi(n) = \frac{1}{2}\frac{n-1}{n+1} - i\int_0^\infty \frac{(1+ti)^n-(1-ti)^n}{e^{2\pi t}-1}\,dt$ .....(7);

so that from Table I. may be very easily deduced a table of the values of the integral in (7), n being of the forms $-\frac{1}{m}$ or $-1-\frac{1}{m}$. The expression which (7) gives for Euler's constant as a definite integral can be found at once by means of (6); for, putting $n=-1-h$, we find, since γ is the limit of $\Phi(-1-h)-\frac{1}{h}$ when $h=0$,

$$\gamma = \frac{1}{2} + 2\int_0^\infty \frac{t\,dt}{(e^{2\pi t}-1)(1+t^2)},$$

agreeing with a result given by De Morgan ("Diff. and Int. Calc.," p. 672).

The form which (7) gives for $\Phi(-2)$ is

$$\frac{3}{2} + 4\int_0^\infty \frac{t\,dt}{(e^{2\pi t}-1)(1+t^2)^2},$$

so that $$\int_0^\infty \frac{t\,dt}{(e^{2\pi t}-1)(1+t^2)^2} = \tfrac{1}{24}\pi^2 - \tfrac{3}{8};$$

and in the same way many similar definite integrals might be evaluated by putting n successively $=-4, -6, \ldots$.

It should be noticed that $\Phi(n)$ is finite whether $1^n+2^n+3^n+\ldots$ is divergent or no.

If n be negative, the sum of the series $1^n-2^n+3^n-\ldots$ is finite in all cases; and as the sum of this series can be made to depend on the value of $\Phi(n)$, we can deduce from Table I. the sum of series of this form when n lies between 0 and -2. We have, if n be negative,

$1^n+2^n+\ldots x^n = \Phi(n) + \frac{x^{n+1}}{n+1}$ + series of terms vanishing when $x=\infty$;

therefore, multiplying by 2^{n+1},

$2\{2^n+4^n\ldots+(2x)^n\} = 2^{n+1}\Phi(n) + \frac{(2x)^{n+1}}{n+1}$ + series of terms vanishing when $x=\infty$.

Writing, then, $2x$ for x in the first series, and subtracting the second from it, we find

$$1^n-2^n+3^n-\ldots = (1-2^{n+1})\,\Phi(n) \quad \ldots\ldots(8),$$

the series extending to infinity.

If then we denote the sum of the series $\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\ldots$ by $\Psi(n)$, we have $$\Psi(n) = (1-2^{-n+1})\,\Phi(-n) \quad \ldots\ldots(9),$$

and Table II. contains the values of $\Psi\left(\frac{1}{m}\right)$ and $\Psi\left(1+\frac{1}{m}\right)$ for $m=1, 2 \ldots 20$, calculated from the formula (9).

The first result in the column for $\Psi\left(\frac{1}{m}\right)$, viz. $\Psi(1)$,

$$= 1 - \frac{1}{2} + \frac{1}{3} - \ldots = \log_e 2;$$

and it is interesting to notice how, as m increases, $\Psi\left(\frac{1}{m}\right)$ approaches the value ·5. It is well known that in analytical investigations $\Psi(0)$ or $1-1+1-1+\ldots$ can usually be replaced by $\frac{1}{2}$; and it is to the fact that the last-written series is the limit of $1^{-1-\frac{1}{m}} - 2^{-1-\frac{1}{m}} + 3^{-1-\frac{1}{m}} - \ldots$, as well as of other series, that this result is due. The value of $\Psi(2)$ or $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$ is well known to be $\frac{\pi^2}{12}$. As m increases, the value of $\Psi\left(1+\frac{1}{m}\right)$ approaches, of course, $\log_e 2$ or ·6931471....

The facility with which the summation of these series is effected affords one of the most striking instances of the power of mathematics as a means of simplifying the results of arithmetical operations. The summation of $\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \ldots$ by direct calculation to three places of decimals would require the calculation of more than the first million terms; while the determination of the value of $\frac{1}{\sqrt[20]{1}} - \frac{1}{\sqrt[20]{2}} + \frac{1}{\sqrt[20]{3}} - \ldots$ correct to the first place of decimals would require the calculation of thousands of billions of terms, and would be truly impossible.

By means of the integral

$$\int_0^\infty e^{-ax^n}\,dx = \frac{1}{n}\,\Gamma\left(\frac{1}{n}\right)\frac{1}{a^{\frac{1}{n}}},$$

the value of $\Psi(n)$ can easily be expressed as a definite integral; for

$$\Psi\left(\frac{1}{n}\right) = \frac{n}{\Gamma\left(\frac{1}{n}\right)}\int_0^\infty e^{-x^n}\left(1 - e^{-x^n} + e^{-2x^n} - \ldots\right)dx$$

$$= \frac{n}{\Gamma\left(\frac{1}{n}\right)}\int_0^\infty \frac{dx}{1+e^{x^n}} \quad \ldots\ldots\ldots\

From (9) and (10) we obtain

$$\Phi(n) = \frac{1}{1-2^{n+1}} \cdot \frac{1}{\Gamma(1-n)} \int_0^\infty \frac{dx}{1+e^{x^{-\frac{1}{n}}}},$$

giving another form for the constant in (2).

By means of the integral (10) we can also express $\frac{1}{1^{1+\frac{1}{m}}} + \frac{1}{2^{1+\frac{1}{m}}} + \dots$ as an integral otherwise; for this series

$$= \frac{1}{\Gamma\left(\frac{2m+1}{m}\right)} \int_0^\infty \frac{dx}{e^{x^q}-1},$$

q being written for $\frac{m}{m+1}$; so that certain values of this integral also can be deduced from the last column of Table I.

Since the series $\frac{1}{1^{1+\frac{1}{m}}} + \frac{1}{3^{1+\frac{1}{m}}} + \frac{1}{5^{1+\frac{1}{m}}} + \dots$ is convergent, it seemed worth while also to calculate a Table (III.) of its values, as they are deducible from $\Phi\left(-1-\frac{1}{m}\right)$ by the formula

$$\Omega\left(1+\frac{1}{m}\right) = \left(1-2^{-1-\frac{1}{m}}\right) \Phi\left(-1-\frac{1}{m}\right),$$

$\Omega\left(1+\frac{1}{m}\right)$ denoting the sum of the series last written.

Expressing Ω as a definite integral, we find

$$\Omega\left(1+\frac{1}{m}\right) = \frac{2^{1+\frac{1}{m}}-1}{2^{\frac{1}{m}}+1} \frac{1}{\Gamma\left(2+\frac{1}{m}\right)} \int_0^\infty \frac{dx}{e^{x^q}-1},$$

q being written for $\frac{m}{m+1}$.

The sum of the series $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\dots$ (n odd) has never, I believe, been exhibited in a finite form analogous to (3), which applies only to the cases when n is even. A table of the numerical values of this series, that is, of $\Phi(-n)$, is given by Legendre ("Traité des Fonctions Elliptiques," vol. ii., p. 432) for $n=1, 2, 3, \dots 35$ to fifteen places of decimals. Legendre remarks that Euler calculated such a Table from $n=1$ to $n=16$ to sixteen places of decimals; but that, on examination of Euler's Table, he found several serious errors, which caused him to re-calculate the Table; at the same time he extended it to $n=35$. Euler's Table is given on p. 456 of this "Differential Calculus." A Table of the same extent as Legendre's is given in De Morgan's "Differential and Integral Calculus," p. 554. It is, no doubt, reprinted from Legendre.

We can, by means of the integral (10), express the series tabulated by Legendre as a definite integral; for since

$$\int_0^\infty e^{-ax^{-\frac{1}{n}}} dx = n\,\Gamma(n)\,a^{-n},$$

we find

$$\frac{1}{\Gamma(n+1)} \int_0^\infty \frac{dx}{e^{x^{-\frac{1}{n}}} \mp 1} = \frac{1}{1^n} \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \ldots \quad \ldots\ldots\ldots(11),$$

$$\frac{1}{\Gamma(n+1)} \int_0^\infty \frac{dx}{e^{x^{-\frac{1}{n}}} \mp e^{-x^{-\frac{1}{n}}}} = \frac{1}{1^n} \pm \frac{1}{3^n} + \frac{1}{5^n} \pm \ldots \quad \ldots\ldots(12),$$

so that from Legendre's Table the values of the integral

$$\int_0^\infty \frac{dx}{e^{x^{-\frac{1}{n}}} - 1}$$

can be easily deduced for the values $n=1, 2, \ldots 35$. Since

$$\frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \ldots = \left(1 - \frac{1}{2^n}\right)\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \ldots\right),$$

the values of

$$\int_0^\infty \frac{dx}{e^{x^{-\frac{1}{n}}} - e^{-x^{-\frac{1}{n}}}}$$

for the same values of n can also be deduced.

If n be odd, the sum of the series

$$\frac{1}{1^n} - \frac{1}{3^n} + \frac{1}{5^n} - \ldots$$

can be expressed by means of a finite series of Bernoulli's Numbers.* We thus find that if n be an odd integer

$$\int_0^\infty \frac{dx}{e^{x^{-\frac{1}{n}}} - e^{-x^{-\frac{1}{n}}}} = \left(\frac{\pi}{2}\right)^n \left[\tfrac{1}{2}(-1)^{\frac{1}{2}(n-1)} + 2^{n-2}\left\{\frac{2^{n-1}-1}{1} B_n - \ldots\right.\right.$$

$$\left.\left. \ldots - \frac{n(n-1)}{1.2} \frac{2^{n-3}-1}{3.2^2} B_{n-1} + \ldots \pm \frac{n \ldots 3}{\Gamma(n-1)} \frac{2^2-1}{(n-1)\,2^{n-2}} B_1 \right\}\right];$$

B_n being, as before, the nth Bernoulli's number.

The Gamma-function has been made use of in the form (10), as the results then take a more elegant form; the ordinary form, however, viz.,

$$\int_0^\infty e^{-ax} x^{n-1} \delta x = \frac{\Gamma(n)}{a^n}$$

would serve to express the results equally well, and perhaps more simply.

The Tables accompanying the paper were, in general, calculated by means of seven-figure logarithms; but in several instances ten-figure logarithms were requisite; as, for instance, in the calculation of the

* Herschel's Examples on the Calculus of Finite Differences, Camb. 1820; p. 95.

term $\frac{m}{m-1}x^{1-\frac{1}{m}}$ in (3). The results have been given to six decimal places, except in the Tables of $\Psi\left(1+\frac{1}{m}\right)$ and $\Omega\left(1+\frac{1}{m}\right)$, in the former of which only five figures, and in the latter six figures, are given; the curtailment of the number of figures in these last two cases is due to the fact that the method of calculation rendered the last two figures obtained uncertain. No pains have been spared by differencing and verification to make the Tables accurate; but it is possible in some cases, though by no means probable, that a result may be inaccurate by as much as 2 in the last figure.

Table IV., though not strictly connected with the subject of the present paper, has yet so much analogy with the other Tables as to properly find a place in connection with them. It gives the values of

$$s_n = \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \ldots,$$

and

$$\sigma_n = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \ldots,$$

and was deduced from Legendre's Table* of

$$S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \ldots,$$

by means of the formulæ

$$s_n = \left(1 - \frac{1}{2^{n-1}}\right) S_n, \quad \sigma_n = \left(1 - \frac{1}{2^n}\right) S_n.$$

If, therefore, Legendre's Table contains any error, there will be corresponding errors in Table IV.; the last figure may be a unit in error.

Expressed as definite integrals,

$$s_n = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} \frac{dx}{e^x+1},$$

$$\sigma_n = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} \frac{dx}{e^x-e^{-x}} = \frac{1}{2\Gamma(n)} \int_0^\infty x^{n-1} \frac{dx}{\sinh x},$$

writing sinh x for the hyperbolic sine of x; by taking $x^n = z$, s_n and σ_n can also be put in the forms

$$s_n = \frac{1}{\Gamma(n+1)} \int_0^\infty \frac{dx}{1+e^{x^{\frac{1}{n}}}},$$

$$\sigma_n = \frac{1}{2\Gamma(n+1)} \int_0^\infty \frac{dx}{\sinh x^{\frac{1}{n}}};$$

so that the values of any of these integrals are easily deduced from Table IV., when $n = 1, 2, \ldots 35$.

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* Traité des Fonctions Elliptiques, t. ii., p. 432.

TABLE I.

m	$\Phi\left(-\frac{1}{m}\right)$	$\Phi\left(-1-\frac{1}{m}\right)$	m	$\Phi\left(-\frac{1}{m}\right)$	$\Phi\left(-1-\frac{1}{m}\right)$
1	[0·577216]	1·644934	11	−0·592658	11·583795
2	−1·460354	2·612375	12	−0·584176	12·583250
3	−0·973360	3·600938	13	−0·577117	13·582788
4	−0·813279	4·595112	14	−0·571150	14·582392
5	−0·733921	5·591582	15	−0·566039	15·582048
6	−0·686582	6·589216	16	−0·561612	16·581748
7	−0·655154	7·587518	17	−0·557743	17·581482
8	−0·632775	8·586241	18	−0·554330	18·581246
9	−0·616033	9·585246	19	−0·551298	19·581035
10	−0·603037	10·584449	20	−0·548587	20·580845

TABLE II.

m	$\Psi\left(\frac{1}{m}\right)$	$\Psi\left(1+\frac{1}{m}\right)$	m	$\Psi\left(\frac{1}{m}\right)$	$\Psi\left(1+\frac{1}{m}\right)$
1	·693147	·82247	11	·520272	·70741
2	·604899	·76515	12	·518602	·70624
3	·571753	·74287	13	·517187	·70525
4	·554487	·73110	14	·515971	·70440
5	·543910	·72383	15	·514916	·70366
6	·536768	·71889	16	·513992	·70301
7	·531623	·71532	17	·513176	·70244
8	·527740	·71262	18	·512449	·70193
9	·524706	·71051	19	·511799	·70147
10	·522270	·70881	20	·511213	·70106

TABLE III.

m	$\Omega\left(1+\frac{1}{m}\right)$	m	$\Omega\left(1+\frac{1}{m}\right)$
1	1·23370	11	6·14561
2	1·68876	12	6·64475
3	2·17191	13	7·14402
4	2·66311	14	7·64340
5	3·15770	15	8·14286
6	3·65406	16	8·64238
7	4·15142	17	9·14196
8	4·64943	18	9·64159
9	5·14788	19	10·1413
10	5·64663	20	10·6410

TABLE IV.

n	s_n	σ_n
1	·69314 71805 599453	∞
2	·82246 70334 241132	1·23370 05501 361698
3	·90154 26773 696957	1·05179 97902 646450
4	·94703 28294 972459	1·01467 80316 041921
5	·97211 97704 469094	1·00452 37627 951397
6	·98555 10912 974351	1·00144 70766 409421
7	·99259 38199 228302	1·00047 15486 523764
8	·99623 30018 526479	1·00015 51790 252961
9	·99809 42975 416053	1·00005 13451 838438
10	·99903 95075 982715	1·00001 70413 630447
11	·99951 71434 980607	1·00000 56660 510900
12	·99975 76851 438581	1·00000 18858 485831
13	·99987 85427 632651	1·00000 06280 554218
14	·99993 91703 459797	1·00000 02092 405192
15	·99996 95512 130992	1·00000 00697 247031
16	·99998 47642 149061	1·00000 00232 371573
17	·99999 23782 920410	1·00000 00077 448395
18	·99999 61878 696101	1·00000 00025 814376
19	·99999 80935 081717	1·00000 00008 604442
20	·99999 90466 115815	1·00000 00002 868077
21	·99999 95232 582156	1·00000 00000 956012
22	·99999 97616 132308	1·00000 00000 318667
23	·99999 98808 013185	1·00000 00000 106221
24	·99999 99403 988924	1·00000 00000 035407
25	·99999 99701 988569	1·00000 00000 011802
26	·99999 99850 992320	1·00000 00000 003934
27	·99999 99925 495505	1·00000 00000 001311
28	·99999 99962 747534	1·00000 00000 000437
29	·99999 99981 373694	1·00000 00000 000146
30	·99999 99990 686822	1·00000 00000 000049
31	·99999 99995 343403	1·00000 00000 000016
32	·99999 99997 671699	1·00000 00000 000005
33	·99999 99998 835849	1·00000 00000 000002
34	·99999 99999 417924	1·00000 00000 000000
35	·99999 99999 708961	1·00000 00000 000000

On the Parallel Surfaces of Conicoids and Conics.

By Samuel Roberts, M.A.

(*Read January 11th*, 1872.)

1. If a straight line of given length moves with one extremity on a given surface, and remains normal to it in all positions, the locus of the other extremity is called a parallel of the given surface, or primitive.

Hence the parallel surface is the locus of the centres of spheres of constant radius which touch the primitive, or it is the envelope of such spheres or of spheres of constant radius which have their centres on the primitive. The idea of parallel curves in plane space is thus extended in a direct manner to space of three dimensions.

In the present paper, I propose to discuss the parallels of quadric surfaces or conicoids; but in several cases, where the results for a general surface of the order m are analogous to those corresponding to them for a conicoid, I have given general expressions, the proofs and developments of which I hope to give on some future occasion.

The parts of the theory already worked out are, as far as I know, contained in Dr. Salmon's "Geometry of Three Dimensions," 2nd Edition, p. 148, and in Professor Cayley's paper on the quartic surfaces $(*\,\Sigma\,\mathrm{U}, \mathrm{V}, \mathrm{W})^2$, ("Quarterly Journal of Mathematics," vol. xi., p. 15 *et seq.*) I have also had occasion to refer to a paper by Professor Clebsch on the problem of normals and the surface of centres (Crelle-Borchardt, t. lxii., p. 64).†

I. *The Parallel of a Central Conicoid.*

2. The equation of a parallel of a central conicoid, represented by

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0$$

† Prof. Clebsch, in the memoir referred to, has considered the surface of centres in a generalized way. Parallel surfaces might be similarly treated. In fact, if $x^2+y^2+z^2+w^2=0$ represents a fixed quadric surface, equal quasi-spheres having a double contact with it along a plane are represented by

$$x^2+y^2+z^2+w^2-(\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z+\mathrm{W}w)^2=0;$$

and X, Y, Z, W are proportional to the coordinates of the quasi-centre.

Forming the condition that these quasi-spheres may touch

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}=0,$$

we get

$$\Sigma\frac{a\mathrm{X}^2}{\theta+a}-1=0.$$

The discriminant with regard to θ must be made homogeneous. I have not found any important advantage in point of symmetry, and therefore confine myself to the special forms.

in rectangular coordinates, is obtained by equating to nothing the discriminant with respect to θ of the quartic equation

$$\frac{\theta x^2}{\theta+a}+\frac{\theta y^2}{\theta+b}+\frac{\theta z^2}{\theta+c}-\theta-k^2=0 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

which may also be arranged thus:

$$\frac{ax^2}{\theta+a}+\frac{by^2}{\theta+b}+\frac{cz^2}{\theta+c}+\theta-x^2-y^2-z^2+k^2=0 \quad \ldots\ldots\ldots (2);$$

or again, if we put $\theta'+d'$ for θ, a' for $a+d'$, b' for $b+d'$, c' for $c+d'$, we have the equivalent and symmetrical form

$$\frac{x^2}{\theta'+a'}+\frac{y^2}{\theta'+b'}+\frac{z^2}{\theta'+c'}-\frac{k^2}{\theta'+d'}-1=0 \quad \ldots\ldots\ldots\ldots (3).$$

The forms which I principally use are (1) and (2). I call k, which is the value of the constant normal distance between the parallel and its primitive, the modulus.

The equation (1) is obtained by forming the discriminant with respect to α, β, γ of

$$\theta\left(\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}-1\right)+\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2-k^2\}=0.$$

The equation of the parallel is thus the condition that a sphere of the radius k shall touch the primitive, the coordinates of the centre being the current coordinates relative to the parallel. If, on the other hand, x, y, z are given constants, we have an expression connecting the squares of the six normal distances from the point whose coordinates are x, y, z to the quadric. Corresponding to these distances, there can be described through the point six parallel surfaces.

It is obvious, especially from the form (3), that x^2, y^2, z^2, $-k^2$ enter the equation in the same way; and since the origin is not generally on the locus, its order is 12.

A similar remark is true of surfaces generally. The order of the parallel surface is equal to twice the number of normals which can be drawn to the primitive from a given point; that is to say, for a general surface of the order m, the order of the parallel is $2(m^3-m^2+m)$.

Limiting Case $k=0$.

3. When the modulus is nothing, the quartic (1) contains θ as a factor. Hence (Salmon's "Higher Algebra," 2nd Edition, Art. 106, p. 86) the equation of the parallel is of the form

$$\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1\right)^2\Phi=0,$$

where $\Phi=0$ represents the imaginary developable which circumscribes the system of confocals

$$\frac{x^2}{\theta+a}+\frac{y^2}{\theta+b}+\frac{z^2}{\theta+c}-1=0 \quad \ldots\ldots\ldots\ldots (a).$$

This developable is generated by planes which touch the primitive and pass through the tangent lines of the imaginary circle at infinity, and the result is most readily extended to surfaces of higher order by the aid of this definition.

If X, Y, Z of the parallel (mod. k) correspond to x, y, z of the primitive, we have

$$X = x + \frac{k\frac{x}{a}}{\left(\Sigma\frac{x^2}{a^2}\right)^{\frac{1}{2}}}, \quad Y = y + \frac{k\frac{y}{b}}{\left(\Sigma\frac{x^2}{a^2}\right)^{\frac{1}{2}}}, \quad Z = z + \frac{k\frac{z}{c}}{\left(\Sigma\frac{x^2}{a^2}\right)^{\frac{1}{2}}}, \quad \Sigma\frac{x^2}{a} - 1 = 0,$$

from which we get (1) by means of

$$X - x = \theta\frac{x}{a}, \quad Y - y = \theta\frac{y}{b}, \quad Z - z = \theta\frac{z}{c}.$$

If $k=0$, we have generally $X=x$, $Y=y$, $Z=z$, except for points on the curve

$$\Sigma\frac{x^2}{a} - 1 = 0, \quad \Sigma\frac{x^2}{a^2} = 0.$$

The developable $\Phi=0$ is the parallel of this curve, mod. 0; and, generally, the corresponding developable is the parallel (mod. 0) of the curve

$$U = 0, \quad \Sigma\left(\frac{dU}{dx}\right)^2 = 0.$$

The surface $\Phi=0$ is of the 8th order, having the focal conics of the primitive and the circle at infinity for double lines, together with a cuspidal curve of the order 12. It will be observed, from the form of the equation, that the sections by planes parallel to the principal planes are plane parallels to an imaginary modulus of the focal conics in those planes. Projected orthogonally on them, the sections in question form systems of parallels, and the loci of their cusps, *i. e.* the corresponding projections of the edge of regression, are the evolutes of the focal conics ("Messenger of Mathematics," August and November, 1871).

In like manner, the section at infinity is the envelope of the conic

$$\frac{x^2}{\theta + a} + \frac{y^2}{\theta + b} + \frac{z^2}{\theta + c} = 0,$$

or four right lines, together with the nodal circle

$$(x^2 + y^2 + z^2)^2 = 0.$$

The general characteristics of the surface are those of a developable touching two given conicoids, or of the reciprocal of a quadriquartic curve.

Sections by the Principal Planes and by the Plane at Infinity.

4. In order to obtain the equation of the section of the parallel by the principal plane $z=0$, we may make $z^2=0$ in (1). The quartic then contains $\theta + c$ as a factor, and the resulting equation is of the form

$$\left\{\text{Discr. of } \frac{\theta x^2}{\theta+a}+\frac{\theta y^2}{\theta+b}-\theta-k^2\right\}\left\{\frac{cx^2}{c-a}+\frac{cy^2}{c-b}+c-k^2\right\}^2=0;$$

that is to say, we have the plane parallel (mod. k) of the principal section of the conicoid by the plane $z=0$ and the conic

$$\frac{cx^2}{c-a}+\frac{cy^2}{c-b}+c-k^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which is a nodal curve, since z enters the equation of the parallel surface as a square.

The sections by the other principal planes are of the same kind.

The section by the plane at infinity is obtained by making the terms in (1) independent of x, y, z vanish. In this case, the quartic may be homogeneously written

$$\frac{\eta\theta x^2}{\theta+a\eta}+\frac{\eta\theta y^2}{\theta+b\eta}+\frac{\eta\theta z^2}{\theta+c\eta}=0.$$

The resulting equation is therefore of the form

$$\{[(b+c)x^2+(c+a)y^2+(a+b)z^2]^2-4(x^2+y^2+z^2)(bcx^2+cay^2+abz^2)\}$$
$$\times(x^2+y^2+z^2)^2(bcx^2+cay^2+abz^2)^2=0.$$

It follows that the section at infinity contains the nodal conics

$$x^2+y^2+z^2=0, \quad bcx^2+cay^2+abz^2=0,$$

and besides the right lines determined by

$$(b-c)^{\frac{1}{2}}x+(c-a)^{\frac{1}{2}}y+(a-b)^{\frac{1}{2}}z=0 \quad \ldots\ldots\ldots\ldots\ldots (5).$$

It will be observed, that since k, a finite length, is evanescent at infinity, the section is the same as for $k=0$.

Similarly, for a surface of the order m, the section of the parallel at infinity contains the corresponding section of the primitive as a nodal curve, the circle at infinity multiple in the degree $m(m-1)^2$ and the $2m(m-1)$ common tangents.

5. We have now to consider the relations of the several nodal conics to the plane parallels with which they are associated.

The plane parallel of the section of the primitive by the plane $x=0$ meets the axis of y in the two double points

$$x=0, \quad y^2=\frac{(c-k^2)(b-c)}{c}, \quad z=0;$$

but these points lie on the conic (4).

In like manner, the same conic passes through the two double points on the axis of x of the plane parallel of the section of the primitive by the principal plane $y=0$.

This determines the conic which has, with the focal conic in its plane, common asymptotes, their points of contact being two of the intersections of the lines (5).

Again (Salmon's "Geometry of Three Dimensions," 2nd Edition,

p. 402), when we make $z^2=0$ in (2), and take the discriminant relative to θ, we in effect find the envelope of the conics

$$\frac{ax^2}{\theta+a}+\frac{by^2}{\theta+b}+\theta-x^2-y^2+k^2=0,\quad z=0 \ldots\ldots\ldots (6),$$

which envelope will touch a conic of the system corresponding to a given value θ, where it meets the consecutive conic

$$\frac{ax^2}{(\theta+a)^2}+\frac{by^2}{(\theta+b)^2}-1=0 \ldots\ldots\ldots (7).$$

For the plane $z=0$, we make $\theta=-c$. The first conic is the double conic of the parallel in that plane; the second conic is the corresponding cuspidal conic of the surface of centres (Clebsch, Crelle-Borchardt, t. lxii., p. 64; Salmon's "Geometry of Three Dimensions," pp. 143 and 402). Hence the double conic of the parallel touches the plane parallel, which makes up with it the complete section by the plane $z=0$ at the points where it meets the corresponding cuspidal conic of the surface of centres.

The coordinates of the points are

$$x=\pm\left(\frac{bk^2-c^2}{b-a}\right)^{\frac{1}{2}}\frac{c-a}{c},\quad y=\pm\left(\frac{ak^2-c^2}{a-b}\right)^{\frac{1}{2}}\frac{c-b}{c};$$

and the common tangents therefore are

$$\left(\frac{bk^2-c^2}{b-a}\right)^{\frac{1}{2}}x+\left(\frac{ak^2-c^2}{a-b}\right)^{\frac{1}{2}}y+c-k^2=0.$$

Similarly, the nodal conic at infinity

$$bcx^2+cay^2+abz^2=0$$

is touched by the lines (5) at points on the surface of centres, and the circle

$$x^2+y^2+z^2=0$$

is touched by the same lines where it meets

$$ax^2+by^2+cz^2=0,$$

the corresponding cuspidal conic of the surface of centres.

6. I investigate here the general intersections of the conic (4) with the associated plane parallel, because they will recur in the sequel.

If we write u for $bx^2+ay^2-ab-k^2(a+b)$, v for $x^2+y^2-a-b-c-k^2$, and $-c$ for θ, the equations (6) and (7) may be replaced by

$$c^2v-cu+2c^3-abk^2=0 \ldots\ldots\ldots (8),$$

$$c^2(c^2-ab)(v+k^2+a+b+c)+c^2(a+b-2c)\{u+ab+k^2(a+b)\}$$
$$-k^2(c-a)^2(c-b)^2=0 \ldots (9),$$

the second equation representing the differential of (4) with respect to c, after dividing by the same quantity.

The discriminant of $\frac{\theta x^2}{\theta+a}+\frac{\theta y^2}{\theta+b}-\theta-k^2=0,$

or

$$\theta^3-(v+c)\theta^2-u\theta+abk^2=0,$$

is $-27a^2b^2k^4+4u^3+4abk^2(v+c)^2+u^2(v+c)^2+18abk^2u(v+c)=0$ ...(10).

If u be eliminated from (8), (9), it will be found that, after division by the factor $c(a+b)-(c^2+ab)$, the result is

$$c^2v+3c^3+abk^2=0.$$

When therefore $c^2v+2c^3-abk^2$ is substituted for cu in (10), the result will be divisible by $(c^2v+3c^3+abk^2)^2$.

Knowing this, we are able to see that the result of the elimination of cu is, when multiplied by c^3,

$$(c^2v+3c^3+abk^2)^2\{c^{\frac{1}{2}}v+2(c^3+a^{\frac{3}{2}}b^{\frac{1}{2}}k)\}\{c^{\frac{1}{2}}v+2(c^3-a^{\frac{3}{2}}b^{\frac{1}{2}}k)\}=0.$$

We have also

$$(cu+c^3+2abk^2)^2\{cu+a^{\frac{1}{2}}b^{\frac{1}{2}}k(2c^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}$$
$$\times\{cu+a^{\frac{1}{2}}b^{\frac{1}{2}}k(2c^{\frac{3}{2}}-a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}=0.$$

Equating to nothing the corresponding factors—that is to say, the two first, the two second, and the two third—we obtain in sets of four the intersections in question.

The Complementary Nodal Lines.

7. Independently of the five nodal conics in the principal planes and the plane at infinity, there is a nodal curve of the order 16, which we shall find to break up into two groups of right lines corresponding to positive and negative values of k. I call these the complementary nodal lines.

The equation (1), when expanded, is

$$\theta^4+(k^2+\Sigma a-\Sigma x^2)\theta^3+\{k^2\Sigma a+\Sigma bc-\Sigma(b+c)x^2\}\theta^2$$
$$+\{k^2\Sigma bc+abc-\Sigma(bcx^2)\}\theta+abck^2=0;$$

where Σa denotes $a+b+c$ &c.

We may write it $\theta^4+B\theta^3+C\theta^2+D\theta+E=0$ ..... (11),

and we must bear in mind that E is constant, and B, C, D are quadric functions.

The conditions that this equation may have two pairs of equal roots, are the equalities

$$\frac{4C-B^2}{8}=\frac{4EC-D^2}{8E}=\frac{D}{B}=\frac{EB}{D}=\frac{8E+BD}{4C}\text{.....(12).}$$

These give seven different equations, which, however, in the present case, are reducible to the two pairs of equations,

$$D-E^{\frac{1}{2}}B=0\text{.....(13)},\qquad D+E^{\frac{1}{2}}B=0\text{.....(15)},$$
$$4C-B^2-8E^{\frac{1}{2}}=0\text{.....(14)},\qquad 4C-B^2+8E^{\frac{1}{2}}=0\text{.....(16)},$$

showing that the nodal curve in question breaks up into two lines of the order 8 corresponding to positive and negative values of k.

It will be convenient to call these by anticipation the two nodal groups.

8. When we make $E=0=k$ in (11), divide by θ and take the discriminant of the residual equation, the result represents, as we know, the imaginary developable circumscribing the system of confocals (a).

The equation is then of the form

$$27\,(abc-\Sigma bcx^2)^2+4\,[\Sigma bc-\Sigma(b+c)x^2]^3$$
$$+4\,(abc-\Sigma bcx^2)(\Sigma a-\Sigma x^2)^2-(\Sigma a-\Sigma x^2)^2[\Sigma bc-\Sigma(b+c)x^2]^2$$
$$-18\,(\Sigma a-\Sigma x^2)[\Sigma bc-\Sigma(b+c)x^2](abc-\Sigma bcx^2)=0\ldots(17).$$

The intersection of this with the primitive is given by the surface

$$[\Sigma bc-\Sigma(b+c)x^2]^2\{4\,[\Sigma bc-\Sigma(b+c)x^2]-(\Sigma a-\Sigma x^2)^2\}=0\ldots(b).$$

The first factor relates to the line of contact

$$U=\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0,$$
$$\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2+\left(\frac{dU}{dz}\right)^2=0.$$

The second factor corresponds to (14) and (16).

I find it of consequence, with respect to the general theory, to remark that the complementary nodal groups when $k=0$ are represented by the intersection (taken twice) of the conicoid with the developable (17) as distinguished from the line of contact. This intersection is known to consist of eight rectilinear generators of the conicoid. The cuspidal curve, which will be presently considered, is represented in the same limiting case by the cuspidal edge of the developable and the line of contact taken three times. Thus a portion of the cuspidal curve is replaced by two united nodal curves, as in plane space we find cusps superseded by pairs of united double points.

9. The equations (13), (14) and (15), (16) may be written, if we attend to the corresponding signs,

$$D\mp E^{\frac{1}{2}}B=0,\ 4\,(C\mp 6E^{\frac{1}{2}})-(B-4E^{\frac{1}{2}})(B+4E^{\frac{1}{2}})=0\ \ldots\ldots(c),$$

or in full,

$$(bc-k\sqrt{abc})x^2+(ca-k\sqrt{abc})y^2+(ab-k\sqrt{abc})z^2$$
$$+k^3\sqrt{abc}-k^2\Sigma bc^2+k\sqrt{abc}\,\Sigma a-abc=0\ldots(18),$$
$$(\Sigma x^2)^2+2\Sigma(b+c-a-k^2)x^2+k^4-2r^2\Sigma a+8k\sqrt{abc}$$
$$-\Sigma a^{\frac{1}{2}}(\Sigma a^{\frac{1}{2}}-2a^{\frac{1}{2}})(\Sigma a^{\frac{1}{2}}-2b^{\frac{1}{2}})(\Sigma a^{\frac{1}{2}}-2c^{\frac{1}{2}})=0\ldots(19).$$

The circle at infinity is a double line on the surface represented by the second equation and the intersections of $\Sigma x^2=0$, $\Sigma bcx^2=0$ count twice on each oft he groups. They are in fact sextuple points on the total nodal curve.

For the plane $z=0$, the equation $D-E^{\frac{1}{2}}B=0$ becomes

$$bcx^2+cay^2-abc-k^2\Sigma bc-k\sqrt{abc}\,(x^2+y^2-\Sigma a-k^2)=0;$$

or, according to the former notation,

$$cu-k\sqrt{abc}\,v-abk^2=0.$$

Substituting in the same way in $B^2-4C+8E^{\frac{1}{2}}=0$, we get

$$v^2+4\,(u+cv+c^2)+8k\sqrt{abc}=0;$$

or, eliminating u, $\{c^{\frac{3}{2}}v+2\,(c^3+a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}^2=0,$

and similarly, $\{cu+a^{\frac{1}{2}}b^{\frac{1}{2}}k\,(2c^3+a^{\frac{1}{2}}b^{\frac{1}{2}}k)\}^2=0.$

The nodal groups therefore meet the principal planes in the points of simple intersection of the corresponding nodal conics and plane parallels (§ 6). These points are eight in number, and may be separated into two sets of four, each set belonging to one of the nodal groups, and each point being a double point thereon. The points are triple on the total nodal curve.

The actual coordinates are given by

$$c\,(a-b)\,x^2+(c-a)(a^{\frac{1}{2}}c^{\frac{1}{2}}-b^{\frac{1}{2}}k)^2=0,$$
$$c\,(b-a)\,y^2+(c-b)(b^{\frac{1}{2}}c^{\frac{1}{2}}-a^{\frac{1}{2}}k)^2=0,$$

and the same with k of the opposite sign.

When $k=0$, the systems of equations give the umbilics real and unreal of the primitive.

The results are, of course, analogous for the other principal planes.

By the foregoing conclusions, and the consideration of the reciprocal surface, I was led to infer that the complementary nodal curve must consist of sixteen right lines. I shall now show that this is really the case.

10. If we eliminate z^2 from

$$\kappa x^2+\lambda y^2+\mu z^2+\nu=0,$$
$$(x^2+y^2+z^2)^2+2\,(Kx^2+Ly^2+Mz^2)+N=0,$$

we have

$$\{(\mu-\kappa)\,x^2+(\mu-\lambda)\,y^2\}^2$$
$$+2\,\{[\kappa\,(\nu-\mu M)+\mu^2K-\mu\nu]\,x^2+[\lambda\,(\nu-\mu M)+\mu^2L-\mu\nu]\,y^2\}$$
$$+(\nu-\mu M)^2-\mu^2\,(M^2-N)=0.$$

Identifying the given pair of equations with (18) and (19),

$$\nu-\mu M=k^3\sqrt{abc}-k^2\Sigma ab+k\sqrt{abc}\,\Sigma a-abc$$
$$-(a+b-c-k^2)(ab-k\sqrt{abc})$$
$$=-(a+b)(\sqrt{ab}-k\sqrt{c})^2,$$

$$M^2-N=(a+b-c-k^2)^2$$
$$-(k^4-2k^2\Sigma a+8k\sqrt{abc}+a^2+b^2+c^2-2ab-2bc-2ca)$$
$$=4\,(\sqrt{ab}-k\sqrt{c})^2;$$

therefore $(\nu-\mu M)^2-\mu^2(M^2-N) = (a-b)^2(\sqrt{ab}-k\sqrt{c})^4$.

Again $\mu^2K-\mu\nu = (b+c-a-k^2)\,ab\,(\sqrt{ab}-k\sqrt{c})^2$
$+(\sqrt{ab}-k\sqrt{c})^2\{c\sqrt{ab}-k\sqrt{c}(a+b)+k^2\sqrt{ab}\}\sqrt{ab}$
$= (\sqrt{ab}-k\sqrt{c})^2\{ab^2-a^2b+2abc-k\sqrt{abc}\,(a+b)\}$,

$\kappa(\nu-\mu M) = -(a+b)(\sqrt{ab}-k\sqrt{c})^2(bc-k\sqrt{abc})$;

therefore $\kappa(\nu-\mu M)+\mu^2K-\mu\nu = b\,(a-b)(c-a)(\sqrt{ab}-k\sqrt{c})^2$.

Similarly the coefficient of y^2 is

$$a\,(a-b)(b-c)(\sqrt{ab}-k\sqrt{c})^2.$$

Hence we have

$\{b\,(c-a)\,x^2-a\,(b-c)\,y^2\}^2$
$+2\{b\,(c-a)\,x^2+a\,(b-c)\,y^2\}(a-b)(\sqrt{ab}-k\sqrt{c})^2$
$+(a-b)^2(\sqrt{ab}-k\sqrt{c})^4 = 0$,

or $\{b\,(c-a)\,x^2+a\,(b-c)\,y^2+(a-b)(\sqrt{ab}-k\sqrt{c})^2\}^2$
$-4ab\,(c-a)(b-c)\,x^2y^2 = 0$;

giving finally

$$\left.\begin{aligned}
&\sqrt{b\,(c-a)}\,x+\sqrt{a\,(b-c)}\,y+\sqrt{b-a}\,(\sqrt{ab}-k\sqrt{c}) = 0.\\
\text{Similarly we get}\quad&\\
&\sqrt{c\,(a-b)}\,y+\sqrt{b\,(c-a)}\,z+\sqrt{c-b}\,(\sqrt{bc}-k\sqrt{a}) = 0,\\
&\sqrt{a\,(b-c)}\,z+\sqrt{c\,(a-b)}\,x+\sqrt{a-c}\,(\sqrt{ca}-k\sqrt{b}) = 0.
\end{aligned}\right\}\ldots(20).$$

Each of these equations represents four planes containing one group of eight right lines. The planes are, of course, tangents to the conicoid represented by (18), which may be written thus :

$$\frac{\sqrt{bc}\,x^2}{(\sqrt{ab}-k\sqrt{c})(\sqrt{ca}-k\sqrt{b})}+\frac{\sqrt{ca}\,y^2}{(\sqrt{ab}-k\sqrt{c})(\sqrt{bc}-k\sqrt{a})}$$
$$+\frac{\sqrt{ab}\,z^2}{(\sqrt{ca}-k\sqrt{b})(\sqrt{bc}-k\sqrt{a})}-1 = 0\ldots.(21);$$

and the right lines are the eight generators of the conicoid passing through the circle at infinity.

We can consequently substitute for the equation (19) a form corresponding to second factor of (b), but derived from the above equation.

The six traces of the planes (20) on the plane at infinity are the six lines joining the points of intersection of

$$\Sigma x^2 = 0,\quad \Sigma(bcx^2) = 0.$$

The planes consequently touch the conicoid (21) at its twelve umbilics, and the nodal group consists of the eight imaginary generators on which the umbilics lie by threes.

If now we eliminate k from

$$\sqrt{b\,(c-a)}\,x-\sqrt{a\,(b-c)}\,y+\sqrt{b-a}\,(\sqrt{ab}-k\sqrt{c})=0,$$

$$\sqrt{c\,(b-a)}\,x-\sqrt{a\,(c-b)}\,z+\sqrt{c-a}\,(\sqrt{ca}-k\sqrt{b})=0,$$

we get

$$\frac{a^{\frac{1}{2}}x}{\sqrt{(a-b)(a-c)}}+\frac{b^{\frac{1}{2}}y}{\sqrt{(b-a)(b-c)}}+\frac{c^{\frac{1}{2}}z}{\sqrt{(c-a)(c-b)}}-1=0\ldots(22),$$

representing eight planes independent of k which contain the whole of the sixteen right lines. In selecting the forms of the equations from which we eliminate, we must take care that both the planes contain the same line of the group.

The order of the total nodal curve is 26.

For a surface of the order m and otherwise general, the order of the nodal curve is $2m^6-4m^5+6m^4-6m^3-14m^2+17m$.

I may remark that each section of the parallel surface by a plane (22) consists of a pair of nodal right lines twice over and a curve of the 8th degree, with eight double points and twelve cusps. In fact, considering one of the pair of nodal right lines, we have seen that it meets a nodal conic and a nodal right line in each of the three principal planes. Moreover, the pair of nodal right lines in the plane meet another pair and an intersection of two nodal conics at infinity. This accounts for sixteen intersections of the plane with the nodal curve. There are eight more, six of them on nodal right lines and two on the nodal conics at infinity. These give the nodes of the section. The cuspidal curve (§ 11) meets the plane in 24 points, but 12 of these are on the nodal right lines and are due to stationary points; the remaining 12 give the cusps of the section.

The section by any plane containing two nodal right lines is of course similarly compound.

The planes (22) touch the circle at infinity, where it is met by the primitive conicoid, and touch the conicoid at the eight points

$$x=\frac{a^{\frac{1}{2}}}{\sqrt{(a-b)(a-c)}},\quad y=\frac{b^{\frac{1}{2}}}{\sqrt{(b-a)(b-c)}},\quad z=\frac{c^{\frac{1}{2}}}{\sqrt{(c-a)(c-b)}},$$

which are remarkable as being points through which no confocal different from the primitive conicoid can be drawn. The planes intersect in these points as triple, and in two points on each of the intersections of the principal planes and the line at infinity as quadruple. The eight points above mentioned coincide with their principal centres of curvature, and therefore lie on the surface of centres—in fact, on the cuspidal conics.

The Cuspidal Curve.

11. The equations of the cuspidal curve, corresponding to the case of three equal roots θ, are obtained by equating to zero the two fundamental invariants of (1); that is to say, we have

$$12E-3BD+C^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23),$$

$$72EC+9BCD-27D^2-27EB^2-2C^3=0 \quad \ldots\ldots\ldots\ldots \quad (24).$$

The curve therefore is of the order 24, and forms the complete intersection of a quartic and a sextic surface.

From the above we get, by elimination of the term BCD,

$$108EC-27D^2-27EB^2+C^3=0 \quad \ldots\ldots\ldots\ldots\ldots \quad (25);$$

and again, by means of (23),

$$\left.\begin{aligned}(C-6E^{\frac{1}{2}})^3-27(D-E^{\frac{1}{2}}B)^2=0\\(C+6E^{\frac{1}{2}})^3-27(D+E^{\frac{1}{2}}B)^2=0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (26),$$

or
$$(C-6E^{\frac{1}{2}})^3-(C+6E^{\frac{1}{2}})^3-27\{(D-E^{\frac{1}{2}}B)^2-(D+E^{\frac{1}{2}}B)^2\}=0,$$
$$(C-6E^{\frac{1}{2}})^3+(C+6E^{\frac{1}{2}})^3-27\{(D-E^{\frac{1}{2}}B)^2+(D+E^{\frac{1}{2}}B)^2\}=0,$$

which are equivalent to (23) and (25).

The cuspidal curve meets the plane at infinity in the points of intersection of $\Sigma(b+c)x^2=0$ with $\Sigma x^2=0$ and $\Sigma(bcx^2)=0$; that is to say, in the eight points of contact of the tangents common to the nodal conics at infinity, which are also the points where the locus of the intersection of three rectangular tangent lines to the primitive meets the primitive and the circle. Each of these points counts as three intersections with the cuspidal curve. They are, in fact, stationary points on the curve.

12. To find the intersections of the cuspidal curve with the principal plane $z=0$, we may employ the equations (26), note being taken of the extraneous curve, counting six times, where $C=0$ meets the plane at infinity.

Making $z^2=0$, we may write those equations, according to the former notation,

$$(u+cv+c^2-6k\sqrt{abc})^3+27(cu-k^2ab+k\sqrt{abc}\,.\,v)^2=0,$$
$$(u+cv+c^2+6k\sqrt{abc})^3+27(cu-k^2ab-k\sqrt{abc}\,.\,v)^2=0.$$

But for the cusps of the plane parallel represented by the discriminant of
$$\theta^3-(v+c)\theta^2-u\theta+abk^2=0,$$
that is to say, of the plane parallel which forms part of the section of the parallel surface by the plane $z=0$, we have

$$v=-3(abck^2)^{\frac{1}{3}}-c, \quad u=-3(abk^2)^{\frac{2}{3}},$$

values which satisfy the equations of the cuspidal curve. Hence, as

might be geometrically anticipated, the twelve cusps of the plane parallel are on the cuspidal curve.

Again, we have seen that the points where the double conic in the plane $z=0$ touches the associated plane parallel are determined by

$$c^2v+3c^3+abk^2=0, \quad cu+c^3+2abk^2=0.$$

The values of v and u thence derived also satisfy the equations of the cuspidal curve. Moreover, from the form of these equations, it appears that the points in question count three times as intersections with the cuspidal curve. They are, in fact, stationary points.

13. Since the elimination of θ from (2), and its first and second differentials with respect to θ, gives the equations of the cuspidal curve of the parallel, and since the like elimination from those differentials gives the surface of centres, we see, what is also apparent geometrically, that the cuspidal curve lies upon the surface of centres. This seems to be generally true. A system of parallels may be regarded as a system of surfaces having a common surface of centres. There will be two branches of the cuspidal curve corresponding to the two sheets of the surface of centres. The general order for a surface of the order m is $12m(m-1)$.

Stationary Points due to four equal roots θ.

14. We have next to consider the points corresponding to the case when (1) has four equal roots.

These points are stationary points of the cuspidal curve, and are in one way obtained by combining (2) with its first, second, and third differentials with regard to θ.

We have then to eliminate θ and obtain the equations of three surfaces whose 32 nett intersections are the points in question.

Thus we may infer that these points are, in fact, the intersections of the cuspidal curve of the parallel with the cuspidal curve of the surface of centres, *i. e.* its eight cuspidal conics not in the principal planes or the plane at infinity.

In fact, the equations (*c*) and (26) have been so written that we see at once that the 32 points determined by

$$\left.\begin{aligned} \mathrm{B} &= 4\omega\, \mathrm{E}^{\frac{1}{4}} \\ \mathrm{C} &= 6\omega^2 \mathrm{E}^{\frac{1}{2}} \\ \mathrm{D} &= 4\omega^3 \mathrm{E}^{\frac{3}{4}} \end{aligned}\right\}, \quad \omega^4=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

are points on the complementary nodal lines and on the cuspidal curve.

The above equations are directly obtained by making (1) a perfect biquadratic or fourth power.

Further, the system of conditions being

$$\Sigma\frac{ax^2}{\theta+a}+\theta-\Sigma x^2+k^2=0,$$

$$\Sigma\frac{ax^2}{(\theta+a)^2}-1=0,\quad \Sigma\frac{ax^2}{(\theta+a)^3}=0,\quad \Sigma\frac{ax^2}{(\theta+a)^4}=0,$$

the three last may also be written

$$\Sigma\frac{ax^2}{(\theta+a)^4}=0,\quad \Sigma\frac{a^2x^2}{(\theta+a)^4}=0,\quad \Sigma\frac{a^3x^2}{(\theta+a)^4}-1=0.$$

From these we get

$$\frac{x^2}{(\theta+a)^4}=\frac{1}{a(b-a)(c-a)},\qquad \frac{y^2}{(\theta+b)^4}=\frac{1}{b(a-b)(c-b)},$$

$$\frac{z^2}{(\theta+c)^4}=\frac{1}{c(a-c)(b-c)};$$

therefore, by the second of the given system of equations,

$$\frac{a^{\frac{1}{2}}x}{\sqrt{(b-a)(c-a)}}+\frac{b^{\frac{1}{2}}y}{\sqrt{(a-b)(c-b)}}+\frac{c^{\frac{1}{2}}z}{\sqrt{(a-c)(b-c)}}-1=0\quad\ldots(22).$$

The points in question lie therefore in the eight planes (22), in which also lie the cuspidal conics of the surface of centres not in the principal planes or the plane at infinity.

15. Actually to determine the points, we may revert to the system (27), which may be written

$$\begin{aligned}
x^2-k^2\quad -\Sigma a\ +4(abck^2)^{\frac{1}{4}}\omega+\quad y^2+\quad z^2&=0,\\
bcx^2-k^2\Sigma bc-abc+4(abck^2)^{\frac{3}{4}}\omega^3+\quad cay^2+\quad abz^2&=0,\\
(b+c)x^2-k^2\Sigma a\ -\Sigma bc+6(abck^2)^{\frac{1}{2}}\omega^2+(c+a)y^2+(a+b)z^2&=0,
\end{aligned}$$

giving

$$\begin{aligned}
&\{x^2-k^2-\Sigma a+4(abck^2)^{\frac{1}{4}}\omega\}\,a^2(c-b)\\
&\quad+\{bcx^2-k^2\Sigma bc-abc+4(abck^2)^{\frac{3}{4}}\omega^3\}\,(c-b)\\
&\quad-\{(b+c)x^2-k^2\Sigma a-\Sigma bc+6(abck^2)^{\frac{1}{2}}\omega^2\}\,a(c-b)=0.
\end{aligned}$$

The coefficient of k^2 is $-bc(c-b)$, and the value of $a^2\Sigma a+abc-a\Sigma bc$ is a^3. Hence we get

$$a^{\frac{1}{2}}x=\pm\frac{\{a-(abck^2)^{\frac{1}{4}}\omega\}^2}{\sqrt{(a-b)(a-c)}},$$

$$b^{\frac{1}{2}}y=\pm\frac{\{b-(abck^2)^{\frac{1}{4}}\omega\}^2}{\sqrt{(b-a)(b-c)}},$$

$$c^{\frac{1}{2}}y=\pm\frac{\{c-(abck^2)^{\frac{1}{4}}\omega\}^2}{\sqrt{(c-a)(c-b)}}.$$

These points lie upon the cuspidal conics of the surface of centres which are determined by the cones

$$a^{\frac{1}{2}}(b-a)^{\frac{1}{2}}(c-a)^{\frac{1}{2}}x^{\frac{1}{2}}-b^{\frac{1}{2}}(a-b)^{\frac{1}{2}}(c-b)^{\frac{1}{2}}y^{\frac{1}{2}}-a+b=0,$$
$$b^{\frac{1}{2}}(a-b)^{\frac{1}{2}}(c-b)^{\frac{1}{2}}y^{\frac{1}{2}}-c^{\frac{1}{2}}(a-c)^{\frac{1}{2}}(b-c)^{\frac{1}{2}}z^{\frac{1}{2}}-b+c=0,$$
$$c^{\frac{1}{2}}(a-c)^{\frac{1}{2}}(b-c)^{\frac{1}{2}}z^{\frac{1}{2}}-a^{\frac{1}{2}}(b-a)^{\frac{1}{2}}(c-a)^{\frac{1}{2}}x^{\frac{1}{2}}-c+a=0.$$

The right lines (13), (14) then intersect the cuspidal conics in sixteen points on two spheres. The points are determined by

$$B=\pm 4E^{\frac{1}{2}},$$
$$D=\pm 4E^{\frac{1}{2}},$$
$$C=\quad 6E^{\frac{1}{2}}.$$

The right lines (15), (16) intersect the cuspidal conics in the sixteen points

$$B=\pm 4E^{\frac{1}{2}}\sqrt{-1},$$
$$D=\pm 4E^{\frac{1}{2}}\sqrt{-1},$$
$$C=-6E^{\frac{1}{2}},$$

also on two spheres. Each sphere passes through one point in each of the eight planes (22), and each plane contains four points which are determined by the cuspidal conics of the surface of centres.

When $k=0$, the thirty-two points fall together in eight points which are the intersections of the primitive, the locus of the intersection of three rectangular tangent lines, and the locus of the intersection of three rectangular tangent planes.

II. *The Parallel of a Paraboloid.*

16. This case requires a separate discussion, though some detail may be dispensed with.

If the primitive is represented by

$$\frac{x^2}{a}+\frac{y^2}{b}-2z=0,$$

the equation of the parallel is obtained by equating to zero the discriminant of

$$\frac{\theta x^2}{\theta+a}+\frac{\theta y^2}{\theta+b}-\theta(2z+\theta)-k^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(28),$$

or

$$\frac{ax^2}{\theta+a}+\frac{by^2}{\theta+b}+\theta(2z+\theta)-x^2-y^2+k^2=0.$$

As before, the equation of the parallel represents also the relation which exists between the squares of the five normal distances from a point to the primitive, and, from the manner in which x^2, y^2, $-k^2$ enter, the order of the parallel is 10.

When $k=0$, we have the paraboloid itself twice over and the developable circumscribing the confocal system

$$\frac{x^2}{\theta+a}+\frac{y^2}{\theta+b}-(2z+\theta)=0.$$

The characteristics of this developable are those of a developable circumscribing two conicoids which touch, or the reciprocal of a quadriquartic curve with a double point. The cuspidal curve is therefore of the order 6. There are two nodal conics, and the line of simple intersection with the primitive is of the order 4. The remark in § 8 holds good in this case also.

Sections by the Principal Planes and the Plane at Infinity.

17. Making $y^2=0$ in (28), we have a factor $\theta+b$, and the section is given by

$$\left\{\text{Disct. of } \frac{\theta x^2}{\theta+a}-(2b+\theta)-k^2\right\}\left\{\frac{bx^2}{b-a}+b(2z-b)-k^2\right\}^2=0,$$

and contains the plane parallel (mod. k) of the parabola which is the section of the primitive by the plane $y=0$, together with a nodal parabola.

The case is similar with respect to the section by $x=0$.

As to the section by the plane at infinity, we may make $k=0$. The section is then determined by

$$(x^2+y^2+z^2)(x^2+y^2)^2\left(\frac{x^2}{a}+\frac{y^2}{b}\right)^2=0.$$

The lines $\frac{x^2}{a}+\frac{y^2}{b}=0$ are nodal lines.

The plane at infinity touches along the lines $x^2+y^2=0$; but they appear not to be nodal.

18. If we write u for $x^2-2az-k^2$, and v for $-2z-a-b$, and proceed as in § 6, we get for the intersections of the nodal conic in the plane $y=0$ with the associated plane parallel

$$(b^2v+3b^3+ak^2)^2\{b^2v+2(b^3+a^2k)\}\{b^2v+2(b^3-a^2k)\}=0,$$
$$b^2v-bu+2b^3-ak^2=0.$$

These expressions give the finite intersections.

These points are singular in the same manner as in the general case; that is to say, the complementary nodal lines which will be next discussed pass in pairs through the four finite simple intersections. The cuspidal curve passes through the two finite points of contact, each of which counts as three intersections.

Two of the points of contact are at infinity, and each of these also counts as three intersections with the cuspidal curve.

The Complementary Nodal Lines.

19. We may still make use of equation (11), but B is now to be taken as of the first order in x, y, z. Consequently the equations (14) and

(16) are only of the second degree, and the nodal curve divides into two lines of the order 4, each corresponding to one value of $\sqrt{k^2}$.

In fact, for the primitive $\frac{x^2}{a}+\frac{y^2}{b}-2z=0$ we have

$$D-E^{\frac{1}{2}}B \equiv bx^2+ay^2-(\sqrt{ab}-k)\{2\sqrt{ab}\,z-k(a+b)\}=0 \quad \ldots(29),$$

$$4C-B^2-8E^{\frac{1}{2}} \equiv x^2+y^2+\left(z-\frac{a+b}{2}\right)^2-(\sqrt{ab}-k)^2=0 \quad \ldots\ldots\ldots\ldots(30).$$

The sphere is one of those parallel (modulus k) to the sphere which passes through the intersection of the primitive with the developable circumscribing the confocal system.

Eliminating y^2 from (29) and (30), we get

$$(b-a)x^2-a\left(z-\frac{a+b}{2}\right)^2-2(\sqrt{ab}-k)\sqrt{ab}\left(z-\frac{a+b}{2}\right)-b(\sqrt{ab}-k)^2=0,$$

or $$(b-a)x^2-\left\{\sqrt{a}\left(z-\frac{a+b}{2}\right)+\sqrt{b}\,(\sqrt{ab}-k)\right\}^2=0,$$

representing two planes.

Similarly we get

$$(a-b)y^2-\left\{\sqrt{b}\left(z-\frac{a+b}{2}\right)+\sqrt{a}\,(\sqrt{ab}-k)\right\}^2=0,$$

representing other two planes. } .....(31).

Hence the nodal line in question consists of four right lines, imaginary generators of the paraboloid (29). The other nodal group also consists of four similar right lines.

The intersections of $\Sigma x^2=0$, $\frac{x^2}{a}+\frac{y^2}{b}=0$ at infinity are intersections of pairs of lines of the nodal groups. The total order of the nodal curve is 14.

The Cuspidal Curve.

20. The surfaces (23) and (24) intersect in more than the cuspidal curve of the parallel. This curve is of the order 18. In fact, the surfaces $12E-3BD+C^2=0$, $96EC+3BCD-27D^2-57EB^2=0$,

respectively of the orders 4 and 5, intersect in the cuspidal curve, and in the curve where $C=0$ meets the plane at infinity; *i.e.*, in the lines $x^2+y^2=0$.

This cuspidal curve also lies on the surface of centres, and it will be observed that its order is twice the order of that surface, as in the general case.

The curve meets each of the principal planes $x=0$, $y=0$, as has been stated in two finite points counting as three, and also passes through the six cusps of the corresponding plane parallel of a parabola. Also in each of these planes the curve passes through the point where the

axis of z meets the plane at infinity, each passage counting as six intersections with those planes. The nature of these intersections is most clearly perceived by reference to the general case where the corresponding section of the primitive is an ellipse.

We now see, from the form of the above equations, that the curve in question meets the plane at infinity in the point, or rather adjacently to the point where the axis of z meets the plane at infinity, and also where $x^2+y^2+z^2=0$ is touched by the lines $x^2+y^2=0$.

21. We have then, at the point where the axis of z meets the plane at infinity, a singular point of high complexity. From the analogy of the general case, I infer that the singularity arises from the union of two stationary points in each of the planes $x=0$, $y=0$, and of four stationary points in the plane at infinity. We shall see subsequently that this estimate of the singularities is consistent with the general characteristics.

In the case of the plane parallel of a parabola, we have two branches having with one another a contact of the second order at infinity; *i.e.*, the corresponding singularity counts as three adjacent, but not coincident, double points. There is evidently a corresponding singularity here, but it is more difficult to conceive clearly, and involves more, probably, than the mutual contact of two sheets of the surface.

Stationary Points due to four equal roots θ.

22. Interpreting the equations (27) in accordance with the forms of B, C, D, E for a paraboloid primitive, we see that there are sixteen such points.

We have, by substitution in these equations,

$$2z+a+b-4\,(abk^2)^{\frac{1}{4}}\,\omega=0,$$

$$x^2+y^2-2\,(a+b)\,z-abk^2+6\,(abk^2)^{\frac{1}{2}}\,\omega^2=0,$$

$$bx^2+cy^2-2abz-(a+b)\,k^2+4\,(abk^2)^{\frac{3}{4}}\,\omega^3=0,$$

$$\omega^4=1;$$

giving
$$x^2=\frac{\{a-(abk^2)^{\frac{1}{4}}\,\omega\}^4}{a\,(b-a)},\quad y^2=\frac{\{b-(abk^2)^{\frac{1}{4}}\,\omega\}^4}{b\,(a-b)},$$

$$z=2\,(abk^2)^{\frac{1}{4}}\,\omega-\frac{a+b}{2}.$$

These values satisfy the equation

$$\frac{a^{\frac{1}{2}}x}{\sqrt{b-a}}+\frac{b^{\frac{1}{2}}y}{\sqrt{a-b}}-z+\frac{a+b}{2}=0 \quad \ldots\ldots\ldots\ldots\ldots (32),$$

representing four planes. We also get these equations by eliminating k from (31).

If now we determine for the surface of centres the cuspidal conics not in the principal planes or the plane at infinity, by means of

$$\frac{ax^2}{(\theta+a)^2}+\frac{by^2}{(\theta+b)^2}-2z-2\theta=0,$$

$$\frac{ax^2}{(\theta+a)^3}+\frac{by^2}{(\theta+b)^3}+1=0,$$

$$\frac{ax^2}{(\theta+a)^4}+\frac{by^2}{(\theta+b)^4}=0,$$

we get

$$\frac{by^2}{(\theta+b)^4}-\frac{1}{(a-b)}=0,$$

$$\frac{ax^2}{(\theta+a)^4}-\frac{1}{(b-a)}=0,$$

$$4\theta+2z+a+b=0;$$

and the cuspidal conics are in the four planes (32). They are also determined by the cylinders

$$b^{\frac{1}{2}}(a-b)^{\frac{1}{2}}y^{\frac{1}{2}}-a^{\frac{1}{2}}(b-a)^{\frac{1}{2}}x^{\frac{1}{2}}+a-b=0,$$

$$b^{\frac{1}{2}}(a-b)^{\frac{1}{2}}y^{\frac{1}{2}}+\frac{z}{2}+\frac{a+b}{4}-b=0,$$

$$a^{\frac{1}{2}}(b-a)^{\frac{1}{2}}x^{\frac{1}{2}}+\frac{z}{2}+\frac{a+b}{4}-a=0.$$

III. *The Parallel of a Central Conicoid of Revolution.*

23. If in the fundamental equation (1) we put $a=b$, the biquadratic contains the factor $\theta+a$, and we have, by taking the discriminant with regard to θ, the extraneous planes $(x^2+y^2)^2=0$, and the parallel of a central conicoid of revolution.

The nett equation of the parallel is thus reduced to the degree 8, and is the discriminant of a cubic equation. From the form

$$\frac{x^2+y^2}{\theta+a}+\frac{z^2}{\theta+c}-1-\frac{k^2}{\theta}=0,$$

it is evident that the parallel is represented by writing x^2+y^2 for x^2 in $\phi(x^2, z^2, -k^2)=0$, the equation of the plane parallel of the conic $\frac{x^2}{a}+\frac{z^2}{c}-1=0$. The surface is therefore generated by the revolution of the plane parallel of a conic about one of its principal axes. The general nature and form of the surface is easily conceived. It is instructive, however, with reference to the general characteristics, to consider the nodal and cuspidal curves.

A section by a plane perpendicular to the axis will consist generally of four circles. For $z=0$, however, we have a double circle

$$\frac{c(x^2+y^2)}{c-a}+c-k^2=0,$$

due, of course, to the revolution of the corresponding pair of double points of the revolving parallel. There will also be a pair of nodal conics at infinity; namely, the circle at infinity and a conic having a double contact with it.

Hence we have a nodal curve of the order 6. We shall also have six cuspidal circles due to the corresponding pairs of cusps of the plane parallel. The order of the cuspidal curve is therefore 12. The double points of the revolving plane parallel situate on the axis give rise to a pair of conical points or cnicnodes.

IV. *The Parallel of a Paraboloid of Revolution.*

24. The order of the parallel is reduced to 6. The finite double point of the parallel on the axis generates a cnicnode. The three corresponding pairs of cusps generate three cuspidal circles. We have, besides, to estimate the singularity generated by the revolution of the three united double points of plane parallel at an infinite distance on or adjacent to the axis. The result seems to be that at infinity the lines $x^2+y^2=0$ are cuspidal. The revolution of two symmetrically placed adjacent double points at infinity gives rise to the lines $x^2+y^2=0$ as nodal on the surface. The addition of a double point on the axis of the generating curve again gives these lines on the surface. The three united double points cause the lines to be cuspidal (art. 30).

V. *The Parallel in space of a Central Conic.*

25. The parallel surface of a curve is a tubular surface. For a curve of the order m and rank r, the order of the tubular surface is, in general, $2(r+m)$, and the class $2r$. When the primitive is an ellipse, the parallel is called the elliptic ring. Using the term "ring" in a free sense, we may similarly speak of the hyperbolic ring, and the parabolic ring.

For the equation of the parallel of the central conic $\frac{x^2}{a}+\frac{y^2}{b}-1=0$, $z=0$, we must take the discriminant, with regard to θ, of

$$\frac{x^2}{\theta+a}+\frac{y^2}{\theta+b}+\frac{z^2-k^2}{\theta}-1=0 \quad \ldots\ldots\ldots\ldots\ldots (33),$$

obtained by putting $c=0$ in (1). The primitive is thus treated as a limiting case of a surface, as conics are included in a system of confocal conicoids. The equation (33) can, however, be obtained directly. (Cayley, Quarterly Journal of Mathematics, xi. p. 19.)

The equation of the parallel surface is therefore obtained by writing z^2-k^2 for $-k^2$ in the equation of the plane parallel of the primitive.

As long as $z^2 < k^2$, the section of the surface by a plane parallel to

that of the primitive, is a plane parallel of the primitive modulus $\sqrt{k^2-z^2}$. The planes $z^2-k^2=0$ touch the surface along conics which are the same as the primitive in form and magnitude. Outside these planes the sections are imaginary. The order of the surface is evidently 8.

If $y=0$, $\theta+b$ becomes a factor of the cubic equation, and the discriminant is of the form

$$\left\{(x^2+z^2-k^2-a)^2+4a(z^2-k^2)\right\}\left\{\frac{x^2}{a-b}-\frac{z^2-k^2}{b}-1\right\}^2=0.$$

If $x=0$, we have

$$\left\{(y^2+z^2-k^2-b)^2+4b(z^2-k^2)\right\}\left\{\frac{y^2}{b-a}-\frac{z^2-k^2}{a}-1\right\}^2=0.$$

For the plane at infinity, the section is determined by the discriminant of

$$\frac{x^2}{\theta+a}+\frac{y^2}{\theta+b}+\frac{z^2}{\theta}=0;$$

or $$\{[bx^2+ay^2+(a+b)z^2]^2-4abz^2(x^2+y^2+z^2)\}(x^2+y^2+z^2)^2=0.$$

We have found three nodal conics, and two torsal conics, indicating nodes on the reciprocal surface.

The equation (33) gives for the cuspidal curve

$$\theta^3+3B\theta^2+3C\theta+D=0,$$

$$C-B^2=0,\quad BD-C^2=0,\quad D-BC=0.$$

Its order is 12.

There are two cnicnodes at infinity where the primitive meets the plane at infinity.

V'.

If, however, the primitive is a circle, the parallel surface is the common tore or anchor ring, generated by the revolution of a circle about an axis in its plane. The surface has as nodal conic the circle at infinity, and four cnicnodes; viz., two on the axis, and two at the points where the primitive meets the plane at infinity, that is to say, on the nodal circle. The order is 4. A good deal has been written on this surface and the general tore.

VI. *The Parallel in space of a Parabola.*

26. The equation of this parallel is obtained in the same manner as that of a central conic by equating to nothing the discriminant of

$$\frac{\theta x^2}{\theta+a}+y^2-(2z+\theta)\theta-k^2=0 \quad \ldots\ldots\ldots\ldots\ldots (34),$$

the result of writing $b=0$ in (28). We may, however, derive the form independently, as in the last case.

As long as $y^2 < k^2$, the section by a plane parallel to the plane of the primitive represented by $\frac{x^2}{a} - 2z = 0$, is a parallel of the primitive.

The planes $y^2 - k^2 = 0$ touch along two parabolas equal and similar to the primitive.

When $x=0$, $\theta + a$ is a factor of (34), and the discriminant is of the form $$(z^2+y^2-k^2)\{y^2+a(2z-a)-k^2\}^2 = 0,$$ showing a nodal parabola.

The order of the surface is evidently 6, and we have a cuspidal curve of the order 6.

There are two adjacent cnicnodes at infinity where the primitive meets the plane at infinity.*

VII. *The Parallel of a Cone of the second order.*

27. Let $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ be the equation of the cone. For the equation of the parallel we must equate to nothing the discriminant of

$$\frac{\theta x^2}{\theta + a} + \frac{\theta y^2}{\theta + b} + \frac{\theta z^2}{\theta + c} - k^2 = 0.$$

The surface is a developable of the order 8, with four double conics and a cuspidal curve of the order 12; the form is similar to that of the developable generated by a plane which touches a conic at a constant inclination to its plane. If, indeed, we write ζ for $\frac{k^2}{\theta}$, A for $\frac{k^2}{a}$, B for $\frac{k^2}{b}$, C for $\frac{k^2}{c}$, X^2 for $\frac{x^2}{a}$, &c., the equation becomes

$$\frac{X^2}{\zeta + A} + \frac{Y^2}{\zeta + B} + \frac{Z^2}{\zeta + C} - 1 = 0;$$

so that the parallel surface is a homographic deformation of the developable circumscribing a system of confocal conicoids, and has the same general characteristics.

I need not dwell on the cases of cylinders. Their double and cuspidal edges correspond to the nodes and cusps of the corresponding plane parallel.

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* A model of this surface, constructed under the direction of Prof. Henrici, is in the possession of the Society.

The Reciprocal Surfaces and General Characteristics.

(I.) *Primitive a Central Conicoid.*

28. The equation of the reciprocal surface may be formed by writing $\frac{1}{\rho}+k$ for $\frac{1}{\rho}$ in the reciprocal of the primitive equation with respect to a concentric sphere of radius unity.

Then the reciprocal of $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0$ being

$$aX^2+bY^2+cZ^2-1=0,$$

we get $$(aX^2+bY^2+cZ^2)-(1+k\rho)^2=0,$$

where $\rho=\sqrt{X^2+Y^2+Z^2}$, for the equation sought.

This is of the form

$$\{(a-k^2)X^2+(b-k^2)Y^2+(c-k^2)Z^2-1\}^2-4k^2(X^2+Y^2+Z^2)=0 \ldots (35).$$

I use the above method because it is applicable to surfaces generally. For the particular case (and many others to which the substitution $\frac{1}{\rho}+k$ for $\frac{1}{\rho}$ is not applicable), Prof. Cayley has given an equally simple process. We have only to form the reciprocal of (1), and take the envelope of that reciprocal relative to θ as parameter. ("Quarterly Journal of Mathematics," vol. xi., pp. 15—25.)

It is not within the scope of the present paper to go into much detail with regard to the reciprocal surfaces, and develope the meaning of the several forms considered as tangential equations of the parallels. I confine myself to a few points bearing on the characteristics.

As quartic surfaces with nodal curves of the second order, the reciprocal surfaces have received much attention from various geometers. I refer to Prof. Cayley's "Sketch of Recent Researches upon Quartic and Quintic Surfaces," "Proceedings," No. 32, p. 186.

Several more recent researches on the subject have, however, now appeared. I find some conclusions as to the case of a quartic surface with two intersecting nodal lines in a paper by Dr. Zeuthen, entitled "Recherche des singularités qui ont rapport à une droite multiple d'une surface," "Math. Ann.," b. iv., p. 18. A memoir by Herr Rudolf Sturm, entitled "Ueber die Flächen mit einer endlichen Zahl von (einfachen) Geraden, vorzugsweise die der vierten und fünften Ordnung" ("Math. Ann.," b. iv., p. 245), contains valuable details relative to quartic surfaces with a nodal conic and cuicnodes. The author determines the most important of the singularities which be-

come point singularities on the reciprocal surface. An earlier acquaintance with this memoir would have materially assisted me in the study of parallels.

For a general surface of the order m, the reciprocal of the parallel is of the order $2m(m-1)^2$, or twice the class of the primitive as in plane curves.

The equation (35) gives for the nodal conic at infinity

$$(a-k^2)\,X^2+(b-k^2)\,Y^2+(c-k)\,Z^2=0,$$

the reciprocal of which represents the corresponding node-couple torse of the parallel surface (Cayley).

The *pinch-points* or cuspidal points on the nodal curve are determined as the intersections at infinity of

$$aX^2+bY^2+cZ^2=0,$$
$$X^2+Y^2+Z^2=0.$$

(II.) *Primitive a Paraboloid.*

29. The reciprocal equation may be written

$$(aX^2+bY^2-2Z)^2-4k^2Z^2(X^2+Y^2+Z^2)=0 \quad \ldots\ldots\ldots (36),$$

obtained either directly or in the manner pointed out in the last article.

The nodal curve now consists of two intersecting right lines. The four pinch-points fall together in the point (X=0, Y=0, Z=0), being determined by

$$aX^2+bY^2 \qquad =0,$$
$$X^2+Y^2+Z^2=0,$$
$$Z=0.$$

The singularity at the origin consists of four pinch-points and an adjacent cnicnode or conical point.

In fact, if we substitute $X'\cos\omega-Y'\sin\omega$ for X, and $X'\sin\omega+Y'\cos\omega$ for Y, and then make $Y'=0$, we get for a general section by a plane through the axis of Z

$$\{(a\cos^2\omega+b\sin^2\omega)\,X'^2-2Z\}^2-4k^2Z^2(X'^2+Z^2)=0.$$

This represents the reciprocal of the plane parallel of a parabola, and corresponds to a tangent cone (cylinder) to the parallel surface from a point at infinity. The section therefore is a unicursal quartic curve having three adjacent double points at the origin; that is to say, two branches have there a contact of the second order, and the axis of X′ is the tangent. Now two of these adjacent double points are due to the intersection of the nodal right lines at the origin. The third double point indicates a cnicnode adjacent thereto.

(III.) *Primitive a Central Conicoid of Revolution.*

30. The equation of the reciprocal surface is

$$\{(a-k^2)(X^2+Y^2)+(c-k^2)Z^2-1\}^2-4k^2(X^2+Y^2+Z^2)=0.$$

It may also be written

$$\left\{(a-k^2)(X^2+Y^2)+(c-k^2)Z^2-\frac{a+k^2}{a-k^2}\right\}^2 - \frac{4k^2}{a-k^2}\left\{(a-c)Z^2+\frac{a}{a-k^2}\right\}=0,$$

showing that we have a nodal conic at infinity and two torsal conics lying on the same surface of the second order as the nodal conic. The pinch-points at infinity are given by

$$a(X^2+Y^2)+cZ^2=0,$$
$$X^2+Y^2+Z^2=0.$$

The two torsal conics correspond to the two cnicnodes on the axis of the parallel surface. We have also two cnicnodes or two points in the nature of cnicnodes; for if a quartic surface is represented by

$$U^2-L^2MN=0,$$

where L, M, N are linear, but U is of the second order, we have in general, besides the nodal conic, two cnicnodes

$$U=0,\quad M=0,\quad N=0,$$

and it may so happen that these points also lie on the conic, their coordinates satisfying L=0. This is the case in the present instance.

The fact that the original parallel and the reciprocal surface are both surfaces of revolution assists us to see the necessity of a singularity of this sort; for, taking the equation of the generating curve as

$$\{(a-k^2)X^2+(c-k^2)Z^2-1\}^2-4k^2(X^2+Z^2)=0,$$

we know that the curve is a quartic, binodal at infinity and symmetrical about the axis of z, which it cuts at right angles. There are then, from geometrical considerations, no proper cnicnodes, which, if any, would be due to double points on the axis. Yet, since the reciprocal is of the order 8, we must have a singularity with the effect of two cnicnodes in reducing the order of the reciprocal.

(IV.) *Primitive a Paraboloid of Revolution.*

31. The reciprocal equation is

$$\{a(X^2+Y^2)-2Z\}^2-4k^2Z^2(X^2+Y^2+Z^2)=0.$$

The pinch-point system

$$X^2+Y^2=0,$$
$$X^2+Y^2+Z^2=0,$$
$$Z=0,$$

shows that the multiple lines have pinch-points all along them. In other words, the lines $X^2+Y^2=0$, $Z=0$ are cuspidal. In fact, the equation may be written in the form

$$\left\{a(X^2+Y^2)-2Z-\frac{2k^2}{a}Z^2\right\}^2-4k^2Z^3\left\{\left(\frac{k^2}{a^2}+1\right)Z+\frac{2}{a}\right\}=0.$$

The effect, then, of the revolution of the three adjacent double points at the origin in the generating plane curve is to cause the nodal lines to be cuspidal (§ 24).

(V.) *Primitive a Central Conic.*

The equation of the reciprocal surface may be written

$$\{(a-k^2)X^2+(b-k^2)Y^2-k^2Z^2-1\}^2-4k^2(X^2+Y^2+Z^2)=0.$$

This is obtained by making $c=0$ in (35).

The equation may also be written in the form

$$\{(a-k^2)X^2+(b-k^2)Y^2-k^2Z^2+1\}^2-4(aX^2+bY^2)=0,$$

showing that, in addition to the nodal conic at infinity, there are two torsal circles lying on the same quadric surface as the nodal conic. These correspond to the two cnicnodes at infinity of the parallel surface. The form also shows that there are two cnicnodes.

(V′.) *Primitive a Circle.*

In this case, the reciprocal surface has the same characteristics as the parallel; that is to say, there is a nodal conic at infinity, and there are four cnicnodes, two on the axis and two on the nodal conic.

The surface is generated by the revolution of a central conic about an axis in its plane and parallel to a principal axis. It is therefore a particular case of the general tore.

(VI.) *Primitive a Parabola.*

The reciprocal equation is

$$(aX^2-2Z)^2-4k^2Z^2(X^2+Y^2+Z^2)=0,$$

obtained by making $b=0$ in (36).

In this case, the nodal lines are coincident (tacnodal). The singularity beyond this is equivalent to four pinch-points and three points in the nature of cnicnodes.

We have seen, in a similar case, that there is a point in the nature of a cnicnode at the origin.

From the nature of the substitution $\phi(X, Y, Z, 1\pm k\rho)=0$ for $\phi(X, Y, Z)=0$, it follows that in general, if there is a cnicnode on

$\phi(X, Y, Z) = 0$, there will be two corresponding cnicnodes on the related surface. In the present case, the parabolic cylinder $aX^2 - 2ZW = 0$ having a node at infinity, $X=0$, $Z=0$, $W=0$, the related surface has the corresponding singularities $X=0$, $Z=0$, $1 \pm k\rho = 0$.

The equation may, in fact, be written in the form

$$a^2X^4 - 4aX^2Z - 4Z^2\{k^2(X^2+Y^2+Z^2) - 1\} = 0,$$

showing that the points $X=0$, $Z=0$, $1 \pm k\rho = 0$ are triple points on the surface, though not triple on the nodal lines.

(VII.) *Primitive a Cone of the Second Order.*

33. The reciprocal curve is a spherical conic determined by

$$X^2+Y^2+Z^2-\frac{1}{k^2} = 0,$$

$$aX^2+bY^2+cZ^2 = 0,$$

corresponding to the primitive

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0.$$

I do not dwell on this and the remaining cases.

34. We are now in a position to consider the general characteristics of the several parallels. For this purpose I make use of Dr. Salmon's notation. (See his chapter on reciprocal surfaces; also Cayley, T. R. S., 1869, Part I., "On Reciprocal Surfaces," and "Corrections and Additions" thereto, R. S. Proceedings, 1871; also Zeuthen, "Mathematische Annalen," Band IV., Heft i., 1871.)

I must content myself here with a list of the singularities to which my results refer, and a few formulæ. Although this is repetition, the list may be useful for reference, when the original sources are not at hand.

Characteristics of a Surface.

n, order of surface.
a, order of tangent cone.
δ, number of its double edges.
κ, number of its cuspidal edges.
b, order of nodal curve.
f, number of its actual double points.
k, number of its apparent double points.
t, number of its triple points.
q, its class.

n', class of surface.
a', class of section by any plane.
δ', number of its double tangents.
κ', number of its inflexions.
b', class of node-couple torse.
f', number of its actual double planes.
k', number of its apparent double planes.
t', number of its triple planes.
q', its order.

ρ, number of points where nodal curve meets curve of contact of tangent cone.

ρ', order of node-couple curve.

j, number of pinch-points.

j', number of pinch-planes.

c, order of cuspidal curve.

c', class of spinode torse.

h, number of its apparent double points.

h', number of its apparent double planes.

r, its class.

r', its order.

σ, number of points where cuspidal curve meets curve of contact of tangent cone.

σ', order of spinode curve.

θ, not defined, but on the cuspidal curve.

θ', not defined.

χ, number of close points.

χ', number of close planes.

β, number of intersections of nodal and cuspidal curves, stationary points on the latter curve.

β', number of common planes of node-couple torse and spinode torse, stationary planes of spinode torse.

γ, number of like intersections, stationary points on nodal curve.

γ', number of common planes, stationary planes of node-couple torse.

i, number of like intersections, not stationary.

i', number of common planes, not stationary planes of either torse.

B, number of binodes.

B′, number of bitropes of surface.

C, number of cnicnodes.

C′, number of cnictropes.

ω, number of off points.

ω', number of off planes.

With regard to these, we have the following fundamental equations:

$$n(n-1) = a+2b+3c,$$
$$a(a-1) = n+2\delta'+3\kappa',$$
$$c-\kappa' = 3(n-a),$$
$$b(b-1) = q+2(k+f)+3\gamma+6t,$$
$$c(c-1) = r+2h+3\beta,$$
$$a(n-2) = \kappa-\mathrm{B}+\rho+2\sigma+3\omega,$$
$$b(n-2) = \rho+2\beta+3\gamma+3t,$$
$$c(n-2) = 2\sigma+4\beta+\gamma+\theta+\omega,$$
$$a(n-2)(n-3) = 2(\delta-\mathrm{C}-3\omega)+3(ac-3\sigma-\chi-3\omega)+2(ab-2\rho-j),$$
$$b(n-2)(n-3) = 4k+(ab-2\rho-j)+3(bc-3\beta-2\gamma-i),$$
$$c(n-2)(n-3) = 6h+(ac-3\sigma-\chi-3\omega)+2(bc-3\beta-2\gamma-i);$$

and the same with accented letters.

35. In the absence of a general theory as to the effect of multiple points existing on the nodal and cuspidal curves, I propose to give the usual characteristics of the several classes of surfaces to which the parallels belong. The specialities of these will then be remarked upon.

There is a certain number of characteristics which may be taken as known. Thus we have those of a general section and of a general tangent cone; or what is the same thing, we know the order of the surface and its reciprocal, and the orders of its nodal and cuspidal curves and of those of its reciprocal. These characteristics form a class by themselves, being independent of isolated point and plane singularities.

Following Dr. Zeuthen, we have $i=i'=0$.

In the table marked (A), under the numbers I., II., &c., I refer to the sections concerning the parallels to which the corresponding columns relate, and give the characteristics of the reciprocal of a quartic surface.

I., with a nodal conic;
II., with a nodal conic and a cnicnode;
III., with a nodal conic and two cnicnodes;
IV., with a cuspidal conic;
V.= III., with a nodal conic and two cnicnodes;
(V′.), with a nodal conic and four cnicnodes;
VI., with a nodal conic and three cnicnodes;

and further I give

VII., the characteristics of the reciprocal of a quadriquartic curve or those of the developable circumscribing two conicoids.

The characteristics I.—VI. may, of course, be regarded as those of the corresponding quartic surfaces, the meanings of corresponding letters being interchanged.

The deficiency expressed by Prof. Cayley's formula,

$$\tfrac{1}{6}(n-1)(n-2)(n-3)-(n-3)(b+c)+\tfrac{1}{2}(q+r)+2t+\tfrac{7}{2}\beta+\tfrac{5}{2}\gamma+i-\tfrac{1}{8}\theta,$$

is nothing in each case, VII. excepted.

	I.	II.	III. =V.	IV.	V′.	VI.	VII.
n	12	10	8	6	4	6	8
a	8	8	8	6	8	8	4
δ	8	8	8	1	8	8	6
κ	12	12	12	8	12	12	
b	26	14	6		2	2	8
k	200	52	12				24
t	40	12					
q	10	6	6		2	2	8
ρ	36	20	12		4	4	16
j					4	2	
c	24	18	12	8		6	12
h	180	96	38	12		6	38
r	36	30	20	32		12	8
σ	16	16	12	8		8	4
β	52	28	12			2	16
χ							
θ				16			
C			2	1	4	2	
n'	4	4	4	4	4	4	
a'	8	8	8	6	8	8	
δ'	4	5	6		8	7	
κ'	12	12	12	8	12	12	
b'	2	2	2		2	2	
k'							
t'							
q'	2	2	2		2	2	
ρ'	4	4	4		4	4	
j'	4	4	4		4	4	
c'				2			
h'							
r'				2			
σ'				2			
β'							
χ'				2			
θ'							
C′		1	2		4	3	

Observations.

36. I. The cuspidal curve being the complete intersection of a quartic and a sextic surface, the number of apparent double points $h=180$. The class $q=10$ of the nodal curve is due to the five nodal conics; the class of the residue is therefore zero, indicating that it consists of 16

right lines. In general, these correspond to the 16 right lines discovered by Prof. Clebsch on the surface of the fourth degree with a nodal conic.

A plane through one of these lines will cut the surface in the line itself and a cubic with a double point. There will be in the section four double points; two of them are due to the nodal conic; the remaining two are points of contact of the plane. Hence the reciprocals of these lines will be double lines on the reciprocal surface. The five nodal conics correspond to Kummer's five bitangent cones on the quartic surface.

In the special case of a parallel, we have seen that the system becomes two groups of eight generators of a conicoid. Four points of quadruple intersection, being also points of intersection of the nodal conics at infinity, become sextuple points of the nodal curve. These points stand for 16 triple points and 12 apparent double points, or each for four triple points and three apparent double points. There are also eight triple points in each of the three principal planes.

$\beta=52$ is made up of four points in each of the principal planes and eight points at infinity, together with the 32 stationary points in the eight special planes.

It might be supposed that the sextuple points on the nodal curve would count for a greater number of triple points than 16. If, however, we consider the case of four planes intersecting in a point, which will be a sextuple point of the nodal curve, the planes being regarded as a single surface, we see that in this case also the sextuple point will represent four triple points and three apparent double points; for the number of apparent double points for six non-intersecting lines is 15. But the number of possible triple points is the number of ways in which four planes can be taken three and three together, *i. e.* four. At the same time, every triple point takes the place of three apparent intersections. Thus we have four triple points substituted for twelve apparent intersections, leaving three unaccounted for. It does not follow, however, that every kind of sextuple point on a nodal curve is of this value. In like manner, we might take m planes intersecting in one point, which would be a point of the order m on the surface, but of the order $\frac{m(m-1)}{2}$ on the nodal curve. In the particular case, the points appear to be quadruple points on the surface, equivalent in each instance to $\frac{4.3.2}{1.2.3}$ triple points.

We may regard the point as one where the tangents form a cone of the order 4, with six double edges touching the nodal curve. For such a point Dr. Zeuthen gives the addition $y(\mu-2)$ to the expression for $h(\mu-2)$. The number of double edges touching the nodal curve is

y, and the order of the conic μ. Putting $y=6$, $\mu=4$, we get a correct result.

It would perhaps be better, in these cases of multiple points, to follow Dr. Zeuthen, in denoting by k the Plückerian number of the double edges of a cone through the double curve, and substituting $k-f-3t+\Sigma\ldots$ for k in the expression for $b(n-2)(n-3)$.

II. The cuspidal curve is the intersection of a quartic and a sextic surface, less the intersection of a quadric and a cubic surface; and $h=96$. The class $q=6$ of the nodal curve is due to three nodal conics, and the class of the residue is zero, indicating that it is made up of eight right lines. In the case of the parallel, the cuspidal curve is reduced to the intersection of a quartic and a quintic surface, less a conic; and $h=96$. The class q is 4, because a nodal conic at infinity consists of right lines, *i.e.* has an actual double point, or $f=1$.

The singular point on the axis at infinity counts as a union of eight points β, two in each of the principal planes and four in the plane at infinity. The number 28 is thus made up of four points finite and in the two principal planes, the eight united points, and the 16 points in the four special planes. There are four triple points t in each of the principal planes and the plane at infinity.

The 'cnictrope' is not a proper one, and corresponds to a point in the nature of a cnicnode on the reciprocal quartic surface.

The order $q'=0$; for, referring to the reciprocal surface, we see that the nodal conic has an actual double point, or $f'=1$.

For the manner in which the complementary nodal lines are reduced in number, see the memoir before alluded to, of Herr Rudolf Sturm, on the quartic surface with one cnicnode and a nodal conic, &c.

When there is a cnicnode, four lines on the quartic surface become 'binary,' or equivalent to two ordinary lines. These 'binary' lines are such that planes through them touch at one point. The eight ordinary lines correspond to the eight double lines of the reciprocal surface.

III. The cuspidal curve is, in general, the intersection of two quartic surfaces, less the intersection of two quadric surfaces; and $h=38$. In the case of a parallel, we have two points at infinity, each of which may be regarded as the intersection of three branches of the nodal curve with six branches of the cuspidal curve; that is to say, we have three nodal conics and six cuspidal circles intersecting in two common points. One nodal conic at infinity is the circle, and one is a conic having a double contact with the circle.

The case of a parallel of revolution is simple geometrically; but, as far as I am aware, does not come within existing general formulæ for characteristics.

IV. The cuspidal curve consists of four conics which intersect in two common points. Hence r is reduced 24 by these, in addition to the reduction by 12 points h. These points also present the singularity $\theta = 16$ which enters the formula for $c\ (n-2)$ in the same way that $3t$ enters the formula for $b(n-2)$. If a is the number of such singular points, the addition is $8a$. In the case of the parallel, r is reduced to 6 because the cuspidal conic at infinity contains an actual double point. In like manner $r'=0$. In place of $j'=4$, we now have $\chi'=2$, on account of the multiple lines of the reciprocal being cuspidal.

V. The torsal conics of the parallel surface pass through the cnic-nodes, and the same is the case in the reciprocal surface. Also $q'=0$, on account of $f'=1$.

V′. For the parallel, we have a particular case of the general tore, and the characteristics are the same as for a surface generated by a conic revolving about an axis in its plane. The speciality is that, considering a section by any plane through the axis, the generating conic and its reflection intersect in two points at infinity.

VI. The cuspidal curve is the intersection of a quadric and a cubic surface, and $h=6$. In the case of the parallel $q'=0$, because the nodal line of the reciprocal surface is made up of two coincident lines or $f'=1$. The three 'cnictropes' are represented by three points in the nature of cnicnodes on or adjacent to the nodal lines.

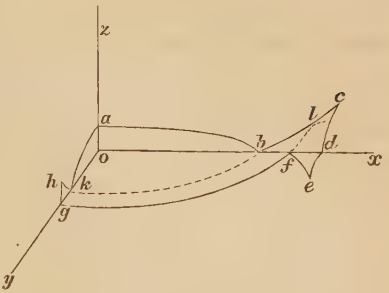
37. It will be remarked that in several instances the characteristics of a tangent cone are the same as those of a general plane parallel of a conic. When the primitive is a central conicoid, having therefore no special relation to the line at infinity, we may determine these characteristics by considering a tangent cone from a point at infinity. This of course is a cylinder, and will be the parallel of the corresponding tangent cone (cylinder) to the primitive. Hence a section of the former cylinder perpendicular to its axis will be a parallel of the corresponding section of the latter. This appears to be generally true for a surface of order m, not specially related to the plane at infinity, and enables us to infer the characteristics of the general tangent cone of the parallel surface.

38. The general form of a parallel surface of a conicoid can be perceived without much difficulty, when we know that the real portion of the nodal curve consists of the real nodal conics.

For the purpose of illustration, I refer to the case of an ellipsoid as primitive.

I will suppose that the modulus is less than the least of the principal semiaxes, but greater than the radii of curvature at the vertices

on the major axis of any one of the three principal sections. Let Oxy, Oxz, Oyz (Fig. 1) represent the principal planes enclosing a quadrant of the primitive. The sections of the parallel by these planes will consist of an outer oval which is not represented and an interior curve. The three outer ovals belong to the exterior sheet of the surface, which presents no visible singularity, its contour being similar to that of an ellipsoid. The interior curves are represented, so far as they lie within the quadrantal space $Oxyz$, by the lines $Oabcd$, $Odefg$, and $Oghka$. Through the points b, k, which are double on the corresponding plane parallels, we have a real nodal ellipse kb; and through f, also double on the corresponding plane parallel, a nodal hyperbola passes, lying in the plane Oxz, and touching the plane parallel, suppose, in l. Beyond this point of contact, which is a stationary point on the cuspidal curve, the hyperbola is the intersection of imaginary sheets.



(Fig. 1.)

If now, with a free hand, we join ch and ce, the figure gives us a general notion of a quadrant of the interior sheet of the parallel surface. The complete sheet will, in its general features, resemble the Figure 2. Outside this, we shall have, as I have said, a surface presenting no visible singularity, but resembling an ellipsoid.

(Fig. 2.)

The form of the interior sheet will vary by the disappearance of (1) the nodal ellipse, (2) the nodal hyperbola, (3) both. And again, the form will vary by the disappearance of (1) two twisted ovals of the cuspidal curve, (2) four. It would seem that we cannot have more than four of such ovals, or more than two actual nodal conics. The general form of the surface, when the primitive is an hyperboloid of two sheets, can be perceived in a similar way. The outer superficies will be hyperbolic, and the two hyperbolic sheets of this portion of the surface may intersect in a real nodal ellipse. The remaining portion of the surface due to negative value of the modulus may exhibit a pair of cuspidal ovals and two hyperbolic sheets.

When the primitive is an hyperboloid of one sheet, the parallel surface may exhibit cuspidal edges on both the principal sheets.

We may have an exterior surface with an elliptic ring bounded by

the real nodal ellipse and a pair of cuspidal ovals. This ring and the exterior surface generally will enclose a surface of the general form of an hyperboloid of one sheet, but with annexed surfaces bounded by the actually nodal part of the nodal hyperbola and two cuspidal ovals. There would be no great difficulty in determining all the possible variations of forms of the surfaces, but the subject requires either drawings or models for its satisfactory discussion.

39. Instead of considering a single parallel surface, we may consider the system of parallels for a given primitive.

It appears that the locus of the cuspidal curves of the parallels is the common surface of centres. Moreover the locus of the stationary points of the parallels will be the cuspidal curves of the surface of centres in the principal planes and in the 8 special planes. The locus of the nodal right lines will be the 8 special planes. I am supposing the primitive to be a central conicoid, but similar conclusions follow in the other cases.

We can see that the discriminant with regard to k^2 of the discriminant with regard to θ of (1) will be of the gross order 60.

This discriminant is made up of the principal planes and the plane at infinity twice over, the 8 special planes twice over, and the surface of centres three times over. It is evident that the curves of curvature and geodesics are intimately connected with the theory of parallels. A system of normals along a line of curvature of the primitive will trace a line of curvature on the parallel; and when in the unwinding of a series of threads from the surface of centres their extremities generate the primitive, we may in precisely the same way generate a parallel surface. The subject, however, is full of interesting matter, which awaits investigation.

40. It is also obvious that, as in the theory of plane parallels, the parallel of the parallel of a surface will break up into two parallels. The discussion of the characteristics in this case belongs however to the general theory with regard to surfaces of a higher order than the second. The order can be determined by the help of the previous results. It appears that the degree of a parallel is diminished by $8p$ when the primitive has the circle at infinity for a multiple curve of the order p. Also the degree is diminished by 4 for a pair of simple contacts with the circle at infinity. The order in the general case is in terms of characteristics of the primitive, twice

$$\text{order} + \text{class} + \text{order of tangent cone.}$$

In the present instance, therefore, we have for the order of the parallel of the parallel of a central conicoid

$$2\,(12+4+8)-8.2-4.2 = 24.$$

For the circle at infinity is a nodal line, and the surface otherwise has 4 contacts with the circle. And generally for a surface of the order m,

$$\begin{aligned}&\text{Order of parallel of parallel}\\ &= 2\{2[m+m(m-1)^2+m(m-1)]+2m(m-1)^2+2m^2(m-1)\}\\ &\qquad -8m(m-1)^2-2\,.\,2m(m-1)\\ &= 4\{m+m(m-1)^2+m(m-1)\}\\ &= 2\times\text{order of parallel.}\end{aligned}$$

February 8th, 1872.

Prof. CAYLEY, V.P., F.R.S., in the Chair.

The Chairman stated that the President had made enquiries at the Home Office as to the mode of procedure requisite for obtaining a Charter for the Society, and that the matter would come on for consideration at the next subsequent meeting of the Society, when members would have an opportunity of stating their views upon the desirability of incorporation.

Mr. J. W. L. Glaisher, B.A., F.R.A.S., was elected a Member of the Society.

Mr. T. Cotterill, M.A., gave an account of his paper "On an Algebraical Form, and the Geometry of its dual connection with a Polygon, plane or spherical."

The Chairman, Dr. Hirst, and Prof. Clifford took part in a discussion on the paper.

The following presents were received :—

"Monatsbericht," Sept., Oct., and Dec., 1871.

"Journal of London Institution," Nos. 10, 11, 12.

The first 22 numbers of the "Bulletin des Sciences Mathématiques et Astronomiques," from the commencement to October, 1871 : from M. G. Darboux, the Editor.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," tome viii., I^{er} et 2^{me} cahier, 1870.

March 14th, 1872.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. W. Paice, M.A. Lond., was proposed for election.

The President made a statement to the effect that it had been deemed desirable to apply for a Charter, and that he had taken the

requisite steps for ascertaining the right mode of procedure; and, in reply to questions by Mr. Sprague and Mr. A. J. Ellis, said that the expense would most likely be small, as there was little prospect at the present time of the application being favourably entertained; and that even if the application were acceded to, it would not be at the Society's expense. The President's proposal—that application should be made to the Council Office for the grant of a Charter, the draft of which had been drawn up by Prof. Cayley—was then agreed to unanimously, and the subject dropped.

The Treasurer then proposed and Prof. Clifford seconded a vote of thanks to Mr. Drach for his present to the Society of two rather rare and interesting early works on Mathematics, by Vieta and Ubaldi respectively.

Prof. Clifford gave a full account of his paper "On a new expression of Invariants and Covariants, by means of alternate numbers." Reference was specially made by the author to the "Vorlesungen über die complexen Zahlen und ihre Functionen," of Dr. Hermann Hankel (1867, part I.).

Mr. Tucker (Hon. Sec.) then read portions of a communication from the Hon. J. W. Strutt, "On the Vibrations of a Gas contained within a Rigid Spherical Case."

Mr. A. J. Ellis, F.R.S., read out the following problem, which had been proposed to him by Prof. Haldeman, of Pennsylvania, United States (who is writing a treatise on English Versification): "The number of lines in a rhymed stanza being given, how many variations of rhyme-distribution does it admit of, supposing no line to be left without a rhyme?" It may be interesting, further, to state, that the Professor remarks, "Of seven-line stanzas, *I have observed in actual use twenty-eight.* The sonnet stanza, of fourteen lines, is very rich; the examples within my reach having given me two hundred and twenty varieties."

The following presents were received:—

"Francisci Vietæ Opera Mathematica": edited by F. A. Schooten, 1646.

"Guidi Ubaldi Mecanicorum liber," Venice, 1615: from Mr. S. M. Drach, F.R.A.S., F.R.G.S.

"Geometria rigorosa di Pietro Dott. Cassani": from A. Stein, Venezia.

"Crelle's Journal," 74 Band, zweites Heft.

"Annali di Matematica," Serie 2ª. tom. v. fasc. 1, Nov. 1871.

"Proceedings of the Royal Society," Vol. xx. No. 132.

"Bulletin des Sciences," Nov. Dec. and Index, 1871, and Jan. 1872.

"Journal of the London Institution," No. 13.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich,

redigirt von Dr. Rudolf Wolf.," 15 Jahrgänge (mostly 4 Hefte in each) from 1863, to the 16th Jahrgang (1871), three parts. Duplicate of 8th Jahrgang viertes Heft.

*On the Vibrations of a Gas contained within a Rigid Spherical Envelope.** By the Hon. J. W. Strutt, M.A.

(*Read March 14th*, 1872.)

Whatever may be the motion of air within a sphere, it may always be resolved into a series of simple vibrations represented analytically by terms involving circular, or imaginary exponential, functions of the time. These exist in perfect independence of one another, so that it is sufficient to consider only one at a time. Moreover, the function containing the time (e^{ikat}) will run through the expressions for the velocity-potential,† and its differentials with respect to space as a simple factor, and may therefore be omitted from the beginning.

If λ be the wave-length, $k = 2\pi \div \lambda$, the conditions to be satisfied are

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\psi + k^2\psi = 0 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

throughout the interior, and at the surface of the sphere

$$\frac{d\psi}{dr} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The main problem before us is the determination of the possible values of k or λ in terms of the radius of the sphere.

Let ψ be expanded in La Place's Series

$$\psi = \psi_0 + \psi_1 + \psi_2 + \ldots + \psi_n + \ldots \quad \ldots\ldots\ldots\ldots (3),$$

then‡

$$\frac{d^2\psi_n}{dr^2} + \frac{2}{r}\frac{d\psi_n}{dr} - \frac{n(n+1)}{r^2}\psi_n + k^2\psi_n = 0 \ldots\ldots\ldots\ldots (4),$$

of which the solution is

$$r\psi_n = S_n e^{-ikr}\left\{1 + \frac{n(n+1)}{2ikr} + \frac{n(n-1)(n+1)(n+2)}{2.4(ikr)^2} + \ldots\right\}$$
$$+ S'_n e^{+ikr}\left\{1 - \frac{n(n+1)}{2ikr} + \frac{n(n-1)(n+1)(n+2)}{2.4(ikr)^2} - \ldots\right\} \ldots.(5),$$

---

* The problem here discussed was referred to in a paper on "the Theory of Resonance," Phil. Trans., 1871. Its publication seems of interest, as it is the only case of the vibration of air within a closed vessel which has hitherto been solved with complete generality, except perhaps that of a rectangular parallelepiped. Considerable alterations and additions have been made since the paper was sent in to the Mathematical Society, partly in accordance with the advice of Sir W. Thomson and Professor Clerk-Maxwell, and partly in consequence of my own further reflection on the subject.

† It is assumed that the motion is irrotational, so that a velocity-potential exists; and, further, that it is so small that the square may be neglected throughout.

‡ Stoke's Phil. Mag. Dec. 1868, or Phil. Trans., 1868.

S_n, S'_n being surface harmonics of order n; or, reversing the series,

$$r\psi_n = S_n e^{-ikr}\frac{1.3.5\ldots(2n-1)}{(ikr)^n}\{1+ikr+\ldots\}$$
$$+S'_n e^{+ikr}\frac{1.3.5\ldots(2n-1)}{(-ikr)^n}\{1-ikr+\ldots\} \quad \ldots\ldots\ldots (6).$$

Besides the condition that the radial velocity must vanish at the surface of the sphere, we have to secure that ψ does not become infinite at the centre. Now from (6), as r diminishes without limit,

$$r\psi_n = 1.3.5\ldots(2n-1)\{S_n(ikr)^{-n}+S'_n(-ikr)^{-n}\}$$
$$= 1.3.5\ldots(2n-1)(ikr)^{-n}\{S_n+(-1)^n S'_n\},$$

showing that the required condition cannot be fulfilled, unless

$$S_n+(-1)^n S'_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

It remains to satisfy the condition expressed by (2). Following Stokes, we may write (5),

$$r\psi_n = S_n e^{-ikr} f_n(ikr)+S'_n e^{+ikr} f_n(-ikr) \quad \ldots\ldots\ldots (8),$$

and

$$\frac{d\psi_n}{dr} = -\frac{S_n}{r^2}e^{-ikr}F_n(ikr)-\frac{S'_n}{r^2}e^{+ikr}F_n(-ikr) \quad \ldots\ldots\ldots (9),$$

where

$$F_n(ikr) = (1+ikr)f_n(ikr)-ikr f'_n(ikr) \quad \ldots\ldots\ldots (10).$$

The value of $\frac{d\psi_n}{dr}$ must vanish over the whole surface of the sphere, when r has the prescribed value. Introducing the ratio of $S'_n : S_n$ given in (7), we obtain as the condition to be satisfied,

$$e^{2ikr}\frac{F_n(-ikr)}{F_n(+ikr)} = (-1)^n \quad \ldots\ldots\ldots\ldots (11).$$

If the numerator of the fraction on the left by a separation of the real and imaginary components be put into the form $\alpha-i\beta$, the denominator is at the same time expressed by $\alpha+i\beta$, and the fraction itself may be written $e^{-2i\theta}$, if $\frac{\beta}{\alpha} = \tan\theta$.

Our condition is therefore

$$e^{2ikr-2i\theta} = \cos n\pi = e^{in\pi},$$

or

$$\theta = kr - n\frac{\pi}{2}+m\pi.$$

Thus

$$\tan\left(kr-n\frac{\pi}{2}\right) = \tan\theta = \frac{\beta}{\alpha} \quad \ldots\ldots\ldots\ldots (12),$$

and

$$\frac{\beta}{\alpha} = -i\frac{F_n(ikr)-F_n(-ikr)}{F_n(ikr)+F_n(-ikr)} \quad \ldots\ldots\ldots\ldots (13);$$

(12) and (13) give the different values of kr possible under the conditions, for each value of n. With respect to the values of $F_n(ikr)$, Professor Stokes gives the following table:—

$$F_0(ikr) = ikr+1,$$

$$F_1(ikr) = ikr+2+\frac{2}{ikr},$$

$$F_2(ikr) = ikr+4+\frac{9}{ikr}+\frac{9}{(ikr)^2},$$

$$F_3(ikr) = ikr+7+\frac{27}{ikr}+\frac{60}{(ikr)^2}+\frac{60}{(ikr)^3},$$

$$F_4(ikr) = ikr+11+\frac{65}{ikr}+\frac{240}{(ikr)^2}+\frac{525}{(ikr)^3}+\frac{525}{(ikr)^4}.$$

Hence, when $n=0$, $\quad \frac{\beta}{\alpha} = kr;$

when $n=1$, $\quad \frac{\beta}{\alpha} = \frac{kr-\frac{2}{kr}}{2};$

when $n=2$, $\quad \frac{\beta}{\alpha} = \frac{kr-\frac{9}{kr}}{4-\frac{9}{k^2r^2}};$

when $n=3$, $\quad \frac{\beta}{\alpha} = \frac{kr-\frac{27}{kr}+\frac{60}{k^3r^3}}{7-\frac{60}{k^2r^2}};$

when $n=4$, $\quad \frac{\beta}{\alpha} = \frac{kr-\frac{65}{kr}+\frac{525}{k^3r^3}}{11-\frac{240}{k^2r^2}+\frac{525}{k^4r^4}}.$

We are now in a position to estimate numerically the periods of the various vibrations corresponding to any term in La Place's series. And first for

Concentric or Radial Vibrations.

From (5) and (7) we find that, when $n=0$,

$$\psi_0 = -2ikS_0\frac{\sin kr}{kr} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14),$$

S_0 being a constant. In this case (12) becomes

$$\tan kr = kr \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15).$$

The first root is $kr=0$, which corresponds to a tone of infinitely low pitch, and has no physical significance. The second root will be when kr is not much short of $\frac{3\pi}{2}$, the third just short of $\frac{5\pi}{2}$, &c. When kr

is large, the roots are approximately of the form

$$kr = (2m+1)\frac{\pi}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16),$$

the difference of consecutive values being π, which agrees with what we know from the theory of vibrations in one dimension. In the motion corresponding to the third and higher roots, the air divides itself into vibrating parts separated by nodal surfaces in the form of concentric spheres. In adjacent segments, the motion is at any given moment oppositely directed.

The roots of equation (15) were calculated by Schwerd for the theory of diffraction.* Omitting the zero root, we have for $kr \div \pi$ the following values:

1·4303, 4·4774,
2·4590, 5·4818,
3·4709, 6·4865;

whence, for the ratio of r to λ,

·7151, 1·2295, 1·7354, 2·2387, 2·7409, 3·2432.

When m is somewhat large, we may use the following approximation to (15):

$$r : \lambda = \frac{2m+1}{4} - \frac{\cdot 1}{(2m+1)\pi^2} \quad \ldots\ldots\ldots\ldots\ldots (17).$$

In the vibration corresponding to the higher roots of (15), any nodal surface may be conceived to be rigid without any alteration in the conditions of the question. We have therefore the solution of the problem of radial vibrations in a spherical shell, provided that the radii of the surfaces are such that both are solutions of (15) with the same value of k. Any cone with vertex at the centre may also be supposed rigid.

The *loops* or places of no variation of density and pressure are given by the roots of the equation $\frac{\sin kr}{kr} = 0$, or $kr = m\pi$, where m is any integer, except zero.

Diametral Vibrations.

This case ($n=1$) is the most interesting. From (5) and (7) we derive

$$\psi_1 = \frac{2S_1}{r}\left\{\cos kr - \frac{\sin kr}{kr}\right\} \quad \ldots\ldots\ldots\ldots\ldots (18).$$

S_1 being an harmonic of order 1, may be represented as proportional to $\cos\theta$, where θ is the angle between any radius and a fixed one, which may be called the axis. The spherical nodes are determined by the equation

$$\tan kr = \frac{2kr}{2-k^2r^2} \quad \ldots\ldots\ldots\ldots\ldots (19).$$

* Verdet, "Leçons d'Optique Physique," t. i., p. 266.

The first root is $kr=0$. Calculating from Trigonometrical Tables by the method of trial and error, I find for the next root, which corresponds to the vibration of most importance within a sphere,

$$\frac{180}{\pi} kr = 119{\cdot}26\,;$$

so that $$r : \lambda = 119{\cdot}26 : 360 = {\cdot}3313.$$

The air sways from side to side in much the same manner as in a doubly-closed pipe. Without analysis, we might anticipate that the pitch would be higher for the sphere than for a closed pipe, whose length is equal to the diameter; because the sphere may be derived from the cylinder with closed ends, by filling up part of the latter with obstructing material, the effect of which must be to sharpen the *spring*, while the mass to be moved remains but little changed.* In fact, for a closed pipe of length $2r$,

$$r : \lambda = {\cdot}25.$$

The sphere is thus about a fourth higher in pitch than the cylinder.

The vibration now under consideration is the gravest mode of which the sphere is capable; it is more than an octave lower than the lowest radial vibration. The next vibration of this type is such that

$$\frac{180}{\pi} kr = 360 - 19{\cdot}65 = 340{\cdot}35,$$

or $$r : \lambda = {\cdot}9454,$$

and is therefore higher than the first radial.

When kr becomes great, we have from (14),

$$kr = m\pi - \frac{2}{m\pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20),$$

corresponding to the mth node. The approximate values of kr for the lower values of m will be found at the end of the paper.

It is to be observed that what are here called nodes are not surfaces of absolutely no motion, as in the case of vibrating plates and membranes; but are defined as surfaces such that at any point the air has only *tangential* motion. In the case of spherical nodes, for instance, there must be no motion parallel to the radius vector. Any nodal surface may be conceived to become rigid, without altering the conditions of the question.

Referring to (18), we see that, since $\frac{d\psi_1}{d\theta}$ vanishes only at the poles, there are no conical nodes with vertex at the centre. Any meridional plane, however, is nodal, and may be supposed rigid. Along any specified radius vector, ψ_1 and $\frac{d\psi_1}{d\theta}$ vanish, and change sign with

* See the paper on "Resonance," Phil. Trans., 1871.

$\cos kr - \frac{\sin kr}{kr}$, that is, when $\tan kr = kr$. The loops in the present case therefore coincide with the nodal surfaces of the concentric vibrations, and, with the exception of the equatorial plane, correspond also to the places of no transverse motion.

$$n = 2.$$

From (5) and (7),

$$\psi_2 = -\frac{2ik\,S_2}{kr}\left\{\sin kr\left(1-\frac{3}{k^2r^2}\right)+\frac{3\cos kr}{kr}\right\} \quad \ldots\ldots (21);$$

and from (12) the spherical nodes are determined by

$$\tan kr = \frac{k^3r^3-9kr}{4k^2r^2-9} \quad \ldots\ldots\ldots\ldots\ldots (22).$$

The first finite root is $kr = 3{\cdot}3422$,

corresponding to $r : \lambda = {\cdot}5319$.

The first vibration of this group is accordingly graver than the first of the radial group. For a tabular statement of all the lower notes, see the end of the paper.

The general harmonic of order 2 may be expressed*

$$S_2 = A_0(\cos^2\theta - \tfrac{1}{3}) + (A_1\cos\phi + B_1\sin\phi)\sin\theta\cos\theta + (A_2\cos 2\phi + B_2\sin 2\phi)\sin^2\theta \ldots\ldots (23),$$

from which we may select for special consideration the following notable cases:

(1.) The zonal harmonic,

$$S_2 = A_0(\cos^2\theta - \tfrac{1}{3}) \quad \ldots\ldots\ldots\ldots (24).$$

Differentiating with respect to θ, $\frac{d\psi_2}{d\theta}$ is seen to be proportional to $\sin 2\theta$, and therefore vanishes when $\theta = \frac{\pi}{2}$. This shows that the equatorial plane is a nodal surface, so that the motion might take place in a closed hemisphere. Also, since S_2 does not involve ϕ, any meridional plane may be regarded as rigid.

(2.) The sectorial harmonic,

$$S_2 = A_2\cos 2\phi\sin^2\theta \quad \ldots\ldots\ldots\ldots (25).$$

Here $\frac{d\psi_2}{d\theta}$ again varies as $\sin 2\theta$, and the equatorial plane is nodal. But

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* Thomson and Tait's "Natural Philosophy," p. 149.

$\frac{d\psi_2}{d\phi}$ varies as $\sin 2\phi$, and therefore does not vanish independently of θ, except when $\sin 2\phi = 0$. It appears accordingly that two, and but two, meridional planes are nodal, and that these are at right angles to one another.

(3.) The tesseral harmonic,

$$S_2 = A_1 \cos\phi \sin\theta \cos\theta \quad \ldots\ldots\ldots\ldots (26).$$

In this case $\frac{d\psi_2}{d\theta}$ vanishes independently of ϕ with $\cos 2\theta$, that is, when $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, which gives a nodal cone of revolution, whose vertical angle is a right angle. $\frac{d\psi_2}{d\phi}$ varies as $\sin\phi$, and there is one meridional nodal plane, and but one.

The consideration of the more complicated nodal surfaces would lead us too far, and is more properly treated as a part of the general theory of surface harmonics. If the harmonic is zonal, no matter of what order, the cones of revolution corresponding to a maximum or minimum value are nodal. Among these is comprised the equatorial plane in the case of an harmonic of even order. Meridional planes also are nodal.

In the progress of our investigation, we have virtually solved problems relating to the vibration of air, when the condition over a part or the whole of the boundary is the constancy of density and pressure. Thus, under the head of radial vibrations, we have a solution of the question of the motion of air in a sphere, such that at the surface the pressure is constant. And our results with regard to diametral vibrations give us a solution for the motion of air in a hemisphere whose curved surface is rigid, while the condition over the plane part is that of no pressure variation. But I have not dwelt in detail on these problems, because they do not possess much physical interest, there being no experimental method of securing that a prescribed surface shall be a loop. Though not leading to anything practical, it may help the imagination to say that the condition in question is tantamount to supposing the vessel open, and neglecting the *inertia* of the air outside.

In the course of an investigation on an allied subject (which I hope before long to communicate to the Society), a very curious relation presented itself between the functions f and F and the well-known zonal harmonic functions of μ, defined by the equation

$$(1-2e\mu+e^2)^{-\frac{1}{2}} = 1+Q_1e+Q_2e^2+\ldots+Q_ne^n \ldots\ldots (27),$$

by means of which the results of the present paper can be very shortly expressed for the general value of n.

The relation in question is expressed, in the case of f, by

$$f(y) = yQ_n\left(1-\frac{d}{dy}\right)\frac{1}{y} \quad \ldots\ldots\ldots\ldots (28),$$

and may be verified as follows:—

Consider the symbolic operator $Q_n\left(1-\frac{d}{dy}\right)$. We know that*

$$Q_n(\mu) = 1-\frac{n}{1}\cdot\frac{n+1}{1}\cdot\frac{1-\mu}{2}+\frac{n(n-1)}{1.2}\cdot\frac{(n+1)(n+2)}{1.2}\cdot\frac{(1-\mu)^2}{2^2}-\ldots;$$

so that

$$Q_n\left(1-\frac{d}{dy}\right) = 1-\frac{n(n+1)}{1.1}\cdot\frac{1}{2}\frac{d}{dy}+\frac{(n-1)\ldots(n+2)}{1.2.1.2}\cdot\frac{1}{2^2}\left(\frac{d}{dy}\right)^2-\ldots.$$

Moreover, $$\left(\frac{d}{dy}\right)^s\frac{1}{y} = (-1)(-2)(-3)\ldots(-s)\,y^{-s-1};$$

from which becomes apparent the identity of $yQ_n\left(1-\frac{d}{dy}\right)\frac{1}{y}$ with the series for $f(y)$ given in (5).

Since

$$Q_n\left(1-\frac{d}{dy}\right)\frac{1}{y} = (-1)^n Q_n\left(\frac{d}{dy}-1\right)\frac{1}{y} = (-1)^n \epsilon^y Q_n\left(\frac{d}{dy}\right)\frac{\epsilon^{-y}}{y},$$

the expression for ψ_n becomes from (8), if $ikr = y$,

$$\psi_n = ik(-1)^n\left\{S_n Q_n\left(\frac{d}{dy}\right)\frac{\epsilon^{-y}}{y}+S_n'(-1)^n Q_n\left(\frac{d}{dy}\right)\frac{\epsilon^{+y}}{y}\right\}\ldots(29),$$

or, in the case when the motion is finite at the pole by (7),

$$\psi_n = -2(-1)^n ik\,S_n Q_n\left(\frac{d}{d.ikr}\right)\frac{\sin kr}{kr} \quad \ldots\ldots (30).$$

In the paper referred to, I prove that the expansion of the quantity on the right, in rising powers of r, is

$$\psi_n = -2(-1)^n ik\,S_n\frac{(ikr)^n}{1.3.5\ldots(2n+1)}\left\{1-\frac{(kr)^2}{2(2n+3)}+\ldots\right\}\ldots(31),$$

which shows that the condition (7), hitherto proved merely to be necessary, is in fact sufficient to secure that ψ_n remains finite at the centre.

When n is considerable, ψ_n not only vanishes with r, but is an infinitesimal of high order.

If $f(ikr) = \alpha' + i\beta'$, we may also express ψ_n in terms of α' and β'. From (7) and (8),

$$\psi_n = -2i^{n+1}\frac{S_n}{r}\left\{\alpha'\sin\left(kr+n\frac{\pi}{2}\right)-\beta'\cos\left(kr+n\frac{\pi}{2}\right)\right\}\ldots(32).$$

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* Thomson and Tait's "Natural Philosophy," p. 623 (quoted from Murphy).

The surfaces of no pressure variation are given by

$$S_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (33),$$

representing cones, having their vertices at the centre; and

$$\tan\left(kr + n\frac{\pi}{2}\right) = \beta' \div \alpha' \quad \ldots\ldots\ldots\ldots \quad (34);$$

or, as it may also be written,

$$Q_n\left(\frac{d}{d.ikr}\right)\frac{\sin kr}{kr} = 0 \quad \ldots\ldots\ldots\ldots \quad (35),$$

representing concentric spheres.

From (10) we see that

$$\frac{F(y)}{y^2} = \left(1 - \frac{d}{dy}\right)\frac{f(y)}{y},$$

so that

$$F(y) = y^2 Q_n\left(1 - \frac{d}{dy}\right)\left(1 - \frac{d}{dy}\right)\frac{1}{y} \quad \ldots\ldots\ldots \quad (36),$$

by means of which, with (7) and (9), or directly from (30), we find

$$\frac{d\psi_n}{dr} = -2(-1)^n ik^2 S_n Q_n\left(\frac{d}{d.ikr}\right)\frac{d}{dkr}\cdot\frac{\sin kr}{kr} \quad \ldots\ldots \quad (37).$$

If $\frac{d\psi_n}{dr}$ vanishes, we have

$$Q_n\left(\frac{d}{d.ikr}\right)\frac{d}{d.kr}\cdot\frac{\sin kr}{kr} = 0 \quad \ldots\ldots\ldots\ldots \quad (38),$$

which, with the value of Q_n,

$$Q_n(\mu) = \frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\mu^{n-4} - \ldots\right\} \ldots (39),$$

may be considered as the complete solution of the main question proposed in the present paper.

In terms of α and β we should have

$$\frac{d\psi_n}{dr} = \frac{2i^{n+1}S_n}{r^2}\left\{\alpha\sin\left(kr + n\frac{\pi}{2}\right) - \beta\cos\left(kr + n\frac{\pi}{2}\right)\right\} \quad \ldots (40).$$

The value of $\frac{d\psi_n}{dr}$ in (40) is of high order in kr, if n be at all considerable.

Expanded in powers of r, the leading term in

$$\alpha\sin\left(kr + n\frac{\pi}{2}\right) - \beta\cos\left(kr + n\frac{\pi}{2}\right)$$

is

$$-(-1)^n\frac{n(kr)^{n+1}}{1.3.5\ldots(2n+1)};$$

so that, as appears in calculating the roots, the functions $\beta \div \alpha$ are more and more approximate expressions for $\tan\left(kr + n\frac{\pi}{2}\right)$ as n increases.

The values of λ for the proper tones of a sphere, whose radius is unity, were calculated from (12), and are given below in Table A.

In order to include the first roots for $n=5$, $n=6$, the table of the functions F given by Prof. Stokes had to be extended. I find

$$F_5(ikr) = ikr + 16 + \frac{135}{ikr} + \frac{735}{(ikr)^2} + \frac{2625}{(ikr)^3} + \frac{5670}{(ikr)^4} + \frac{5670}{(ikr)^5},$$

$$F_6(ikr) = ikr + 22 + \frac{252}{ikr} + \frac{1890}{(ikr)^2} + \frac{9765}{(ikr)^3} + \frac{34020}{(ikr)^4} + \frac{72765}{(ikr)^5} + \frac{72765}{(ikr)^6};$$

so that

$$\left(\frac{\beta}{\alpha}\right)_5 = \frac{kr - \frac{135}{kr} + \frac{2625}{k^3r^3} - \frac{5670}{k^5r^5}}{16 - \frac{735}{k^2r^2} + \frac{5670}{k^4r^4}},$$

$$\left(\frac{\beta}{\alpha}\right)_6 = \frac{kr - \frac{252}{kr} + \frac{9765}{k^3r^3} - \frac{72765}{k^5r^5}}{22 - \frac{1890}{k^2r^2} + \frac{34020}{k^4r^4} - \frac{72765}{k^6r^6}}.$$

Table A,

Giving the values of λ for a sphere of unit radius.

Number of Internal Spherical Nodes.	Order of Harmonic. 0	1	2	3	4	5	6
0	1·3983	3·0186	1·8800	1·392	1·113	·9300	·8002
1	·81334	1·0577	·86195	·7320	·6385		
2	·57622	·68251	·59208	·5248			
3	·44670	·50653	·45380				
4	·36485	·40330					
5	·30833	·33523					

For convenience of comparison, I have also calculated in a second table the pitch of each proper tone referred to the gravest of the whole system. The table is extended far enough to include all that lie within an interval of *two octaves*.

TABLE B.

Pitch of each Tone, referred to Gravest.	Order of Harmonic.	Number of Internal Spherical Nodes.
1·0000	1	0
1·6056	2	0
2·1588	0	0
2·169	3	0
2·712	4	0
2·8540	1	1
3·2458	5	0
3·5021	2	1
3·7114	0	1
3·772	6	0

The method of the present paper is easily applied to the more general case obtained by replacing the hollow sphere by a spherical shell.

From (9), if $\frac{d\psi_n}{dr}$ vanish for the radii r_1 and r_2,

$$\epsilon^{2ikr_1}\frac{F_n(-ikr_1)}{F_n(+ikr_1)} = \epsilon^{2ikr_2}\frac{F_n(-ikr_2)}{F_n(+ikr_2)} \quad \ldots\ldots\ldots (41);$$

whence we obtain

$$\tan k(r_1-r_2) = \frac{\left(\frac{\beta}{\alpha}\right)_1 - \left(\frac{\beta}{\alpha}\right)_2}{1+\left(\frac{\beta}{\alpha}\right)_1\left(\frac{\beta}{\alpha}\right)_2} \quad \ldots\ldots\ldots (42)$$

as the equation giving the roots in all cases.

If r_2 be now made to vanish, we shall fall back on our previous result. In the paper referred to, it is proved that, when r_2 is vanishingly small,

$$F_n(ikr_2) = 1.3.5 \ldots (2n-1)(n+1)(ikr_2)^{-n}\{1+ikr_2+\ldots\} \ldots (43);$$

so that, when n is even, $\left(\frac{\beta}{\alpha}\right)_2 = 0$,

and when n is odd, $\left(\frac{\beta}{\alpha}\right)_2 = \infty$.

In the first case, then, (42) becomes

$$\tan kr_1 = \frac{\beta}{\alpha},$$

and in the second, $$\tan kr_1 = -\frac{\alpha}{\beta},$$

which are together equivalent to

$$\tan\left(kr \pm n\frac{\pi}{2}\right) = \frac{\beta}{\alpha}.$$

April 11th, 1872.

Prof. CAYLEY, F.R.S., V.P., in the Chair.

Mr. W. Paice, M.A. Lond., was elected a member; and Mr. J. W. L. Glaisher, B.A., was admitted into the Society.

Prof. Cayley, having called upon Mr. Samuel Roberts, Vice-President, to occupy the chair, proceeded to give an account of his paper "On the Mechanical Description of Certain Sextic Curves." Mr. Roberts afterwards exhibited an apparatus which was arranged to show, in connection with Prof. Cayley's paper, curves traced by a fixed pencil on a plane, two points of which were constrained to move in fixed circles. These curves are similar in character to those described by different points of the moving plane relative to a fixed plane. He further drew attention to an analogous manner in which certain surfaces of the fourth degree may be generated.

Prof. Crofton called the attention of the meeting to the fact that a certain proof of the theorem, that any motion of a plane figure in its plane is equivalent to a rotation round some point, might appear at first sight to be capable of extension to three dimensions, and to show that any motion of a solid is equivalent to a rotation—a result known not to be true.

Some conversation followed, in which Prof. Cayley and Mr. Merrifield suggested the probable explanation of the point; the following seems to be the result.

Let A, B be any two points of a plane figure, and let A′, B′ be the positions of the points after any displacement of the figure. By bisecting AA′ and BB′ at right angles, we can always find a *single* point O equidistant from A, A′, and from B, B′, excepting the case when the bisectors coincide, and O may range over a straight line; and in this case it is easy to see that the displacement is equivalent to a rotation round the point of intersection of A′B′ with AB. In any other case, O is the centre of rotation. For the triangles OAB, OA′B′ being equal in all respects, let OA revolve till it falls on OA′, carrying with it the triangle OAB; this triangle will now fall on the same side of OA′ as OA′B′ (and therefore must coincide with it); for if it fell on the opposite side, it is easily shown that we should have the above excepted case. Thus any motion of a plane figure, which is always determined by that of any two of its points, is equivalent to a rotation.

Now it would appear at first sight that, speaking generally, and without going minutely into the exceptional cases, if A′, B′, C′ are the new positions of any three points A, B, C of a solid, supposed to receive any possible displacement in space, as we can always find a point

O, equidistant from A, A′, from B, B′, and from C, C′,—that the triangle ABC could be carried into the position A′B′C′ by a rotation round O.

The explanation of the difficulty consists in the two tetrahedra OABC, OA′B′C′, which are always equal by symmetry, not being always superposable, the displacement is equivalent to a rotation round O. It also follows that, if so, O is not a single point, but may range over a straight line; for as the motion is a rotation round some axis through O, the planes bisecting AA′, BB′, CC′ at right angles meet in a line—viz., the axis. A fuller examination would no doubt be instructive: thus, it does not appear at once why the case when the tetrahedra are superposable is a less general one than that where they are not. Probably it will appear that the point O has some curious relation to the real motion of the solid, when that is not a simple rotation.

Further discussion followed upon some other questions, in which the Chairman (Prof. Cayley), Messrs. Cotterill, Sprague, and others, took part.

The following presents were received:—

A Carte de Visite of Herr B. Riemann: from Herr Stein.

First and Second Annual Reports of the Association for the Improvement of Geometrical Teaching.

On the Induction of Electric Currents in an Infinite Plane Sheet of Uniform Conductivity: by Prof. J. Clerk-Maxwell, F.R.S. (Proceedings of the Royal Society, No. 132, 1872): from the Author.

On the Mechanical Description of Certain Sextic Curves.

By A. Cayley.

[*Read April 11th*, 1872.]

The curves in question might be taken to be those described by a point C rigidly connected with points A and B, each of which describes a circle: but the construction is considered under a somewhat more general form. I consider a quadrilateral, the sides of which are a, b, c, d, and the inclinations of these to a fixed line $\alpha, \beta, \gamma, \delta$. This being so, if a, b, c, d, and one of the angles, say δ, are constant, then we have between the three variable angles the relations

$$a\cos\alpha + b\cos\beta + c\cos\gamma + d\cos\delta = 0,$$
$$a\sin\alpha + b\sin\beta + c\sin\gamma + d\sin\delta = 0,$$

giving rise to a single relation between any two of the variable angles; and we consider a curve such that the coordinates x, y of any point thereof are given linear functions of the sines and cosines of the three variable angles, or what is the same thing, of the sines and cosines of

any two of these angles. We thus unite together what would otherwise be distinct cases; for everything is symmetrical in regard to the sides a, b, c and the corresponding variable angles α, β, γ, irrespectively of the order of succession of these sides: and we can thus, in the discussion of the curve, employ any two at pleasure, say α, β, of the variable angles, without determining whether the sides a, b are contiguous or opposite.

Eliminating, then, the variable angle γ, we obtain between α, β a relation which, if we write therein $\tan \frac{1}{2}\alpha = u$, $\tan \frac{1}{2}\beta = v$, takes the form $(*)\,(u, 1)^2\,(v, 1)^2 = 0$; viz., either of the variables u, v is expressible rationally in terms of the other of them and of the root of a quartic function thereof; say v is a rational function of u and $\sqrt{U}$. And hence a curve for which the coordinates x, y are rational functions of u, v, is a curve having a deficiency $D=1$, or what is the same thing, having a number of dps. less by unity than the maximum number $[=\frac{1}{2}(n-1)(n-2)$, if n be the order of the curve].

It will further appear that the relation $(*)\,(u, 1)^2\,(v, 1)^2 = 0$ is satisfied by the values $u=v=i$ and $u=v=-i$ (if, as usual, $i=\sqrt{-1}$).

In the curve in question, the coordinates (x, y) are given linear functions of the sines and cosines of α, β; and if we make the curve meet an arbitrary line $Ax+By+C=0$, we obtain between the sines and cosines of α, β a linear relation which, substituting therein the expressions in terms of u, v, takes the form

$$(*)\,(u, 1)^2.\,(1+v^2)+(*)\,(v, 1)^2.\,(1+u^2) = 0,$$

viz., this is a relation of the form $(\dagger)\,(u, 1)^2\,(v, 1)^2 = 0$, such that it is satisfied by the four sets of values $u=\pm i$, $v=\pm i$, and therefore in particular by the values $u=v=i$, and $u=v=-i$.

Hence, considering the intersections of the curve by the arbitrary line, the values of (u, v) are given by the two equations $(*)\,(u, 1)^2\,(v, 1)^2 = 0$, $(\dagger)\,(u, 1)^2\,(v, 1)^2 = 0$; these, regarding for a moment u, v as ordinary rectangular coordinates, represent each of them a quartic curve having two dps. at infinity on the axes $u=0$, $v=0$ respectively: each of these points reckons therefore as 4 intersections, and the number of the remaining intersections therefore is $4.4-2.4$, $=8$. But, by what precedes, the two quartic curves have also in common the points $u=v=i$ and $u=v=-i$; and rejecting these, there remain $8-2$, $=6$ intersections.

The conclusion is, that the curve is a sextic curve of deficiency 1, that is, having 9 dps. The reasoning may be presented under a slightly different form as follows: regarding u, v as coordinates, we have the curve $(*)\,(u, 1)^2\,(v, 1)^2 = 0$, a binodal quartic curve, and having therefore the deficiency 1; the curve passes, as above mentioned, through the points $u=v=i$, and $u=v=-i$. The required curve is obtained as a transformation of the quartic curve by formulæ of the form

$x:y:z\,(=1)=\mathrm{P}:\mathrm{Q}:\mathrm{R}$, where P, Q, and R $[=(1+u^2)(1+v^2)]$ are quartic functions of the coordinates u, v, such that P=0, Q=0, R=0 are each of them a quartic curve passing twice through each of the nodes and once through each of the before-mentioned points, $(u=v=i)$ and $(u=v=-i)$, of the binodal quartic curve. Hence the curve in question is a curve of the order $4.4-2.4-2.1, =6$, and having the same deficiency as the binodal quartic, that is, the deficiency is $=1$.

I observe that the sextic curve does not, in general, pass through the circular points at infinity, but it intersects the line at infinity in three distinct pairs of points; one of these, or all three of them, (but not two pairs only,) may coincide with the circular points at infinity, the circular points at infinity being, in the latter case, triple points, or the curve being tricircular: this will appear presently.

To obtain the foregoing equation $(*)\,(u,1)^2\,(v,1)^2=0$, the elimination of γ gives

$$(a\cos\alpha+b\cos\beta+d\cos\delta)^2+(a\sin\alpha+b\sin\beta+d\sin\delta)^2=c^2,$$

that is,

$$a^2+b^2-c^2+d^2+2ab\cos(\alpha-\beta)+2ad\cos(\alpha-\delta)+2bd\cos(\beta-\delta)=0;$$

or, substituting herein the values

$$\cos\alpha=\frac{1-u^2}{1+u^2},\quad \sin\alpha=\frac{2u}{1+u^2},\quad \cos\beta=\frac{1-v^2}{1+v^2},\quad \sin\beta=\frac{2v}{1+v^2},$$

this is easily found to be $0=$

	1	$2u$	u^2
1	$(a+b)^2-2(a+b)d\cos\delta+d^2-c^2$	$2ad\sin\delta$	$(a-b)^2+2(a-b)d\cos\delta+d^2-c^2$
$2v$	$2bd\sin\delta$	$2ab$	$2bd\sin\delta$
v^2	$(a-b)^2-2(a-b)d\cos\delta+d^2-c^2$	$2ad\sin\delta$	$(a+b)^2+2(a+b)d\cos\delta+d^2-c^2$

or writing, for greater convenience, $\delta=0$, that is, taking α, β, γ to be the inclinations of the sides a, b, c to the fixed side d, this is $0=$

	1	$2u$	u^2
1	$(a+b-d)^2-c^2$	0	$(a-b+d)^2-c^2$
$2v$	0	$2ab$	0
v^2	$(a-b-d)^2-c^2$	0	$(a+b+d)^2-c^2$

Or say this is $(A+Bu^2)+8abuv+v^2(C+Du^2)=0$,
where, writing for shortness

$$\begin{aligned} a+d-b-c &= \lambda, & -a+b+c+d &= \lambda', \\ b+d-c-a &= \mu, & -b+c+d+a &= \mu', \\ c+d-a-b &= \nu, & -c+d+a+b &= \nu', \\ a+b+c+d &= \sigma, & -d+a+b+c &= \rho'; \end{aligned}$$

then $A=-\nu\rho'$, $B=\lambda\mu'$, $C=\lambda'\mu$, $D=\nu'\sigma$.

We at once verify that the equation is satisfied by $u=v=\pm i$, viz., this will be so if only $A-B-C+D-8ab=0$,
and we have

$$\begin{aligned} A+D &= 2(a+b)^2+2d^2-2c^2, \\ B+C &= 2(a-b)^2+2d^2-2c^2; \end{aligned}$$

whence the relation in question.

Writing $u=i$, we have

$$\begin{aligned} \{v(C-D)+4abi\}^2 &= -16a^2b^2-(A-B)(C-D) \\ &= -\tfrac{1}{4}(A-B+C-D)^2, \end{aligned}$$

whence

$$\begin{aligned} v(C-D) &= \pm\tfrac{1}{2}i\{\mp 8ab+A-B+C-D\} \\ &= i(C-D) \text{ or } -i(A-B); \end{aligned}$$

so that, corresponding to $u=i$, we have the values $v=i$ and $v=-i\dfrac{A-B}{C-D}$; and similarly, corresponding to $u=-i$, the values $v=-i$ and $v=i\dfrac{A-B}{C-D}$.

And in like manner, corresponding to $v=i$, we have the values $u=i$ and $u=-i\dfrac{A-C}{B-D}$; and to $v=-i$, the values $u=-i$ and $u=i\dfrac{A-C}{B-D}$.

It is easy to show that, if $u=i+\epsilon$, $v=i+\zeta$ are the values consecutive to $u=v=i$, then $a\epsilon+b\zeta=0$: in fact, substituting the foregoing values in the relation between u, v, and writing for $8ab$ its value, $=A-B-C+D$, we have

$$\begin{aligned} A+B(-1+2i\epsilon)+(A-B-C+D)\{-1+i(\epsilon+\zeta)\} \\ +C(-1+2i\zeta)+D\{1-2i(\epsilon+\zeta)\}=0, \end{aligned}$$

which is $\epsilon(A+B-C-D)+\zeta(A-B+C-D)=0$;
or finally $a\epsilon+b\zeta=0$. And similarly, if $u=-i+\epsilon$, $v=-i+\zeta$ are the values consecutive to $u=v=-i$, then we have the same relation $a\epsilon+b\zeta=0$.

The points at infinity on the sextic curve are those for which $1+u^2$ or $1+v^2$, or both of these, are $=0$; viz., the values of u, v for the six points are

$$\begin{aligned} u &= i+\epsilon, \ -i-\epsilon, \quad i \quad , -i \quad , -i\left(\frac{A-C}{B-D}\right), \ i\left(\frac{A-C}{B-D}\right); \\ v &= i+\zeta, \ -i-\zeta, \ -i\frac{A-B}{C-D}, \ i\frac{A-B}{C-D}, \quad i \quad , -i, \end{aligned}$$

where, instead of $u=v=i$ and $u=v=-i$, I have written down the consecutive values of u, v, and as before ϵ, ζ are infinitesimals such that $a\epsilon+b\zeta=0$.

Suppose that the coordinates x, y of a point on the sextic curve are

$$x = \mathrm{a}\cos\alpha+\mathrm{c}\sin\alpha+\mathrm{b}\cos\beta+\mathrm{d}\sin\beta,$$
$$y = \mathrm{a}'\cos\alpha+\mathrm{c}'\sin\alpha+\mathrm{b}'\cos\beta+\mathrm{d}'\sin\beta;$$

then, if $u = \pm(i+\epsilon)$, $\cos\alpha = \frac{-i}{\epsilon}$, $\sin\alpha = \frac{\pm 1}{\epsilon}$, and similarly if $v = \pm(i+\zeta)$, then $\cos\beta = -\frac{i}{\zeta}$, $\sin\beta = \frac{\pm 1}{\zeta}$. Hence the points at infinity of the sextic curve are as follows:—

1°. $u = \pm(i+\epsilon)$, v not $= \pm(i+\zeta)$,

$$x = \frac{-\mathrm{a}i\pm\mathrm{c}}{\epsilon}, \quad y = \frac{-\mathrm{a}'i\pm\mathrm{c}'}{\epsilon},$$ first pair of points;

2°. $v = \pm(i+\zeta)$, u not $= \pm(i+\epsilon)$,

$$x = \frac{-\mathrm{b}i\pm\mathrm{d}}{\zeta}, \quad y = \frac{-\mathrm{b}'i\pm\mathrm{d}'}{\zeta},$$ second pair of points;

3°. $u = \pm(i+\epsilon)$, $v = \pm(i+\zeta)$,

$$x = \frac{-\mathrm{a}i\pm\mathrm{c}}{\epsilon}+\frac{-\mathrm{b}i\pm\mathrm{d}}{\zeta}, \quad y = -\frac{\mathrm{a}'i\pm\mathrm{c}'}{\epsilon}+\frac{-\mathrm{b}'i\pm\mathrm{d}}{\zeta};$$

$a\epsilon+b\zeta=0$, as above, third pair of points;

which six points are in general distinct from each other, and from the circular points at infinity.

The foregoing values of x, y may be said to be "circular *quoad* α," if $\mathrm{a}=\mathrm{c}'$, $\mathrm{a}'=-\mathrm{c}$; and similarly to be "circular *quoad* β," if $\mathrm{b}=\mathrm{d}'$, $\mathrm{b}'=-\mathrm{d}$.

And we see at once that if the values are circular *quoad* α, then the first pair of points coincide with the circular points at infinity; and that, in like manner, if the values are circular *quoad* β, then the second pair of points coincide with the circular points at infinity; but if the values are circular *quoad* α and β respectively, then each of the three pairs of points coincides with the circular points at infinity: so that these are then triple points on the curve; or the curve is tricircular, having besides the two triple points, 3 dps.

The relation between u, v gives

$$\{v(\mathrm{C}+\mathrm{D}u^2)+4abu\}^2 = 16a^2b^2.u^2-(\mathrm{A}+\mathrm{B}u^2)(\mathrm{C}+\mathrm{D}u^2),$$

and it thus appears that if any one of the functions A, B, C, D is $=0$, the function under the radical sign is a mere linear function of u^2, say it is $\mathrm{L}+\mathrm{M}u^2$; introducing a new parameter θ such that $u = \sqrt{\left(\frac{\mathrm{L}}{\mathrm{M}}\right)}\frac{2\theta}{1+\theta^2}$, we have $\sqrt{\mathrm{L}+\mathrm{M}u^2} = \sqrt{\mathrm{L}}\,\frac{1+\theta^2}{1-\theta^2}$, and consequently u, v are each of them a rational function of θ. Hence, when any one of the relations in

question is satisfied, or say, when $a+d=b+c$, $b+d=c+a$, or $c+d=a+b$, the curve becomes unicursal: there is no diminution of the order, and the curve is consequently a unicursal sextic, or sextic with 10 dps.

It would at first sight appear that the curve might become unicursal in a different manner; viz., it would be unicursal if

$$16a^2b^2u^2-(A+Bu^2)(C+Du^2)$$

was a perfect square; but this is only the case when one of the four sides a, b, c, d is $=0$. The condition in fact is

$$AD+BC-16a^2b^2=2\sqrt{ABCD};$$

that is $$\lambda\lambda'\mu\mu'-\nu\nu'\rho'\sigma-16a^2b^2=2\sqrt{-\lambda\lambda'\mu\mu'\rho'\sigma};$$

where, putting for shortness $M=d^2-a^2-b^2-c^2$, we have

$$\begin{aligned}\lambda\lambda'&=M-2bc+2ca+2ab,\\ \mu\mu'&=M+2bc-2ca+2ab,\\ \nu\nu'&=M+2bc+2ca-2ab,\\ -\rho'\sigma&=M-2bc-2ca-2ab;\end{aligned}$$

and thence $$\lambda\lambda'\mu\mu'=M^2-4b^2c^2-4c^2a^2+4a^2b^2+4abM+8c^2ab,$$

$$-\nu\nu'\rho\sigma'=M^2-4b^2c^2-4c^2a^2+4a^2b^2-4abM-8c^2ab,$$

and the equation thus becomes

$$M^2-4b^2c^2-4c^2a^2-4a^2b^2=\sqrt{(M^2-4b^2c^2-4c^2a^2+4a^2b^2)^2-16a^2b^2(M+2c^2)^2},$$

viz., putting for a moment $X=M^2-4c^2(a^2+b^2)$, this is

$$(X-4a^2b^2)^2=(X+4a^2b^2)^2-16a^2b^2(M+2c^2)^2,$$

that is, $$16a^2b^2\{X-(M+2c^2)^2\}=0;$$

or, substituting for X its value, the equation is

$$64a^2b^2c^2(M+a^2+b^2+c^2)=0,$$

that is, $a^2b^2c^2d^2=0$.

We may have, simultaneously, 1°, $a=d$, $b=c$; 2°, $b=d$, $a=c$; 3°, $c=d$, $a=b$; the three cases are really equivalent, but the results present themselves in different forms.

1°. Here $A=0$, $B=4a(a-b)$, $C=0$, $D=4a(a+b)$; the relation between u, v contains the factor u, and throwing this out, and also the constant factor $4a$, it is

$$u[(a-b)+(a+b)v^2]+2bv=0,$$

viz., u is given as a rational function of v.

2°. Here $A=0$, $B=0$, $C=4b(b-a)$, $D=4b(b+a)$; the equation contains the factor v, and throwing out this and also the constant factor $4b$, the equation is $$v[(b-a)+(b+a)u^2]+2au=0,$$

viz., v is given as a rational function of u.

3°. Here $A=4a(a-c)$, $B=0$, $C=0$, $D=4a(a+c)$; or, dividing by $4a$, the equation is $$(a-c)+2auv+(a+c)u^2v^2=0;$$

viz., this is $(uv+1)[(a+c)uv+a-c] = 0,$
which may be reduced to $(a+c)uv+a-c = 0,$
giving u or v each a rational function of the other.

I do not discuss the theory in detail, but only remark that in each case there is a conic thrown off, and that in place of the sextic we have a unicursal (or trinodal) quartic curve.

May 9th, 1872.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

The Rev. C. Taylor, M.A., Fellow, and Mr. R. W. Genese, B.A., Scholar, of St. John's College, Cambridge, were proposed for election.

Mr. J. W. L. Glaisher gave an account of his paper "On Functions with Recurring Derivatives;" and Mr. Tucker (Hon. Sec.) read portions of communications from Prof. J. Clerk-Maxwell "On the Interpretation of Lagrange's and Hamilton's Equations of Motion," and "On the Condition that, in the transformation of any figure by curvilinear co-ordinates in three dimensions, every angle in the new figure shall be equal to the corresponding angle in the original figure." Conversation ensued on the papers, in which the President, Profs. Cayley, Clifford, Dr. Hirst, and others took part.

Prof. Clifford made some remarks on a theory of the exponential function derived from the equation $\frac{du}{dt} = pu$. After showing that a quantity which is equally multiplied in equal times must always increase at a rate proportional to itself, or which is the same thing, satisfy the equation $\frac{du}{dt} = pu$, we may call the ratio p of the growth to the growing quantity the *intrinsic rate*. We may then define as follows:—e^{px}. u is the result of making u grow at the intrinsic rate p for x seconds. This applies so far to pure quantity only; but if we regard u as—

(a) a vector on a line, *i.e.*, a signed magnitude;
(b) a vector in a plane, *i.e.*, a complex magnitude; or
(c) a vector in space;

then, p being the ratio of two such vectors, is respectively

(a) a signed ratio; justifying e^{-px}.
(b) a complex ratio; from which it becomes obvious that
$$e^{i\theta} = \cos\theta + i\sin\theta.$$
(c) a quaternion; from which we get Hamilton's theory of exponential functions of quaternions.

A semi-justification of the symbolic form of Taylor's Theorem (independently of the series) is also obtained by this method.

The series for $e^{px}.u$ in all cases where it holds good, may then be either established by Taylor's theorem; or shown to be a solution, and the only one, of the equation $\frac{du}{dt} = pu$ between values 0 and x of t.

Prof. Cayley, Dr. Hirst, Mr. S. Roberts, and Mr. Perigal took part in subsequent remarks in connexion with *the degenerate forms of curves*, and the possibility of extending Mr. Perigal's construction for describing a portion of a parabola.

The following presents were received:—

"Journal of the London Institution," Nos. 14, 15.

"Bulletin des Sciences," Feb.—June, 1872. (5 Nos.)

"Monatsbericht," Jan., Feb., 1872.

"On the Analogues in the Theory of Quadrics to some known properties in the Theory of Conics," and "On the attraction of the Ellipsoid for the law of the inverse fourth power of the distance" (from the Quarterly Journal of Mathematics): presented by the author, the Rev. R. Townsend, F.R.S.

"Sur les théorèmes d'Ivory relatifs aux Surfaces Homofocales du second degré," par G. Darboux: from the author.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux, Tome iv. (3 pts.), 1866; Tome v. (5 pts.), 1867; Tome vi. (3 pts.), 1868; Tome vii., 1869.

"Mémoire sur le mouvement des Nœuds de la Lune et sur l'inégalité en latitude qui donne la mésure de l'aplatissement de la Terre," par M. G. Lespiault; Paris, 1861.

"Note sur les petites Planètes situées entre Mars et Jupiter," par M. G. Lespiault; Bordeaux, 1861. Both extracted from the above Bordeaux Society Memoirs.

"Tables diverses pour la décomposition des Nombres en leurs facteurs premiers," par V. A. Le Besgue; Paris, 1864.

"Tables donnants pour la moindre racine primitive d'un nombre premier, ou puissance d'un nombre premier: 1°, les nombres qui correspondent aux indices; 2°, les indices des nombres premiers et inférieurs au module," par V. A. Le Besgue.

"Tables Arithmétiques pour servir d'appendice à l'introduction à la Théorie des nombres de M. Le Besgue," par J. Hoüel; Paris, 1866.

"On Cyclides and Sphero-Quartics," by John Casey, LL.D.: from the author.

"Proceedings of the Royal Society," No. 133.

Six Geometrical Memoirs by Dr. Emil Weyr, communicated to the Academy of Sciences of Vienna, viz.:—"Construction des Krümmungs-

kreises für Fusspunktcurven" (21st Jan. 1869); "Ueber Curvenbüschel" (13th Jan. 1870); "Zur Vervollständigung der Involutionen höherer Ordnung" (28th April, 1870); "Geometrische Mittheilungen" (1°, 12th May, 2°, 19th May, 1870); "Ueber ähnliche Kegelschnitte" (23rd June, 1870): presented by Dr. Hirst, F.R.S.

On Functions with Recurring Derivatives. By J. W. L. Glaisher, B.A., F.R.A.S., Fellow of Trinity College, Cambridge.

[*Read May 9th*, 1872.]

In the "Edinburgh Transactions" for 1867 (vol. xxiv.), Mr. Sang has, under the above title, investigated at considerable length the properties of the functions that satisfy the differential equation

$$\frac{d^n u}{dx^n} = u \quad \ldots\ldots\ldots\

$$e^{\theta x} = \phi_1 x + \theta\phi_2 x + \theta^2\phi_3 x\,;$$

and similarly $$e^{\theta y} = \phi_1 y + \theta\phi_2 y + \theta^2\phi_3 y\,;$$

whence, multiplying together, and forming the like equation for $(x+y)$, we have

$$\begin{aligned}\phi_1(x+y) + \theta\phi_2(x+y) + \theta^2\phi_3(x+y) \\ = \quad (\phi_1 x\,\phi_1 y + \phi_2 x\,\phi_3 y + \phi_3 x\,\phi_2 y) \\ + \theta\,(\phi_1 x\,\phi_2 y + \phi_2 x\,\phi_1 y + \phi_3 x\,\phi_3 y) \\ + \theta^2(\phi_1 x\,\phi_3 y + \phi_2 x\,\phi_2 y + \phi_3 x\,\phi_1 y)\,;\end{aligned}$$

and observing that this equation is satisfied for *each* root of the equation $\theta^3 = 1$, it is clear that the terms in θ^0, θ^1, θ^2 must separately vanish; that is, the equation gives the values of the ϕ_1, ϕ_2, ϕ_3 of $x+y$ in terms of those of x, y.

It may be added that we have

$$\begin{aligned}1 = \quad & e^x.\,e^{\theta x}.\,e^{\theta^2 x} \\ = \quad & (\phi_1 x + \ \phi_2 x + \ \phi_3 x) \\ & \times (\phi_1 x + \theta\,\phi_2 x + \theta^2\phi_3 x) \\ & \times (\phi_1 x + \theta^2\phi_2 x + \theta\,\phi_3 x),\end{aligned}$$

that is, $$1 = (\phi_1 x)^3 + (\phi_2 x)^3 + (\phi_3 x)^3 - 3\phi_1 x\,\phi_2 x\,\phi_3 x,$$

the relation corresponding to $\cosh^2 x - \sinh^2 x$.

The paper was written without the Author being acquainted with that of Olivier; and as the chief results—viz., those relating to the functions of the third order—are to be found there, it was thought that an abstract, giving the results, not obtained by Olivier, for the nth order, was all that was required.

Consider the differential equation $\dfrac{d^n u}{dx^n} = u$. It is evidently satisfied by $u = \phi_1(x), \phi_2(x) \ldots \phi_n(x)$, defined by the equations

$$\left.\begin{aligned}\phi_1(x) &= 1 + \frac{x^n}{[n]} + \frac{x^{2n}}{[2n]} + \ldots \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \\ \phi_n(x) &= x^{n-1} + \frac{x^{2n-1}}{[2n-1]} + \frac{x^{3n-1}}{[3n-1]} + \ldots\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3).$$

These may be called the n fundamental series with recurring derivatives of the nth order, since it is clear that no derivative under the nth will be the same as the primary series. We also observe that the functions are so connected that

$$\frac{d}{dx}\phi_i(x) = \phi_{i-1}(x)\,;$$

thus ϕ_1 is the derivative of ϕ_2, ϕ_n of ϕ_1, ϕ_{n-1} of ϕ_n, &c.

From the definitions (3) we have at once, by Taylor's theorem, or by Clivier's method,

$$\left.\begin{array}{l} \phi_1(x+y) = \phi_1 x\,\phi_1 y + \phi_n x\,\phi_2 y \ldots + \phi_2 x\,\phi_n y \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ \phi_i(x+y) = \phi_i x\,\phi_1 y + \phi_{i-1} x\,\phi_2 y \ldots + \phi_{i+1} x\,\phi_n y \\ \cdots \quad \cdots \quad \cdots \quad \cdots \end{array}\right\} \ldots\ldots\ldots (4).$$

These relations are, of course, generalizations of the formulæ

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y, \quad \sinh(x+y) = \&c.,$$

in hyperbolic trigonometry. The law of formation in (4) is evident; the functions of one argument (y) move forwards from ϕ_1 to ϕ_n, while the functions of x begin from $\phi_i x$ [if $\phi_i(x+y)$ is the expression under consideration], and proceed backwards to $\phi_{i-1}, \phi_{i-2} \ldots \phi_{i+1}$.

In (2), put $y=-x$; and since $\phi_1(0)=1$, $\phi_2(0)=0$, $\phi_3(0)=0$, &c., we have

$$\begin{array}{l} \phi_1 x\,\phi_1(-x) + \phi_n x\,\phi_2(-x) \ldots + \phi_2 x\,\phi_n(-x) = 1, \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ \phi_2 x\,\phi_1(-x) + \phi_1 x\,\phi_2(-x) \ldots + \phi_3 x\,\phi_n(-x) = 0; \\ \cdots \quad \cdots \quad \cdots \quad \cdots \end{array}$$

whence

$$\phi_1(-x) = \frac{\Phi_1(x)}{\mathrm{M}}, \quad \phi_2(-x) = \frac{\Phi_n(x)}{\mathrm{M}}, \quad \ldots \quad \phi_i(-x) = \frac{\Phi_{n-i+2}(x)}{\mathrm{M}}, \quad \&c.,$$

where

$$\mathrm{M} = \left|\begin{array}{ccccc} \phi_1, & \phi_n, & \phi_{n-1}, & \ldots & \phi_2 \\ \phi_2, & \phi_1, & \phi_n, & \ldots & \phi_3 \\ \phi_3, & \phi_2, & \phi_1, & \ldots & \phi_4 \\ \cdots & \cdots & \cdots & & \cdots \\ \phi_n, & \phi_{n-1}, & \phi_{n-2}, & \ldots & \phi_1 \end{array}\right|^{*} \ldots\ldots\ldots\ldots\ldots (5);$$

and $\Phi_1(x)$, $\Phi_n(x)$, $\Phi_{n-1}(x)$ ... are the minors of ϕ_1, ϕ_n, ϕ_{n-1} ... in the top line. It will now be shown that $\mathrm{M}=1$ identically, so that

$$\phi_1(-x) = \Phi_1(x), \quad \phi_2(-x) = \Phi_n(x), \quad \&c.$$

Let $1, \omega_1, \ldots \omega_{n-1}$ be the n nth roots of unity, then

$$\left.\begin{array}{l} e^{x} = \phi_1 + \phi_2 \ldots + \phi_n \\ e^{\omega_1 x} = \phi_1 + \omega_1 \phi_2 \ldots + \omega_1^{n-1} \phi_n \\ \cdots \quad \cdots \quad \cdots \quad \cdots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (6);$$

and therefore

$$(\phi_1 + \phi_2 \ldots + \phi_n) \ldots (\phi_1 + \omega_{n-1}\phi_2 \ldots + \omega_{n-1}^{n-1}\phi_n) = e^{x(1+\omega_1+\omega_2\ldots+\omega_{n-1})} = 1 \ldots\ldots\ldots (7).$$

Now $\phi_1 + \omega_i\phi_2 + \omega_i^2\phi_3 \ldots + \omega_i^{n-1}\phi_n$ is a factor of the determinant M; for on multiplying the second, third, &c. rows by ω_i, ω_i^2, &c., and adding them to the top row, its constituents become

$$\phi_1 + \omega_i\phi_2 \cdots + \omega_i^{n-1}\phi_n, \quad \phi_n + \omega_i\phi_1 \ldots + \omega_i^{n-1}\phi_{n-1}, \quad \&c.,$$

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* ϕ_1, ϕ_2, ... are written for $\phi_1 x$, $\phi_2 x$, ..., for the sake of clearness, whenever there is no risk of ambiguity.

the second of which is equal to the first multiplied by ω_i, the third to the first multiplied by ω_i^2, &c.; whence the conclusion follows; for

$$M = k(\phi_1+\omega_1\phi_2+\ldots)\ldots(\phi_1+\omega_{n-1}\phi_2+\ldots);$$

and observing that the coefficient of ϕ_1^n in M must be unity, we have $k=1$, and therefore, from (7), $M=1$; that is, writing the rows as columns, $\phi_1, \phi_2, \ldots$ are connected by the identical relation

$$\begin{vmatrix} \phi_1, & \phi_2, & \phi_3, \ldots & \phi_n \\ \phi_n, & \phi_1, & \phi_2, & \phi_{n-1} \\ \phi_{n-1}, & \phi_n, & \phi_1, & \phi_{n-2} \\ \ldots & \ldots & \ldots & \ldots \\ \phi_2, & \phi_3, & \phi_4, \ldots & \phi_1 \end{vmatrix} = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

If $n=2$, this takes the form

$$\cosh^2 x - \sinh^2 x = 1.$$

We may notice that, since $\phi_1+\phi_2\ldots+\phi_n = e^x$, by adding all the rows to the top row, and dividing by this quantity, we have

$$\begin{vmatrix} 1, & 1, & 1, \ldots & 1 \\ \phi_n, & \phi_1, & \phi_2, & \phi_{n-1} \\ \phi_{n-1}, & \phi_n, & \phi_1, & \phi_{n-2} \\ \ldots & \ldots & \ldots & \ldots \\ \phi_2, & \phi_3, & \phi_4, & \phi_1 \end{vmatrix} = e^{-x};$$

and thus the sum of the minors of the constituents of either the top row (or first column) of the determinant in (8) is equal to e^{-x}.

If we denote by $\psi_1(x), \psi_2(x), \ldots$ the series corresponding to (3) for the differential equation $\dfrac{d^n u}{dx^n} = -u,$

and which only differ from $\phi_1, \phi_2, \ldots$ by having the alternate signs negative, the relations corresponding to (5) are written down at once; and it will be found that the relation corresponding to (8) is

$$\begin{vmatrix} \psi_1, & -\psi_n, & -\psi_{n-1}, & \ldots & -\psi_2 \\ \psi_2, & \psi_1, & -\psi_n, & \ldots & -\psi_3 \\ \psi_3, & \psi_2, & \psi_1, & \ldots & -\psi_4 \\ \ldots & \ldots & \ldots & & \ldots \\ \psi_n, & \psi_{n-1}, & \psi_{n-2}, & \ldots & \psi_1 \end{vmatrix} = 1,$$

all the elements to the right of the diagonal being negative. If n be odd, it is easily seen that this becomes identical with (8) on writing $\psi_1, \psi_3, \ldots$ for $\phi_1, \phi_3, \ldots$, and $-\psi_2, -\psi_4, \ldots$ for $\phi_2, \phi_4, \ldots$, as is indeed evident from the differential equations; but if n be even, there is no such transformation: in fact, to take the case of $n=2$, no change of sign can transform $\phi_1^2-\phi_2^2=1$ into $\psi_1^2+\psi_2^2=1$.

On the Condition that, in the Transformation of any Figure by Curvilinear Coordinates in Three Dimensions, every Angle in the new Figure shall be equal to the corresponding Angle in the original Figure. By J. Clerk Maxwell, F.R.S.

[*Read May 9th*, 1872.]

In the corresponding problem in two dimensions, the only condition is

$$x+\sqrt{-1}y=f(\xi+\sqrt{-1}\eta) \quad \ldots\ldots\ldots\ldots (1),$$

where x, y are the coordinates in one system, and ξ, η in the other.

In three dimensions the solution is more restricted.

Let x, y, z be functions of ξ, η, ζ; then the point in the system x, y, z, for which ξ is constant, will be in a certain surface, and by giving ξ a series of values we obtain a series of such surfaces. There will be two other series of surfaces, corresponding to η and ζ respectively. If in the second system ξ, η, ζ are rectangular coordinates, the surfaces corresponding to ξ, η, ζ in the first system will intersect at right angles.

The condition of this is

$$\frac{dx}{d\eta}\frac{dx}{d\zeta}+\frac{dy}{d\eta}\frac{dy}{d\zeta}+\frac{dz}{d\eta}\frac{dz}{d\zeta}=0 \quad \ldots\ldots\ldots\ldots (2),$$

with two other equations in ζ and ξ, and in ξ and η.

If we now write

$$\left.\begin{aligned}\alpha^2&=\left(\frac{dx}{d\xi}\right)^2+\left(\frac{dy}{d\xi}\right)^2+\left(\frac{dz}{d\xi}\right)^2\\ \beta^2&=\left(\frac{dx}{d\eta}\right)^2+\left(\frac{dy}{d\eta}\right)^2+\left(\frac{dz}{d\eta}\right)^2\\ \gamma^2&=\left(\frac{dx}{d\zeta}\right)^2+\left(\frac{dy}{d\zeta}\right)^2+\left(\frac{dz}{d\zeta}\right)^2\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3);$$

and $d\xi$ is the intercept of the line (η=const., ζ=const.) cut off between the surfaces ξ and $\xi+d\xi$.

The angle e, at which a line whose coordinates are functions of p cuts a line whose coordinates are functions of q, is found from the equation

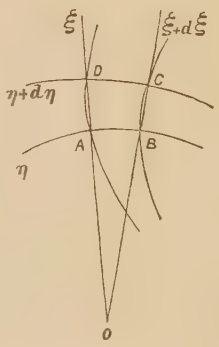
$$\frac{dx}{dp}\frac{dx}{dq}+\frac{dy}{dp}\frac{dy}{dq}+\frac{dz}{dp}\frac{dz}{dq}$$
$$=\cos e\left\{\left(\frac{dx}{dp}\right)^2+\left(\frac{dy}{dp}\right)^2+\left(\frac{dz}{dp}\right)^2\right\}^{\frac{1}{2}}\left\{\left(\frac{dx}{dq}\right)^2+\left(\frac{dy}{dq}\right)^2+\left(\frac{dz}{dq}\right)^2\right\}^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (4).$$

Expressing this in terms of ξ, η, and ζ, it becomes

$$\alpha^2\frac{d\xi}{dp}\frac{d\xi}{dq}+\beta^2\frac{d\eta}{dp}\frac{d\eta}{dq}+\gamma^2\frac{d\zeta}{dp}\frac{d\zeta}{dq}$$
$$=\cos e\left\{\alpha^2\left(\frac{d\xi}{dp}\right)^2+\beta^2\left(\frac{d\eta}{dp}\right)^2+\gamma^2\left(\frac{d\zeta}{dp}\right)^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\alpha^2\left(\frac{d\xi}{dq}\right)^2+\beta^2\left(\frac{d\eta}{dq}\right)^2+\gamma^2\left(\frac{d\zeta}{dq}\right)^2\right\}^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (5).$$

In order that the angle e, at which these lines intersect in the system (x, y, z), should be always equal to the angle ϵ, at which the corresponding lines in the system (ξ, η, ζ) intersect, it is necessary and sufficient that $\alpha = \beta = \gamma$ ..... (6),

for these are the conditions that equations (4) and (5) should be of the same form.



Now consider the quadrilateral ABCD, cut off from the surface (ζ = const.) by the surfaces ξ, $\xi+d\xi$, η, and $\eta+d\eta$.

$$\mathrm{AB} = \alpha d\xi,$$
$$\mathrm{AD} = \beta d\eta,$$
$$\mathrm{CD} = \left(\alpha + \frac{d\alpha}{d\eta}\, d\eta\right) d\xi,$$
$$\mathrm{BC} = \left(\beta + \frac{d\beta}{d\xi}\, d\xi\right) d\eta.$$

Since the three sets of surfaces are orthogonal, their intersections are lines of curvature, by Dupin's Theorem. Hence AB is a line of curvature of the surface η, drawn parallel to α. Hence the normals at A and B will intersect at some point O. Let us call OA, *the radius of curvature of η in the plane of α*, $\mathrm{R}_{\eta\alpha}$, then

$$\mathrm{AB} : \mathrm{DC} :: \mathrm{R}_{\eta\alpha} : \mathrm{R}_{\eta\alpha} + \mathrm{AD}.$$

Hence $$\mathrm{R}_{\eta\alpha} = \frac{\alpha\beta}{\frac{d\alpha}{d\eta}}; \quad \text{similarly} \quad \mathrm{R}_{\eta\gamma} = \frac{\beta\gamma}{\frac{d\gamma}{d\eta}}.$$

These values of the radii of curvature are true for any system of orthogonal surfaces, but since in this case $\alpha=\gamma$, we find

$$\mathrm{R}_{\eta\alpha} = \mathrm{R}_{\eta\gamma};$$

or the principal radii of curvature of the surface η are equal to each other at every point of the surface. Hence the surfaces η must be spherical.

Since $\alpha=\beta=\gamma$, the other families of surfaces ξ and ζ must also be spherical.

Now take one of the points where three spherical surfaces meet at right angles for the origin, and invert the system. These spheres when inverted become three planes intersecting at right angles, and the other spheres become spheres, each intersecting at right angles two of these planes. Hence their centres lie in the lines of intersection of the planes. Let a, b, c be the distances, from the point of intersection of the three planes, of the centres of spheres belonging each to one of the three systems, and let their radii be p, q, r respectively. Then the conditions that these spheres intersect at right angles are $q^2+r^2 = b^2+c^2$, $r^2+p^2 = c^2+a^2$, $p^2+q^2 = a^2+b^2$,

whence $p^2=a^2,\quad q^2=b^2,\quad r^2=c^2;$

or all the spheres pass through the origin, and each system has a common tangent plane. Hence if we take these three planes as coordinate planes, and write $x^2+y^2+z^2=r^2,$

$$\xi^2+\eta^2+\zeta^2=\rho^2;$$

we have $r\rho=\mathrm{R}^2$, a constant quantity,

$$\frac{x}{\xi}=\frac{y}{\eta}=\frac{z}{\zeta};$$

whence $$x=\xi\frac{\mathrm{R}^2}{\rho^2},\quad y=\eta\frac{\mathrm{R}^2}{\rho^2},\quad z=\zeta\frac{\mathrm{R}^2}{\rho^2};$$

$$\xi=x\frac{\mathrm{R}^2}{r^2},\quad \eta=y\frac{\mathrm{R}^2}{r^2},\quad \zeta=z\frac{\mathrm{R}^2}{r^2};$$

$$\alpha=\beta=\gamma=\frac{\mathrm{R}^2}{\rho^2}=\frac{r^2}{\mathrm{R}^2}=\frac{r}{\rho}.$$

It appears from this that the only method of transforming a figure of three dimensions, so that every angle in the system shall remain of the same magnitude, is the method of inversion or of reciprocal radii vectores.

[Reference may be made to "Méthodes de Transformation en Géométrie et en Physique mathématique," par J. N. Haton de Goupillière, especially § vii.; "*Journal de l'École Polytechnique*," tom. xxv., p. 188.]

June 13th, 1872.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Sir Charles Wheatstone, F.R.S., was amongst the visitors present.

Mr. W. Paice, M.A., was admitted into the Society; and the Rev. C. Taylor, M.A., and Mr. R. W. Genese, B.A., were ballotted for in the usual manner, and elected members.

Prof. Cayley, V.P., gave an account of his paper "On the Surfaces divisible into Squares by their Curves of Curvature."

Sir W. Thomson, Mr. Merrifield, and Prof. Clifford put a few questions to the Author upon points suggested by his paper.

Mr. S. Roberts, V.P., gave some illustrative details of his communication "On Prof. Cremona's Transformation between Two Planes, and Tables relating thereto."

Dr. Hirst briefly described a few results he had obtained bearing upon "a Manifold Correspondence of Two Planes." Prof. Clifford made a few remarks on Dr. Hirst's communication.

Sir W. Thomson then explained the object of his short paper on "the Simultaneous Reduction of Two Polynomial Quadratics to Sums of Squares." The Author cited particularly Cayley, "Cambridge and Dublin Mathematical Journal," Feb. 1849, and "Quarterly Mathematical Journal," 1858; and the celebrated problem of Cauchy's, treated in vol. iv. of his "Exercices," under the title "Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des planètes." The Hon. J. W. Strutt referred the Author to a memoir bearing on the subject of his paper.

On the Surfaces divisible into Squares by their Curves of Curvature.

By A. Cayley, F.R.S.

[*Read 13th June*, 1872.]

Prof. Cayley gave an account of an investigation recently communicated by him to the Academy of Sciences at Paris. The fundamental theorem is that, if the coordinates x, y, z of a point on a surface are expressed as functions of two parameters p, q (such expressions, of course, replacing the equation of the surface); and if these parameters are such that $p = \text{const.}$, $q = \text{const.}$ are the equations of the two sets of curves of curvature respectively; then (writing for shortness $\frac{dx}{dp} = x_1$, $\frac{dx}{dq} = x_2$, $\frac{d^2x}{dp^2} = x_3$, $\frac{d^2x}{dp\,dq} = x_4$, $\frac{d^2x}{dq^2} = x_5$, and the like for y, z), the coordinates x, y, z, considered always as functions of p, q, satisfy the equations

$$x_1x_2 + y_1y_2 + z_1z_2 = 0,$$

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_4, & y_4, & z_4 \end{vmatrix} = 0.$$

The last equation is equivalent to

$$\begin{aligned} x_4 + \mathrm{A}x_1 + \mathrm{B}x_2 &= 0, \\ y_4 + \mathrm{A}y_1 + \mathrm{B}y_2 &= 0, \\ z_4 + \mathrm{A}z_1 + \mathrm{B}z_2 &= 0; \end{aligned}$$

and if in the notation of Gauss we write

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 &= \mathrm{E}, \\ x_2^2 + y_2^2 + z_2^2 &= \mathrm{G}, \end{aligned}$$

then adding the equations multiplied by x_1, y_1, z_1 respectively, and also adding the equations multiplied by x_2, y_2, z_2 respectively, we find

$$\mathrm{A} = -\frac{1}{2}\frac{1}{\mathrm{E}}\frac{d\mathrm{E}}{dq}, \quad \mathrm{B} = -\frac{1}{2}\frac{1}{\mathrm{G}}\frac{d\mathrm{G}}{dq}$$

and the equations thus become

$$2x_4 - \frac{1}{\mathrm{E}}\frac{d\mathrm{E}}{dq}x_1 - \frac{1}{\mathrm{G}}\frac{d\mathrm{G}}{dq}x_2 = 0,$$

&c. &c. &c.,

which, in fact, agree with the equations (10 bis) in Lamé's "Leçons sur les Coordonnées curvilignes," Paris (1859), p. 89. The surface will be divisible into squares if only E : G is the quotient of a function of p by a function of q, or say if

$$\mathrm{E} = \Theta\mathrm{P}, \quad \mathrm{G} = \Theta\mathrm{Q},$$

where Θ is any function of (p, q), but P and Q are functions of p and q respectively; we then have

$$\frac{1}{\mathrm{E}}\frac{d\mathrm{E}}{dq} = \frac{1}{\Theta}\frac{d\Theta}{dq}, \quad \frac{1}{\mathrm{G}}\frac{d\mathrm{G}}{dp} = \frac{1}{\Theta}\frac{d\Theta}{dp},$$

and the equations for x, y, z are

$$2x_4 - \frac{1}{\Theta}\frac{d\Theta}{dq}x_1 - \frac{1}{\Theta}\frac{d\Theta}{dp}x_2 = 0,$$

&c. &c. &c.,

viz., x, y, z being functions of p, q such that $x_1x_2 + y_1y_2 + z_1z_2 = 0$, and which besides satisfy these equations, or say which each of them satisfy the equation

$$2u_4 - \frac{1}{\Theta}\frac{d\Theta}{dq}u_1 - \frac{1}{\Theta}\frac{d\Theta}{dp}u_2 = 0,$$

then the values of x, y, z in terms of (p, q) determine a surface which has the property in question.

On Professor Cremona's Transformation between Two Planes, and Tables relating thereto. By Samuel Roberts, M.A.

[*Read June 13th*, 1872.]

In the transformation theory of Prof. Cremona, we have a point-to-point correspondence between two planes. To a right line in one plane corresponds a unicursal curve of the order n in the other plane, and *vice versâ*. To the right lines in one plane (say P) corresponds a reseau of unicursal curves of the order n in the other plane (say P′). Any two curves of the reseau intersect in a certain number of fundamental points giving the equivalent of $n^2 - 1$ intersections, and these points are the same for all the curves of the reseau. The two curves in question meet besides in one point, which corresponds to the intersection of the two right lines, to which those curves themselves correspond. All this holds good when P and P′ are interchanged.

The fundamental or principal points in P may be divided into groups

$$\alpha_p,\ \alpha_q,\ \alpha_r,\ \&c.,$$

where α_p denotes the number of such points which are multiple in the degree p, and so on. In like manner, the principal points in P′ may be divided into groups $\alpha'_{p'},\ \alpha'_{q'},\ \alpha'_{r'},\ \&c.$

The systems satisfy respectively the equations

$$\begin{aligned}
\Sigma p\ \alpha_p &= 3n-3,\\
\Sigma p^2\alpha_p &= n^2-1,\\
\Sigma p'\alpha'_{p'} &= 3n-3,\\
\Sigma p'^2\alpha'_{p'} &= n^2-1.
\end{aligned}$$

Some solutions of these equations are, however, usually inadmissible; and the problem of determining for a given integer n the systems of principal points has not been completely solved. Prof. Cremona has determined them for $n=1, 2, 3 \ldots 10$, and has also given some of the forms for general values of n.

In many instances, two conjugate systems are the same in form; that is to say, we may write

$$\begin{aligned}
&\alpha_p=\alpha'_{p'},\quad \alpha_q=\alpha'_{q'},\quad \alpha_r=\alpha'_{r'},\quad \&c.,\\
&p=p',\quad q=q',\quad r=r',\quad \&c.
\end{aligned}$$

When this is not the case, the systems still possess this property, that to each group α_p in one system corresponds an equal group $\alpha'_{p'}$ in the other system, and *vice versâ*; that is to say, we may write

$$\alpha_p=\alpha'_{p'},\quad \alpha_q=\alpha'_{q'},\quad \alpha_r=\alpha'_{r'},\quad \&c.,$$

but *not* necessarily $p=p',\quad q=q',\quad r=r',\quad \&c.$

This relation, following from the symmetry of the two systems, was inferred by Prof. Cremona, and rigorously established by Prof. Clebsch (Math. Annalen, Band IV., p. 490).

But when two conjugate systems are determined (and it is to be observed that for $n>3$ there are several conjugate systems), we have yet to determine the singular correspondents of the principal points themselves.

To the principal points in P corresponds the Jacobian of the reseau which corresponds to right lines in P, and this Jacobian is made up of certain curves which are of the order p, q, r, &c., and are unicursal; that is to say, it is made up of α_p curves C_p, α_q curves C_q, α_r curves C_r, &c., and to each constituent point of the group α_p corresponds a curve C_p, to each constituent point of the group α_q corresponds a curve C_q, and so forth.

We have next to determine the order of multiplicity of the principal points in P′ through which these singular correspondents respectively pass.

This problem being solved, we are able to construct tables which in fact express the character of the Jacobians.

I do not wish to occupy space with more details than are necessary to make the remarks I have to offer intelligible. The general theory will be found in the original memoirs by Prof. Cremona, and in a paper by Prof. Cayley on the rational transformation between two spaces. (Proc. Lond. Math. Soc., Vol. III., p. 127 *et seq*.) I am concerned at present with the form of the tables which for $n = 2, 3 \ldots 6$, and some general cases, are given in the last-named memoir.

In the first place, let us suppose that a table or a pair of tables, where the conjugates are dissimilar, is given, so that we know the corresponding conjugate systems and the character of the Jacobians.

If we take any right line in P′, its correspondent in P is a unicursal curve of the order n, passing p times through each point of the group α_p, q times through each point of the group α_q, and so on. If, however, the line passes through a point of the group $\alpha'_{p'}$, the reduced correspondent is of the order $n-p'$, and the reduction takes place in consequence of the point of the group $\alpha'_{p'}$ having for its singular correspondent a unicursal curve of the order p'. The gross correspondent of the right line still remains of the order n, and passes p times through each point of the group α_p, &c., but breaks up into the nett correspondent $C_{n-p'}$ and the curve $C_{p'}$, which, as corresponding to a point of the group $\alpha'_{p'}$, is invariable for the pencil of right lines passing through that point. What is to be observed is that, if we know the relation of the curve $C_{p'}$ to the principal points in P, we can determine the relation of the curve $C_{n-p'}$ to the same system.

And if the primitive line is a curve $C_{k'}$, and passes through the principal points in P′ in a stated manner, we can in the same way deduce the relation of the correspondent C_k in P to the principal points in P, when we know the nature of the correspondent if $C_{k'}$ does not pass through any of the principal points, and know also the nature of the correspondents of the principal points in P′.

To fix the ideas, suppose that

$$\alpha_p, \quad \beta_q, \quad \gamma_r,$$
$$\alpha_s, \quad \beta_t, \quad \gamma_u,$$

are conjugate systems, and that C_s, corresponding to a point of the group α_s, passes

a times through A points of the group α_p,
b times through B points of the group α_q,
c times through C points of the group α_r.

Then to a right line through a point of the group α_s will correspond a curve C_{n-s} which passes

p times through α_p—A points of the group α_p,
$p-a$ times through A points of the group α_p,
q times through β_q—B points of the group β_q,
$q-b$ times through B points of the group β_q,
r times through γ_r—C points of the group γ_r,
$r-c$ times through C points of the group γ_r.

I will now make use of these considerations in one or two cases of transformation.

For $n=2$, we have the table (Proc., vol. iii. p. 148)

$$\begin{array}{c|c|} & a_1 = 3 \\ \hline a'_1 = 3 & 2 \\ \hline \end{array}.$$

Let the curve $C_{k'}$ pass a'_1 times through a point of the group a'_1, b'_1 times through another point of the same group, and c'_1 times through the remaining point.

Let the correspondent C_k in P pass a_1 times through a point of the group a_1, b_1 times through another point of the same group, and c_1 times through the remaining point.

If the curve $C_{k'}$ did not pass through any principal point, we should have $a_1 = b_1 = c_1 = k'$.

We have then

$$k = 2k' - a'_1 - b'_1 - c'_1,*$$
$$k' = 2k - a_1 - b_1 - c_1;$$
$$a_1 = k' - b'_1 - c'_1,$$
$$b_1 = k' - a'_1 - c'_1,$$
$$c_1 = k' - a'_1 - b'_1.$$

And since the conjugate systems are similar

$$a'_1 = k - b_1 - c_1,$$
$$b'_1 = k - a_1 - c_1,$$
$$c'_1 = k - a_1 - b_1.$$

These are the known formulæ for a quadric transformation.

For the cubic transformation we have

$$\begin{array}{c|c|c|} & a_1 = 4 & a_2 = 1 \\ \hline a'_1 = 4 & 1 & 1 \\ \hline a'_2 = 1 & 4 & 1 \\ \hline \end{array}$$

* The general formula being

$$k = k'n - \Sigma(a'_1 + 2a'_2 + \ldots),$$
$$k' = kn - \Sigma(a_1 + 2a_2 + \ldots).$$

Let $C_{k'}$ in P′ pass a_1' times, b_1' times, c_1' times, d_1' times through the respective points of the group α_1', and a_2' times through the sole point of the group α_2'. In like manner let k, a_1, b_1, c_1, d_1, a_2 refer to the correspondent C_k in P_1. Then we have

$$k = 3k' - \Sigma a_1' - 2a_2',$$
$$k' = 3k - \Sigma a_1 - 2a_2,$$

with
$$a_1 = k' - a_1' - a_2',$$
$$b_1 = k' - b_1' - a_2', \quad a_2 = 2k' - \Sigma a_1' - a_2',$$
$$c_1 = k' - c_1' - a_2',$$
$$d_1 = k' - d_1' - a_2';$$

and conversely
$$a_1' = k - a_1 - a_2,$$
$$b_1' = k - b_1 - a_2, \quad a_2' = 2k - \Sigma a_1 - a_2,$$
$$c_1' = k - c_1 - a_2,$$
$$d_1' = k - d_1 - a_2.$$

These are the formulæ for the cubic transformation. In like manner we can write down the formulæ of transformation in any case where the Jacobian tables are completely given.

Thus, for an example of dissimilar conjugates take the last pair of tables for $n=6$.

Using the same notation as before (Proc., vol. iii. p. 149), we have

for $C_{k'}$ and its correspondent
$$k = 6k' - \Sigma a_1' - 2\Sigma a_2' - 4a_4',$$
$$k' = 6k - \Sigma a_1 - 2a_2 - 3\Sigma a_3,$$

$$a_1 = k' - a_2' - a_4', \qquad a_2 = 2k' - \Sigma a_2' - 2a_4',$$
$$b_1 = k' - b_2' - a_4', \qquad a_3 = 3k' - b_1' - c_1' - \Sigma a_2' - 2a_4',$$
$$c_1 = k' - c_2' - a_4', \qquad b_3 = 3k' - a_1' - c_1' - \Sigma a_2' - 2a_4',$$
$$d_1 = k' - d_2' - a_4', \qquad c_3 = 3k' - a_1' - b_1' - \Sigma a_2' - 2a_4';$$

and conversely

$$a_1' = k - b_3 - c_3, \qquad a_2' = 2k - a_1 - a_2 - \Sigma a_3,$$
$$b_1' = k - a_3 - c_3, \qquad b_2' = 2k - b_1 - a_2 - \Sigma a_3, \quad a_4' = 4k - \Sigma a_1 - a_2 - 2\Sigma a_3,$$
$$c_1' = k - a_3 - b_3, \qquad c_2' = 2k - c - a_2 - \Sigma a_3,$$
$$d_2' = 2k - d_1 - a_2 - \Sigma a_3.$$

For a more general case I take the second pair of dissimilar conjugates for $n=3p$, and obtain in a similar manner

$$k = 3pk' - \Sigma_4 a_1' - 2a_2' - 3\Sigma_{2p-2} a_3' - (3p-3)\, a_{3p-3}',$$
$$k' = 3pk - \Sigma_{2p-2} a_1 - (p-1)\, a_{p-1} - p\Sigma_4 a_p - 2p a_{2p},$$

$$a_1 = k' - a'_3 - a'_{3p-3}, \qquad a_p = pk' - a'_1 - a'_2 - \sum_{2p-2} a'_3 - (p-1)\, a'_{3p-3},$$

$$b_1 = k' - b'_3 - a'_{3p-3}, \qquad b_p = pk' - b'_1 - a'_2 - \sum_{2p-2} a'_3 - (p-1)\, a'_{3p-3},$$

$$\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots$$

$2p-2$ equations,

$$a_{p-1} = (p-1)\, k' - \sum_{2p-2} a'_3 - (p-2)\, a'_{3p-3},$$

4 equations,

$$a_{2p} = 2pk' - \sum_{4} a'_1 - a'_2 - 2 \sum_{2p-2} a'_3 - (2p-2)\, a_{3p-3};$$

where the suffixes under Σ denote the numbers of the letter, which the symbols cover.

And we also have

$$a'_1 = k - a_p - a_{2p}, \qquad a'_3 = 3k - a_1 - a_{p-1} - \sum_{4} a_p - 2a_{2p},$$

$$b'_1 = k - b_p - a_{2p}, \qquad b'_3 = 3k - b_1 - a_{p-1} - \sum_{4} a_p - 2a_{2p},$$

$$\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots$$

4 equations,

$$a'_2 = 2k - \sum_{4} a_p - a_{2p},$$

$2p-2$ equations,

$$a'_{3p-3} = (3p-3)\, k' - \sum_{2p-2} a_1 - (p-2)\, a_{p-1} - (p-1) \sum_{4} a_p - (2p-2)\, a_{2p}.$$

These results might be tabulated in the following manner. The last system gives

a'_1	$k - a_p - a_{2p}$
a'_2	$2k - \sum_4 a_p - a_{2p}$
a'_3	$3k - a_1 - a_{p-1} - \sum_4 a_p - 2a_{2p}$
a_{3p-3}	$(3p-3)\, k - (p-2)\, a_{p-1} - (p-1) \sum_4 a_p - (2p-2)\, a_{2p}$

But, in fact, it is unnecessary to tabulate what is so directly derivable from the Jacobian tables.

It will be observed that if in each of the conjugate systems of principal points there are M groups containing m constituents in all, we have m equations linear in a'_1, &c., to express the values of m quantities, a_1, &c., and *vice versâ*. The one set of equations is derivable from the other, and we need not, in fact, make use of two conjugate tables, since either of them will give the relations sought by solving a linear system.

With an ulterior purpose, I now take an extreme case $k' = 0$, so that $C_{k'}$ represents a point or points.

We shall have in general $k = 0$, since there is a point to point correspondence between P and P′.

But let us suppose that $C_{k'}$ is in fact a constituent of the group a_p. Looking at the manner in which the expressions for k, k', a_p &c., a'_3 &c., are obtained, we see that, retaining the general formulæ, we shall get, with negative signs, the values of a_p &c. for the singular corres-

pondent of a point of the group $a'_{p'}$, when we put $a'_{p'}=1$, and erase the rest of the accented letters.

For in the case of a general curve $C_{k'}$ we have been subtracting for the effect of the singular correspondents of principal points. We now have $k'=0$, while the singular correspondent $C_{p'}$, which in the general case is deemed extraneous, is no longer so. Thus, in the cubic transformation, putting $a'_1=1$, $b_1=\&c.=k'=0$, we get

$$-a_1=1, \quad -a_2=1;$$

indicating that the right line corresponding to a point of the group a'_1 passes once through a point of the group a_1, and once through a point of the group a_2. Since a knowledge of this was assumed in obtaining the formulæ, we do not gain anything directly by this consideration, except that it becomes plain how we can obtain from a given table its conjugate by the solution of a linear system. In addition to this, however, the results show a remarkable symmetry, which may be regarded from a more general point of view.

I observe that the individual constituents of any group a_p have no preeminence among themselves. From this simple consideration, it follows that if a curve $C_{k'}$ in P′ passes through each point of each group of principal points in P′ a number of times, which does not vary for the same group, then the correspondent C_k in P also passes through each point of each group of principal points in P a number of times which does not vary for the same group. And further, since the number "zero" is included, we may express the conclusion thus,—

If a curve $C_{k'}$ in P′ passes through each point of a certain number of groups of principal points in P′, a number of times which does not vary for the same group, and not otherwise, then the correspondent C_k in P passes through each point of the groups of principal points in P, or some of them, a number of times which does not vary for the same group, and not otherwise.

In what precedes, we have supposed that the different groups of principal points are of different orders of multiplicity. I observe further, however, that this supposition is not intrinsically necessary. For if we divide a group (say $a'_{p'}$) into two or more sub-groups, and also divide the equal corresponding group (say a_p) into two or more corresponding equal groups, the conclusion remains unaltered. To illustrate this, consider the case

$$n=4p+2,$$

$$\begin{aligned} a'_1 &= 3, & a_1 &= 2p-1, \\ a'_2 &= 3, & a_p &= 3, \\ a'_4 &= 2p-1, & a_{p+1} &= 1, \\ a'_{4p-2} &= 1, & a_{2p+1} &= 3. \end{aligned}$$

For $p=1$, these become

$$\begin{vmatrix} a'_1=3, & a_1=1 \\ a'_2=3, & a_1=3 \\ a'_4=1, & a_2=1 \\ a'_2=1, & a_3=3 \end{vmatrix} = \begin{vmatrix} a'_1=3, & a_1=4 \\ a'_2=4, & a_2=1 \\ a'_4=1, & a_3=3 \end{vmatrix}$$

And it is evidently immaterial whether we treat this as a case of four groups or a case of three groups.

Hence we may conclude that what has been stated above with reference to a curve passing through the constituents of groups of different orders of multiplicity, is also true when the groups, or some of them, are of the same degree of multiplicity.

It follows, for instance, that if $C_{k'}$ passes an equal number of times through a certain number K of the constituents of $a'_{p'}$, and we will suppose a different but equal number of times through the remaining $a'_{p'}-K$ constituents, and also through each of the points of the other groups an equal number of times not varying for the same group, then the correspondent C_k in P will pass through K of the constituents of the corresponding group a_p an equal number of times, and through the remainder of the group a_p an equal but generally different number of times, and through the other groups a number of times not varying for the same group. For we may divide $a_p = a'_{p'}$ into a group K and a group $a'_{p'}-K$, so that we still have equal corresponding groups. Applying this reasoning still further, we arrive at the following conclusion:—

If $C_{k'}$ *in* P′ *passes an equal number of times through* K *constituents of any group of principal points, then the correspondent* C_k *in* P *passes an equal number of times through* K *constituents of the corresponding equal group.*

And this result is borne out by the formulæ which have been given.

Falling back, then, on groups of different multiplicity, and the formulæ of transformation for that case, we gather that the expressions for a_p, b_p, &c. must contain $a'_{p'}$, $b'_{p'}$, &c. either altogether or not at all; and that they must contain $a'_{p'}$, $b'_{p'}$, &c., relating to the corresponding equal group in such combinations of l letters that

$$\frac{a_p \cdot a_p-1 \ldots a_p-l+1}{1.2.3 \ldots l} = a_p;$$

that is to say, $l=1$ or $l=a_p-1$.

Let us now apply this to the case when $k'=0$, and $C_{k'}$ represents a point of the group $a'_{p'}$. The correspondent will then represent the singular correspondent of a point of the group $a'_{p'}$, except, as has been explained, that the signs will be negative, and this discrepancy of sign can of course be readily removed.

From what has been said, the following conclusion results : –

The numbers (exclusive of indices, according to Prof. Cayley's notation) which enter a table are the numbers of the groups of the principal system or zero, except as to the squares of the table which relate to corresponding equal groups. The numbers in these squares will be either unity, or the number of the corresponding group less 1, *or both of these numbers.*

A difficulty arises when each system contains two or more equal groups. In such a case, I do not know how to determine, *a priori*, the corresponding equal group. It is not difficult, however, practically to determine this. Meanwhile we know that in such cases the numbers in the corresponding square are either zero, the number of the group, unity, or the number of the group less 1. In the case of corresponding equal groups, *both* unity and the number of the group less 1 may occur, with different indices.

Although the indices remain undetermined, a step has been gained, and for values of n of moderate magnitude the tables can be completed without difficulty. The solution is unique when the conjugate systems are given. Otherwise we should have more than one possible correspondent to a principal point.

I have employed this method to complete the tables up to $n=10$, and also for the general forms given by Prof. Cremona.

Since the corresponding equal groups may be so arranged that a diagonal of the table belongs to them (and may be so arranged in various ways), I have adopted this plan so that the numbers on either side of the diagonal are the numbers of the groups. The letters and numbers at the top of the several tables denote the principal groups in one plane; those at the left-hand margin indicate the fundamental groups in the other plane, the correspondents of which pass through the first-mentioned system in the manner indicated by the numbers in the tables. These numbers are the numbers of the points through which the correspondents pass; their indices give the multiplicity.

It is unnecessary to repeat the tables given by Prof. Cayley.

TABLES $n=7$.

	a_1 12	a_6 1
a_1 12	1	1
a_6 1	12	1^5

	a_1 2	a_2 3	a_3 2	a_4 1
a_3 2	1	3	2	1^2
a_2 3		2	2	1
a_1 2			1	1
a_4 1	2	3	2^2	1^2

*

	a_2 3	a_3 4
a_2 3	1	4
a_3 4	3	$1^2,3$

	a_1 5	a_3 3	a_4 1
a_2 5	1	3	1
a_1 3		1	1
a_5 1	5	3^2	1^3

	a_1 3	a_2 5	a_5 1
a_3 3	1	5	1^2
a_1 5		1	1
a_4 1	3	5	1^3

TABLES $n=8$.

	a_1 14	a_7 1
a_1 14	1	1
a_7 1	14	1^6

	a_1 3	a_2 2	a_3 3	a_5 1
a_3 3	1	2	3	1^2
a_2 2		1	3	1
a_1 3			1	1
a_5 1	3	2	3^2	1^3

	a_1 1	a_2 3	a_3 2	a_4 2
a_1 1				2
a_2 3		1	2	2
a_4 2	1	3	$1^2,1$	2^2
a_3 2		3	2	$1^2,1$

	a_3 7
a_3 7	$6,1^2$

	a_1 3	a_2 3	a_4 3
a_4 3	2	3	3^2
a_2 3		2	3
a_1 3			2

	a_1 3	a_2 6	a_6 1
a_4 3	2	6	1^3
a_1 6		1	1
a_3 1		6	1^2

	a_1 6	a_3 1	a_4 3
a_2 6	1	1	3
a_6 1	6	1^2	3^3
a_1 3			2

* It will be observed that this table is sibi-reciprocal only in a restricted sense, since the suffixes of the corresponding groups are not the same. I have noticed this in other cases where there are several equal groups.

TABLES $n=8$, *continued.*

	a_2 5	a_3 2	a_5 1
a_3 5	4	2	1^2
a_1 2		1	1
a_4 1	5	2^2	1^2

	a_1 2	a_3 5	a_4 1
a_3 2	1	5	1^2
a_2 5		4	1
a_5 1	2	5^2	1^2

TABLES $n=9$.

	a_1 16	a_8 1
a_1 16	1	1
a_8 1	16	1^7

	a_1 4	a_2 1	a_3 4	a_6 1
a_3 4	1	1	4	1^2
a_2 1			4	1
a_1 4			1	1
a_6 1	4	4	4^2	1^4

	a_1 2	a_2 3	a_3 1	a_4 2	a_5 1
a_4 2	1	3	1	2^2	1^2
a_2 3		1	1	2	1
a_3 1		3	1	2	1^2
a_1 2				1	1
a_5 1	2	3	1^2	2^2	1^3

	a_2 4	a_4 4
a_2 4	1	4
a_4 4	4	$1, 3^2$

	a_1 3	a_2 7	a_7 1
a_4 3	1	7	1^3
a_1 7		1	1
a_5 1	3	7	1^4

	a_1 7	a_4 3	a_5 1
a_2 7	1	3	1
a_1 3		1	1
a_7 1	7	3^3	1^4

TABLES $n=9$ *continued.*

	a_1 1	a_2 4	a_3 3	a_6 1
a_4 1	1	4	3	1^3
a_3 4		3	3	1^2
a_1 3			1	1
a_5 1	1	4	3^2	1^3

	a_1 3	a_3 4	a_4 1	a_5 1
a_3 3	1	4	1	1^2
a_2 4		3	1	1
a_1 1			1	1
a_6 1	3	4^2	1^3	1^3

	a_2 3	a_3 3	a_4 1	a_5 1
a_3 3	2	3	1	1^2
a_4 3	3	$1^2,2$	1^2	1^2
a_1 1			1	1
a_2 1		3	1	1

	a_1 1	a_2 1	a_3 3	a_4 3
a_4 1	1	1	3	3^2
a_5 1	1	1	3^2	3^2
a_2 3			2	3
a_3 3		1	3	$1^2,2$

TABLES $n=10$.

	a_1 18	a_9 1
a_1 18	1	1
a_9 1	18	1^8

	a_1 5	a_3 5	a_7 1
a_3 5	1	5	1^2
a_1 5		1	1
a_7 1	5	5^2	1^5

	a_1 1	a_2 4	a_4 2	a_5 2
a_1 1				2
a_2 4		1	2	2
a_4 2		4	$1,1^2$	2^2
a_5 2	1	4	2^2	$1^2,1^3$

	a_2 2	a_3 2	a_4 3	a_5 1
a_3 2	1	2	3	1^2
a_2 2		1	3	1
a_4 3	2	2	$2^2,1$	1^2
a_5 1	2	2^2	3^2	1^3

	a_3 7	a_6 1
a_3 7	6	1^2
a_6 1	7^2	1^3

TABLES $n=10$ (*continued*).

	a_1 3	a_2 8	a_3 1
a_5 3	2	8	1^4
a_1 8		1	1
a_4 1		8	1^3

	a_1 8	a_4 1	a_5 3
a_2 8	1	1	3
a_3 1	8	1^3	3^4
a_1 3			2

	a_1 2	a_2 3	a_3 4	a_7 1
a_4 2	1	3	4	1^3
a_3 3		2	4	1^2
a_1 4			1	1
a_6 1	2	3	4^2	1^4

	a_1 4	a_3 3	a_4 2	a_6 1
a_3 4	1	3	2	1^2
a_2 3		2	2	1
a_1 2			1	1
a_7 1	4	3^2	2^3	1^4

	a_1 1	a_2 3	a_3 2	a_4 2	a_6 1
a_2 1			2	2	1
a_3 3		2	2	2	1^2
a_5 2	1	3	$1^2, 1$	2^2	1^3
a_1 2				1	1
a_4 1		3	2	2^2	1^2

	a_1 2	a_2 1	a_3 3	a_4 1	a_5 2
a_4 2	1	1	3	1^2	2^2
a_1 1					2
a_2 3			2	1	1
a_6 1	2	1	3^2	1^2	2^3
a_3 3		1	3	1	$1, 1^2$

	a_1 3	a_2 3	a_4 3	a_6 1
a_5 3	2	3	3^2	1^3
a_2 3		1	3	1
a_1 3			1	1
a_3 1		3	3	1^2

	a_1 3	a_2 3	a_3 1	a_5 3
a_4 3	1	3	1	3^2
a_2 3		1	1	3
a_6 1	3	3	1^2	3^3
a_1 3				2

TABLES $n=10$ (*continued*).

	a_1 3	a_4 6
a_3 3	2	6^2
a_2 6		5

	a_2 6	a_5 3
a_4 6	5	3^2
a_1 3		2

	a_2 1	a_3 5	a_5 2
a_1 1			2
a_4 5	1	$1^2, 4$	2^2
a_3 2		5	$1, 1^2$

	a_1 1	a_3 2	a_4 5
a_2 1			5
a_5 2	1	$1, 1^2$	5^2
a_3 5		2	$1^2, 4$

TABLES $n=3p+1$.

	a_1 $2p-2$	a_p 3	a_{p+1} 2	a_{2p} 1
a_3 $2p-2$	1	3	2	1^2
a_2 3		2	2	1
a_1 2			1	1
a_{3p-2} 1	$2p-2$	3^{p-1}	2^p	1^{2p-2}

	a_1 2	a_3 3	a_3 $2p-2$	a_{3p-2} 1
a_{p+1} 2	1	3	$2p-2$	1^p
a_p 3		2	$2p-2$	1^{p-1}
a_1 $2p-2$			1	1
a_{2p} 1	2	3	$(2p-2)^2$	1^{2p-2}

	a_1 $2p-1$	a_p 5	a_{2p-1} 1
a_3 $2p-1$	1	5	1^2
a_1 5		1	1
a_{3p-2} 1	$2p-1$	5^{p-1}	1^{2p-1}

	a_1 5	a_3 $2p-1$	a_{2p-1} 1
a_p 5	1	$2p-1$	1^{p-1}
a_1 $2p-1$		1	1
a_{2p-1} 1	5	$2p-1$	1^{2p-1}

TABLES $n=3p+2$.

	a_1 3	a_2 2	a_3 $2p-1$	a_{3p-1} 1
a_{p-1} 3	1	2	$2p-1$	1^2
a_p 2		1	$2p-1$	1^{p-1}
a_1 $2p-1$			1	1
a_{2p-1} 1	3	2	$2p-1$	1^{2p-1}

	a_1 $2p-1$	a_p 2	a_{p+1} 3	a_{2p-1} 1
a_3 $2p-1$	1	2	3	1^2
a_2 2		1	3	1
a_1 3			1	1
a_{3p-1} 1	$2p-1$	2^{p-1}	3^p	1^{2p-1}

TABLES $n = 3p+2$ (*continued*).

	α_2 5	α_3 $2p-2$	α_{3p-1} 1
α_{p+1} 5	4	$2p-2$	1^p
α_1 $2p-2$		1	1
α_{2p} 1	5	$(2p-2)^2$	1^{2p-2}

	α_1 $2p-2$	α_{p+1} 5	α_{2p} 1
α_3 $2p-2$	1	5	1^2
α_2 5		4	1
α_{3p-1} 1	$2p-2$	5^p	1^{2p-2}

TABLES $n = 4p$.

	α_1 1	α_2 3	α_3 2	α_4 $2p-3$	α_{4p-4} 1
α_{p+1} 1		3	2	$2p-3$	1^p
α_p 3		1	2	$2p-3$	1^{p-1}
α_{2p} 2	1	3	$1, 1^2$	$(2p-3)^2$	1^{2p-2}
α_1 $2p-3$				1	1
α_{2p-1} 1		3	2	$(2p-3)^2$	1^{2p-3}

	α_1 $2p-3$	α_p 3	α_{p+1} 1	α_{2p-1} 1	α_{2p} 2
α_4 $2p-3$	1	3	1	1^2	2^2
α_2 3		1	1	1	2
α_1 1					2
α_{4p-4} 1	$2p-3$	3^{p-1}	1^p	1^{2p-3}	2^{2p-2}
α_3 2		3	1	1	$1, 1^2$

	α_1 2	α_3 5	α_4 $2p-4$	α_{4p-4} 1
α_{p+1} 2	1	5	$2p-4$	1^p
α_p 5		4	$2p-4$	1^{p-1}
α_1 $2p-4$			1	1
α_{3p-1} 1	2	5^2	$(2p-4)^3$	1^{3p-4}

	α_1 $2p-4$	α_p 5	α_{p+1} 2	α_{3p-1} 1
α_4 $2p-4$	1	5	2	1^3
α_3 5		4	2	1^2
α_1 2			1	1
α_{2p-4} 1	$2p-4$	5^{p-1}	2^p	1^{3p-4}

	α_1 3	α_2 3	α_4 $2p-2$	α_{4p-4} 1
α_{2p} 3	2	3	$(2p-2)^2$	1^{2p-2}
α_p 3		2	$2p-2$	1^{p-1}
α_1 $2p-2$			1	1
α_{p-1} 1			$2p-2$	1^{p-2}

	α_1 $2p-2$	α_{p-1} 1	α_p 3	α_{2p} 3
α_4 $2p-2$	1	1	3	3^2
α_{4p-4} 1	$2p-2$	1^{p-2}	3^{p-1}	3^{2p-2}
α_2 3			2	3
α_1 3				2

TABLES $n=4p$ (*continued.*)

	α_1 6	α_3 1	α_4 $2p-2$	α_{4p-4} 1
α_p 6	1	1	$2p-2$	1^{p-1}
α_{p-1} 1			$2p-2$	1^{p-2}
α_1 $2p-2$			1	1
α_{3p} 1	6	1^2	$(2p-2)^2$	1^{3p-3}

	α_1 $2p-2$	α_{p-1} 1	α_p 6	α_{3p} 1
α_4 $2p-2$	1	1	6	1^3
α_3 1			6	1^2
α_1 6			1	1
α_{4p-4} 1	$2p-2$	1^{p-2}	6^{p-1}	1^{3p-3}

TABLES $n=4p+1$.

	α_2 3	α_3 3	α_4 $2p-3$	α_{4p-3} 1
α_{p+1} 3	2	3	$2p-3$	1^p
α_{2p} 3	3	$1^2,2$	$(2p-3)^2$	1^{2p-2}
α_1 $2p-3$			1	1
α_p 1		3	$2p-3$	1^{p-1}

	α_1 $2p-3$	α_p 1	α_{p+1} 3	α_{2p} 3
α_4 $2p-3$	1	1	3	3^2
α_{4p-3} 1	$2p-3$	1^{p-1}	3^p	3^{2p-2}
α_2 3			2	3
α_3 3		1	3	$1^2,2$

	α_1 2	α_2 2	α_3 1	α_4 $2p-2$	α_{4p-3} 1
α_{2p} 2	1	3	1	$(2p-2)^2$	1^{2p-2}
α_p 3		1	1	$2p-2$	1^{p-1}
α_{p+1} 1		3	1	$2p-2$	1^p
α_1 $2p-2$				1	1
α_{2p+1} 1	2	3	1^2	$(2p-2)^2$	1^{2p-1}

	α_1 $2p-2$	α_p 3	α_{p+1} 1	α_{2p} 2	α_{2p+1} 1
α_4 $2p-2$	1	3	1	2^2	1^2
α_2 3		1	1	2	1
α_3 1		3	1	2	1^2
α_1 2				1	1
α_{4p-3} 1	$2p-2$	3^{p-1}	1^p	2^{2p-2}	1^{2p-1}

TABLES $n=4p+1$ (*continued*).

	α_1 3	α_3 4	α_4 $2p-3$	α_{4p-3} 1
α_{p+1} 3	1	4	$2p-3$	1^p
α_p 4		3	$2p-3$	1^{p-1}
α_1 $2p-3$			1	1
α_{3p} 1	3	4^2	$(2p-3)^3$	1^{3p-3}

	α_1 $2p-3$	α_p 4	α_{p+1} 3	α_{3p} 1
α_4 $2p-3$	1	4	3	1^3
α_3 4		3	3	1^2
α_1 3			1	1
α_{4p-3} 1	$2p-3$	4^{p-1}	3^p	1^{3p-3}

	α_1 7	α_4 $2p-1$	α_{2p-3} 1
α_p 7	1	$2p-1$	1^{p-1}
α_1 $2p-1$		1	1
α_{3p+1} 1	7	$(2p-1)^3$	1^{3p-2}

	α_1 $2p-1$	α_p 7	α_{4p+1} 1
α_4 $2p-1$	1	7	1^3
α_1 7		1	1
α_{4p-3} 1	$2p-1$	7^{p-1}	1^{3p-2}

TABLES $n=4p+2$.

	α_3 7	α_4 $2p-4$	α_{4p-2} 1
α_{p+1} 7	6	$2p-4$	1^p
α_1 $2p-4$		1	1
α_{3p} 1	7^2	$(2p-4)^3$	1^{3p-3}

	α_1 $2p-4$	α_{p+1} 7	α_{3p} 1
α_4 $2p-4$	1	7	1^3
α_3 7		6	1^2
α_{4p-2} 1	$2p-4$	7^p	1^{3p-3}

	α_1 1	α_2 3	α_3 2	α_4 $2p-2$	α_{4p-2} 1
α_p 1			2	$2p-2$	1^{p-1}
α_{p+1} 3		2	2	$2p-2$	1^p
α_{2p+1} 2	1	3	$1,1^2$	$(2p-2)^2$	1^{2p-1}
α_1 $2p-2$				1	1
α_{2p} 1		3	2	$(2p-2)^2$	1^{2p-2}

	α_1 $2p-2$	α_p 1	α_{p+1} 3	α_{2p} 1	α_{2p+1} 2
α_4 $2p-2$	1	1	3	1^2	2^2
α_1 1					2
α_2 3			2	1	2
α_{2p+2} 1	$2p-2$	1^{p-1}	3^p	1^{2p-2}	2^{2p-1}
α_3 2		1	3	1	$1,1^2$

TABLES $n=4p+2$ (*continued*).

	a_1 3	a_2 3	a_4 $2p-1$	a_{4p-2} 1
a_{2p+1} 3	2	3	$(2p-1)^2$	1^{2p-1}
a_p 3		1	$2p-1$	1^{p-1}
a_1 $2p-1$			1	1
a_{p+1} 1		3	$2p-1$	1^p

	a_1 $2p-1$	a_p 3	a_{p+1} 1	a_{2p+1} 3
a_4 $2p-1$	1	3	1	3^2
a_2 3		1	1	3
a_{4p-2} 1	$2p-1$	3^{p-1}	1^p	3^{2p-1}
a_1 3				2

	a_1 4	a_3 3	a_4 $2p-2$	a_{4p-2} 1
a_{p+1} 4	1	3	$2p-2$	1^p
a_p 3		2	$2p-2$	1^{p-1}
a_1 $2p-2$			1	1
a_{3p+1} 1	4	3^2	$(2p-2)^3$	1^{3p-2}

	a_1 $2p-2$	a_p 3	a_{p+1} 4	a_{3p+1} 1
a_4 $2p-2$	1	3	4	1^3
a_3 3		2	4	1^2
a_1 4			1	1
a_{4p-2} 1	$2p-2$	3^{p-1}	4^p	1^{3p-2}

TABLES $n=4p+3$.

	a_2 3	a_3 3	a_4 $2p-2$	a_{4p-1} 1
a_{p+1} 3	1	3	$2p-2$	1^p
a_{2p+1} 3	3	$1^2, 2$	$(2p-2)^2$	1^{2p-1}
a_1 $2p-2$			1	1
a_{p+2} 1	3	3	$2p-2$	1^{p+1}

	a_1 $2p-2$	a_{p+1} 3	a_{p+2} 1	a_{2p+1} 3
a_4 $2p-2$	1	3	1	3^2
a_2 3		1	1	3
a_{4p-1} 1	$2p-2$	3^p	1^{p+1}	3^{2p-1}
a_3 3		3	1	$1^2, 2$

	a_1 1	a_3 6	a_4 $2p-3$	a_{4p-1} 1
a_{p+2} 1	1	6	$2p-3$	1^{p+1}
a_{p+1} 6		5	$2p-3$	1^p
a_1 $2p-3$			1	1
a_{3p+1} 1	1	6^2	$2p-3$	1^{3p-2}

	a_1 $2p-3$	a_{p+1} 6	a_{p+2} 1	a_{3p+1} 1
a_4 $2p-3$	1	6	1	1^3
a_3 6		5	1	1^2
a_1 1			1	1
a_{4p-1} 1	$2p-3$	6^p	1^{p+1}	1^{3p-2}

TABLES $n = 4p+3$ (*continued*).

	α_1 2	α_2 3	α_3 1	α_4 $2p-1$	α_{4p-1} 1
α_{2p+1} 2	1	3	1	$(2p-1)^2$	1^{2p-1}
α_{p+1} 3		2	1	$2p-1$	1^p
α_p 1			1	$2p-1$	1^{p-1}
α_1 $2p-1$				1	1
α_{2p+2} 1	2	3	1^2	$(2p-1)^2$	1^{2p}

	α_1 $2p-1$	α_p 1	α_{p+1} 3	α_{2p+1} 2	α_{2p+2} 1
α_4 $2p-1$	1	1	3	2^2	1^2
α_3 1		1	3	2	1^2
α_2 3			2	2	1
α_1 2				1	1
α_{4p-1} 1	$2p-1$	1^{p-1}	3^p	2^{2p-1}	1^{2p}

	α_1 5	α_3 2	α_4 $2p-1$	α_{4p-1} 1
α_{p+1} 5	1	2	$2p-1$	1^p
α_p 2		1	$2p-1$	1^{p-1}
α_1 $2p-1$			1	1
α_{3p+2} 1	5	2^2	$(2p-1)^2$	1^{3p-1}

	α_1 $2p-1$	α_p 2	α_{p+1} 5	α_{3p+2} 1
α_4 $2p-1$	1	2	5	1^3
α_3 2		1	5	1^2
α_1 5			1	1
α_{4p-1} 1	$2p-1$	2^{p-1}	5^p	1^{3p-1}

On an Algebraical Form, and the Geometry of its Dual Connexion with a Polygon, plane or spherical. By T. COTTERILL, M.A.

[*Read February 8th*, 1872.]

The following is a *résumé* of some of the results and the method employed in a paper on this subject, read before the Society, Feb. 8, 1872. A portion of the MS. having been mislaid, and some improvements having suggested themselves, the whole paper will be given hereafter.

A polygon, plane or spherical, can be denoted in two ways—either by its angular points, or its lines equal in number (say n), taken in a

definite order of sequence. The m lines intersect again in $\frac{m(m-3)}{2}$ points, called polygon points of the curve, or, for shortness, P points, and determine a curve of the order $(m-3)$ passing through them, the order curve of the polygon. In the same manner, the $\frac{m(m-3)}{2}$ diagonals determine the class curve of the polygon. The two figures being correlative, it is sufficient to consider only one case—for instance, the order curve. The lines are denoted by numerals or italics, or both; and when placed within a ( ) in a fixed order, represent a polygon or its corresponding curve. Thus $(123abc) = (23abc1) = \&c.$ is a hexagon with consecutive sides 1, 2, 3, a, b, c, or the unique cubic which passes through the nine intersections of its non-consecutive lines. A symbol, as (a, x), is the point of intersection of the line-pair a and x; such P points are divided into sets, a point of the first set being the intersection of two lines separated by one line, and of the second set when it is separated by two lines, and so on. The minor of a curve of a polygon is the curve obtained by striking out any number of consecutive lines of the original polygon.

Thus (1234) is the right line connecting the P points (1, 3) and (2, 4).

The curve (12345) of a pentagon is the conic through the five P points (1, 3), (2, 4), (3, 5), (4, 1), and (5, 2). It follows, from Pascal's Theorem, that the tangent at a point, as (2, 4), passes through the intersection of the lines 3 and (1425).

Cubics through the P points of the conic (12345), cutting the lines 2, 3, 4 on the line 6, form a faisceau touching the conic at the point (2, 4). The cubic of this system, which passes through the point (1, 5), is the cubic of the hexagon (123456), and touches a minor conic, as (34561), at (4, 6), a point of the first set. The tangent at a point of the second set, as (2, 5), passes through the intersection of the lines (2516) and (2534). It is shown in the analysis that these constructions suffice for determining the tangent at any P point of the curve of a polygon. The three pairs of points (1, 3), (4, 6) and (2, 4), (5, 1) and (3, 5), (6, 2) correspond. in the meaning attached to the words by Prof. Cayley; so that the cubic is the locus of points at which the three pairs (as well as any other pair) of corresponding points subtend angles in involution.

The three P points (6, 4), (4, 1), (1, 5), on four consecutive lines 4, 5, 6, 1, form a triplet such that one conic of the reseau through them is inscribed in the cubic, and any other cuts the cubic again in a triplet of points of the same nature. The opposite of any such triplet and a fourth point on the curve, as d, is d', the corresponding point to d. From these and other properties suggested by the analysis there is no difficulty in tracing the curve.

The nine P points of the cubic are on any four consecutive lines, as 2, 3, 4, 5. Taking the four intersections of these lines with a fresh line 7, we have thirteen points lying on two broken quartics [the cubic (123456) and line 7] and (the four lines 2, 3, 4, 5). Any other quartic of the faisceau through these thirteen points touches the cubic in the three points (4, 2), (2, 5), (5, 3), and amongst them is the quartic of the heptagon (1234567), determined by passing through the point (1, 6), touched by each minor conic in one point and by each minor cubic in three points.

Proceeding in this manner, we see that the $\frac{n(n+3)}{3}$ P points of a polygon of $(n+3)$ lines determine one curve (order n), passing through the $\frac{r(r+3)}{2}$ P points of each minor curve (order r), and touching it in $\frac{r(r-1)}{2}$ of these points.

An inflexion or multiplicity at a P point is obtained by supposing one or more of the lines of the polygon to pass through it, besides the pair which determine it. Other properties are easily obtained geometrically; but the clearest insight into these curves, and their connexion with certain associated curves, is gained from the following theorem in determinants, which gives the quantic of the curve of a polygon referred to its lines in many different forms, as well as those of the associated curves just alluded to :—

Theorem in Determinants.

If a triad of letters in ( ), as (abc), represent the determinant of the third order $\begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix}$, the suffix numbers being the same in each determinant, then

$$\frac{1}{(txy)}\left\{\frac{(tab)}{(axy)(bxy)}+\frac{(tbc)}{(bxy)(cxy)}+\ldots\ldots+\frac{(tmn)}{(mxy)(nxy)}+\frac{(tna)}{(nxy)(axy)}\right\}$$

is independent of t, so that this function $\frac{1}{(txy)}\Sigma\frac{(tab)}{(axy)(bxy)}$ has a constant value, which can be obtained by substituting for t one of the fixed letters $a, b \ldots m, n$, in which case two of the terms disappear. Let this be denoted by I. Then $\mathrm{I}.(txy)=\Sigma\frac{(tab)}{(axy)(bxy)}$.

If I vanish, then $\Sigma\frac{(tab)}{(axy)(bxy)}=0$ identically. But if I is finite, and $\Sigma\frac{(tab)}{(axy)(bxy)}=0$, then $(txy)=0$.

The form $\Sigma\frac{(tab)}{(axy)(bxy)}$ has evidently the same properties as the

expression for the area of a plane polygon in terms of the areas of the triangles and polygons into which it can be divided, and denoted in the usual manner by means of their vertices.

Substituting p for t, it is at once manifest that in the form $\frac{1}{(pxy)}\Sigma\frac{(pab)}{(axy)(bxy)}$ the triads may represent the areas of the triangles formed from the vertices of the polygon and the three points p, x, y. But in order to obtain the order curve of the polygon, we suppose, in the form $\frac{1}{(txy)}\Sigma\frac{(tab)}{(axy)(bxy)}$, the italics to represent lines, and a triad, as (tab), the alternating and therefore cyclical function correlative to twice the area denoted by the product of the sine of any angle of a triangle and its distance from the opposite side. In this case, if an italic by itself be the perpendicular from a point of the plane on the line denoted by the italic, we shall have $\frac{1}{t}\Sigma\frac{(tab)}{a.b}$ independent of the position of t in the plane. Let Π denote the product of the symbols $a, b \ldots m, n$; then $\frac{\Pi}{t}\Sigma\frac{(tab)}{a.b} = \mathrm{U} = (ab \ldots mn)$, the simple multiform quantic of the order curve of the polygon, assuming as many shapes as the polygon can be divided into its own triangles and polygons. Thus, if the lines of the polygon be a single line a and p consecutive lines P, a single line x and q consecutive lines Q, and P and Q by themselves denote the perpendiculars from a point on the consecutive lines, then the identity $(a\mathrm{P}x\mathrm{Q}) = (a\mathrm{P}x).\mathrm{Q} + (x\mathrm{Q}a).\mathrm{P}$ proves the properties of the minors already mentioned, and that two complementary minors $(a\mathrm{P}x)$, $(x\mathrm{Q}a)$ intersect on the curve of the polygon. An important variety of the form is $(ta\mathrm{P}x)$, in which t is variable, forming a reseau through p^2-p+1 points; any two of the curves intersecting again in $(p-1)$ points collinear with (a, x), and determined by two of the curves breaking up and containing curves of the order $(p-1)$. From this form we also find sets of $\frac{p(p-1)}{2}$ points on a curve of a polygon order n, through which a curve order $(p-1)$ can pass touching the primitive at the points; the sets of $\frac{n(n-1)}{2}$ points being such that a curve of the order $(n-1)$ through them cuts the primitive again in $\frac{n(n-1)}{2}$ points of the same nature.

Again, $t.\mathrm{U} = \Pi\Sigma\frac{(tab)}{a.b}$, the quantic of the curve and an arbitrary line. Thus, $t.(a\mathrm{P}x\mathrm{Q}) = (ta\mathrm{P}x).\mathrm{Q} + (tx\mathrm{Q}a).\mathrm{P}$, showing that, if we take a line t, the curves $(ta\mathrm{P}x)$ and $(tx\mathrm{Q}a)$, having the P point (a, x) in common, generate the primitive curve by their intersections.

Geometrical meanings can sometimes be attached to the forms containing more terms.

The fundamental expression is also proved by comparing the area of a triangle with the corresponding correlative function of the polars of its vertices to a conic of the plane, and in a form showing the analogue in any space possessing an absolute. The Author is happy to add that the subject in space has attracted the attention of Prof. Clifford, who has laid the results of his investigations before the Society.

The following presents have been received in the recess:—

"Memorie del Reale Istituto Lombardo," vol. xii. della serie iii., fasc. ii., iii., iv., 1872.

"Rendiconti del R. I. Lombardo," serie ii., vol. iii., fasc. i.—viii. and xvi.—xx.; vol. iv., fasc. i.—xx.; vol. v., fasc. i.—vii.

"Crelle," Band 74, drittes und viertes Heft; Band 75, erstes Heft, 1872.

"Monatsbericht," April, 1872.

"Annali di Matematica," serie ii., tom. v., fasc. ii., May, 1872.

"Bulletin des Sciences," vol. iii., July to October, 1872; and Index to tom. ii.

"Smithsonian Report," 1870. (Washington, 1871.)

"Rappresentazione piana di alcune superficie algebriche dotate di curve cuspidali," nota del Prof. L. Cremona, 1872; and "Le figure reciproche nella statica grafica," L. Cremona, Milano, 1872: both from the Author.

"Le proprietà cardinali degli strumenti ottici anche non centrati," F. Casorati: from the Author.

NINTH SESSION, 1872—3.

November 14th, 1872.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

The Reports of the Treasurer and Secretaries were read and adopted.

From the Report of the Secretaries, it appeared that the number of Members, since the last General Meeting, held November 9th, 1871, had decreased from 114 to 113.

The Obituary of the Society during the Session comprised the names of Mr. E. A. L. B. Smith, Fellow of Christ's College, Cambridge, and Mr. Knowles, A.I.A.

The communications made to the Society during the past Session had been as follows :—

"On the Partition of an Even Number into Two Primes :" by Dr. J. J. Sylvester, F.R.S.

"On the Theorem that an Arithmetical Progression which contains more than one contains an infinite number of Prime Numbers :" by the same.

"On the Surfaces the Loci of the Vertices of the Cones which satisfy Six Conditions :" by Prof. Cayley, F.R.S.

"On the Constants that occur in Certain Summations by Bernoulli's Series :" by Mr. J. W. L. Glaisher.

"On the Parallel Surfaces of Conicoids and Conics :" by Mr. S. Roberts.

"On an Algebraical Form, and the Geometry of its Dual Connexion with a Polygon, plane or spherical :" by Mr. T. Cotterill.

"On a new expression of Invariants and Covariants by means of Alternate Numbers :" by Prof. W. K. Clifford.

"Vibrations in a Sphere :" by Hon. J. W. Strutt.

"On the Mechanical Description of Certain Sextic Curves :" by Prof. Cayley.

"On Functions with Recurring Derivatives :" by Mr. J. W. L. Glaisher.

"On Equations of Motion;" and "On the Transformation of Solids:" by Prof. Clerk Maxwell.

"On the Surfaces divisible into Squares by their Curves of Curvature:" by Prof. Cayley.

"On Prof. Cremona's Transformation between Two Planes, and Tables relating thereto:" by Mr. S. Roberts.

"A Manifold Correspondence of Two Planes:" by Dr. Hirst.

"On the Simultaneous Reduction of Two Polynomial Quadratics to Sums of Squares:" by Prof. Sir W. Thomson.

Shorter communications were made by Profs. Cayley, Clifford, Mr. Roberts, Mr. W. B. Davis, Mr. A. J. Ellis, Prof. Crofton, and Mr. Sprague.

The same Mathematical Journals had been subscribed for as in the preceding Session.

The Proceedings had been sent to the Royal Societies of London and of Edinburgh; the Royal Irish Academy; the Cambridge Philosophical Society; the Philosophical Society of Manchester; the Institute of Actuaries; the British Museum; l'Académie des Sciences, Paris; l'École Polytechnique; la Société Philomathique; Akademie der Wissenschaften, Berlin; il Istituto Lombardo, Milan; the Smithsonian Institution, Washington, U.S.; l'Académie Royale de Belgique; Société des Sciences Physiques et Naturelles de Bordeaux; Society of Natural Sciences of Zurich; Prof. Luigi Cremona, an Editor of the "Annali di Matematica"; Herr Borchardt, Berlin, Editor of "Crelle's Journal"; M. Darboux, Editor of the "Bulletin des Sciences Mathématiques et Astronomiques"; and the Editor of "Nature."

In return for these, the Society had received the Proceedings of the Royal Society; the Journal of the Institute of Actuaries; the Annali di Matematica; the Monatsberichte for 1871, 2, from the Akademie der Wissenschaften; Crelle's Journal, from Herr Borchardt; "Nature," weekly, from the Editor; the Transactions of the Royal Irish Society; Annuaire et Bulletins de l'Académie Royale des Sciences, des Lettres, et des Beaux-Arts de Belgique; Journal de l'École Polytechnique; Report and Proceedings of the Smithsonian Institution, Washington; Memorie and Rendiconti del Reale Istituto Lombardo; Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux; Bulletins des Sciences Mathématiques et Astronomiques; Vierteljahrschrift der Naturforschenden Gesellschaft in Zurich.

The following presents had been made to the Society:—

"Nautical Almanac" for 1875.

Five Memoirs from M. Darboux.

"Theory of Heat," by Prof. Clerk-Maxwell: from the Author.

Nine Memoirs from the Royal University of Norway (p. 10 *supra*).

"Journal of London Institution."

"Vietæ opera mathematica" (1646); and "Ubaldi mecanicorum liber": from Mr. S. M. Drach.

"Geometria rigorosa," by Dr. Cassani; and a Carte de Visite of Herr Riemann: from Herr A. Stein.

First and Second Annual Reports of the Association for the Improvement of Geometrical Teaching.

"On the Induction of Electric Currents in an Infinite Plane Sheet of Uniform Conductivity," by Prof. Clerk-Maxwell: from the Author.

Two papers from the "Quarterly Journal," by the Rev. Prof. Townsend: from the Author. (p. 112 *suprà*.)

Five memoirs from M. Hoüel. (p. 112 *suprà*.)

Six Geometrical Memoirs by Dr. Emil Weyr: from Dr. Hirst. (p. 112 *suprà*.)

"On Cyclides and Sphero-Quartics," by Dr. Casey: from the Author.

"Rappresentazione piana di alcune superficie algebriche dotate di curve cuspidali," nota del Prof. L. Cremona; "Le figure reciproche nella statica grafica," L. Cremona: from the Author.

"Le proprietà cardinali degli strumenti ottici anche non centrati," F. Casorati: from the Author.

After the Reports had been read, and the Rev. R. Harley, F.R.S., elected Auditor, the meeting proceeded to the election of the new Council.

The following gentlemen were declared by the Scrutators (Messrs. Paice and Savage) duly elected:—

President—Dr. Hirst, F.R.S. Vice-Presidents—Prof. Crofton, F.R.S.; Samuel Roberts, Esq., M.A.; W. Spottiswoode, Esq., LL.D., F.R.S. Treasurer—Samuel Roberts, Esq., Vice-President. Secretaries—M. Jenkins, Esq, M.A.; R. Tucker, Esq., M.A. Other Members of the Council—Prof. Cayley, F.R.S.; Prof. Clifford, M.A.; Thos. Cotterill, Esq., M.A.; *J. W. L. Glaisher, Esq., B.A., F.R.A.S.; *Rev. R. Harley, F.R.S.; Prof. Henrici, Dr. Phil.; C. W. Merrifield, Esq., F.R.S.; Prof. H. J. S. Smith, F.R.S.; Jas. Stirling, Esq., M.A.; J. J. Walker, Esq., M.A. (The gentlemen marked * were not on the preceding Council list.)

Dr. Hirst, having taken the chair, expressed in very feeling terms his regret at the loss the Mathematical world and the Society had sustained by the very recent death of Dr. Clebsch, who had been elected a Foreign Member in December last.

Mr. R. Pendlebury, B.A., Fellow of St. John's College, Cambridge, was proposed for election; and Mr. C. R. Hodgson, B.A., was admitted into the Society.

Dr. Spottiswoode stated to the meeting that the application for a Royal Charter had for the present failed; and then proceeded to read

his paper entitled "Remarks on some Recent Generalizations of Algebra." Prof. Clifford and Mr. Harley spoke upon the subject.

Prof. Henrici exhibited a series of models of cubic surfaces, and pointed out their several singularities, and explained how the models were constructed.

Prof. Clifford next read a paper "On a Theorem relating to Polyhedra analogous to Mr. Cotterill's Theorem on Plane Polygons," and exhibited several illustrative models.

A paper by the Hon. J. W. Strutt was, in his absence, taken as read. Its title was, "Investigation of the Disturbance produced by a Spherical Obstacle on the Waves of Sound."

The Secretary read an abstract of M. Hermite's paper, "Sur l'Intégration des Fonctions Circulaires," which had been drawn up by Prof. Cayley.

Mr. Roberts gave an account, in the Author's absence, of Prof. Cayley's communication on "The Mechanical Description of a Cubic Curve."

The following presents were received:—

"Bulletin des Sciences," Nov. 1872.

"Monatsbericht," Mai, Juni, 1872.

"Crelle," 75 Band, 2es Heft.

"Journal of London Institution," No. 17.

"Ueber Systeme von Functionen mehrer Variabeln," von L. Kronecker (from Monatsbericht); two parts, März, Aug., 1869: from the Author.

"On the Law of Facility of Errors of Observations, and on the Method of Least Squares," by J. W. L. Glaisher, B.A., F.R.A.S.: from the Author.

Remarks on some Recent Generalizations of Algebra. By William Spottiswoode, M.A., F.R.S., LL.D.*

The first generalization of Arithmetic was that of ordinary Algebra, which, like its parent, involved only real quantities, and was founded upon a single unit as its basis. The operations of this calculus, however, soon brought the analyst into contact with an expression of which the quantities first contemplated could give no account. The problem so arising, although appearing under many forms, is always reducible to the solution of the equation $x^2+1=0$. Whether this equation admitted of a solution in any intelligible sense, was for a long time doubted;

* Read on vacating the office of President of the Society.

and the expressions occurring in attempted solutions were consequently called "impossible." The importance attached to the possibility or impossibility, as it was called, of a solution is, however, only one instance out of many of an exaggerated estimate of some particular aspect of a question during the progress of a controversy. What meaning can be attached to a root of this equation in the special applications of Algebra to Geometry or Physics, or whether any meaning at all can be given to it, is, after all, a question of only secondary importance to the pure analyst. To him the main question is, In what relation does an expression capable of serving as a root of this equation stand to other and previously known expressions of Algebra? As a first step, it was soon seen that if i were an expression for such a root, it must satisfy the relation $i^2=-1$; and on that account it was said that $\sqrt{-1}$ was a root of the equation in question. In fact, it was considered that the positive and negative square roots of negative unity were the two and only two roots sought. This is perhaps to be regretted, as it tended to fix the attention upon a particular view of the problem of interpretation, and to that extent retarded the general admission of such expressions into analysis, and impeded the progress of generalization. Thus much, however, became clear, that the new quantity i, or $\sqrt{-1}$, was so far distinct in kind from ordinary quantities that, if any equation such as $a+ib=0$ held good, it involved, as a necessary consequence, the two following, viz., $a=0$, $b=0$. The expression $a+ib$ is now, as is very well known, called a complex quantity; and the property above enumerated would now be expressed thus:—that no linear equation, involving one or more complex quantities of the form $a+ib$, can subsist unless the sum of the terms independent of i, and likewise the sum of the terms multiplied by i, simultaneously vanish.

But this is by no means all that is involved in the assumption that $\pm i$ are roots of the equation $x^2+1=0$. For in order that we may have

$$x^2+1=(x-i)(x+i)=0,$$

it is assumed, first, that the development of the products is effected according to the usual laws of algebraical multiplication, viz., that

$$(x-i)(x+i)=x^2+xi-ix-i^2;$$

in other words, that the distributive principle holds good; and secondly, that

$$xi-ix=0;$$

i.e., that the commutative principle holds good also. Lastly, if, as is commonly the case, it be considered that the equation being of the second degree can have only two roots, and that $\pm i$ represent them; then we, in fact, assume that a product can vanish only when one of its factors vanishes.

The consequence which, as mentioned above, was found to flow from

an equation of the form $a+ib=0$ (viz., $a=0, b=0$) proved that no value for i, such as that derived by usual processes, $i=-b:a$, could properly be obtained. And as the new symbol i does not stand in any ordinary quantitative relation to common quantities, it is considered as constituting a new and independent unit in Algebra. And it was of importance to see clearly how far the laws of ordinary Algebra were maintained in respect of the new unit i; because it is in the number of units, and the laws according to which they combine, that Algebras mainly differ.

The recognition of a geometrical interpretation of the symbol i in questions arising out of plane figures naturally led to the enquiry whether a system of similar symbols could not be constructed which should have an analogous application to Geometry in space. But as we are here less concerned with the geometrical than with the algebraical aspect of the subject, I pass on to the consideration of Algebras involving more than one of these new units, and of the subdivisions under which particular systems may be grouped.

The quantities occurring in such Algebras are called Higher Complex Numbers, which may be defined as linear functions of the units, of the form $\alpha+\iota_1\beta+\iota_2\gamma+\ldots$

where $\alpha, \beta, \ldots$ are either real or ordinary complex numbers of the form $a+ib$; and $\iota_1, \iota_2, \ldots$ are the new units. And in all systems hitherto proposed it has been assumed that in the process of addition the commutative and associative principles obtain; in other words, that the addition and subtraction of the units and their combinations are performed as in ordinary Arithmetic or Algebra.

It is consequently in the process of multiplication, *i.e.*, in the laws according to which the multiplication of the units is effected, that different systems of complex numbers are distinguished from one another.

In the majority of systems proposed, the distributive and associative principles have been adopted;* and starting from this basis, a variety of laws of multiplication might be laid down. The following apparently comprise the principal systems now in use.

(1) The Commutative principle might be adopted, so that $\iota_1, \iota_2, \ldots$ being the units, $\iota_1\iota_2=\iota_2\iota_1$; and the actual value of such a product might be the subject of any other arbitrary assumptions. Such an Algebra might be called Commutative.

(2) The Commutative principle being suspended, the following relation might be adopted: $\iota_1\iota_2=-\iota_2\iota_1$, expressive of what might be called the Alternative principle. Quantities involving units so related have been termed Alternate Numbers by Grassmann and by Hankel.

* Exceptions will be found in H. Scheffler's "Situationscalcul," 1851, and Kirkman and Cayley on "Pluquaternions, &c.," Phil. Mag., Dec. 1848, and March 1845.

(3) In connection with either (1) or (2), or independently of either of them, it might be assumed that the product of any two of the units could be expressed as a linear function of some or all of the units; thus,

$$\iota_1\iota_2 = \alpha + i_1\beta + \iota_2\gamma + \dots$$

A particular case of this would be that occurring in quaternions, viz., $jk=i$, $ki=j$, $ij=k$. Such an algebra might be called a Linear Algebra. The systems proposed by Prof. Peirce are of this nature, and, as they all involve the associative principle, they have been in consequence called by him Linear Associative Algebras.

(4) It may, or may not, be assumed that a product shall vary with each of its factors. And it is very important that it should be clearly understood whether this assumption is made or not, because upon it depends whether a product vanishes with each of its factors, and also whether the process of division shall be a definite one or not.

It will perhaps be worth while to examine some of the consequences which would follow from these assumptions.

If we were to adopt the general form of (3), and add to it the assumption (4), we should, by selecting the products which involve one of the units, say ι_1, obtain a system of equations of the following form:

$$\begin{aligned} \iota_1\iota_1 &= a_{11} + a'_{11}\iota_1 + a''_{11}\iota_2 + \dots \\ \iota_1\iota_2 &= a_{12} + a'_{12}\iota_1 + a''_{12}\iota_2 + \dots \\ &\dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

where the coefficients are real, or ordinary complex numbers. Eliminating, as we are able to do in virtue of (4), the $(n-1)$ quantities $\iota_2, \iota_3, \dots \iota_n$, from these n equations, we obtain

$$\begin{vmatrix} a_{11} + a'_{11}\iota_1 - \iota_1^2, & a''_{11} \dots \\ a_{12} + a'_{12}\iota_1 \quad , & a''_{12} - \iota_1 \dots \\ \dots \quad \dots & \dots \quad \dots \end{vmatrix} = 0;$$

which, when developed, would be of the form

$$(1, b, b_1, \dots)(\iota_1, 1)^{n+1} = 0 = (\iota_1 - c_1)(\iota_2 - c_2) \dots,$$

where $c_1, c_2, \dots$ are real or ordinary complex numbers. But, the principle (4) being supposed to obtain, this equation can be satisfied only by the vanishing of one of its factors, say $\iota_1 = c_1$. In other words, ι_1 is reduced to a real or ordinary complex number, which is contrary to the hypothesis.

It appears, therefore, that in any system wherein the products and powers of the units are themselves linear functions of the units, or *limited system* as it may be called, the assumption (4) must necessarily be excluded.

Of such a system quaternions are an example. For a quaternion may be expressed in the form

$$q = a_0 + a_1\iota_1 + a_2\iota_2 + a_3\iota_3,$$

where ι_1, ι_2, ι_3 are "imaginary square roots of -1," or "units" in the same sense as that in which that term was used above. These satisfy the equations

$$\iota_2\iota_3 = -\iota_3\iota_2 = \iota_1, \quad \iota_1^2 = -1,$$
$$\iota_3\iota_1 = -\iota_1\iota_3 = \iota_2, \quad \iota_2^2 = -1,$$
$$\iota_1\iota_2 = -\iota_2\iota_1 = \iota_3, \quad \iota_3^2 = -1.$$

The system is therefore a limited one. But if q_1 be another quaternion it does not follow from such an equation as $qq_1 = 0$, that either $q = 0$, or $q_1 = 0$. In other words, the assumption (4) is not made. Thus if q be a quaternion and a an ordinary complex number, and if $q^2 = a$; then since the units ι_1, ι_2, ι_3 are commutative with the ordinary unit $i\,(=\sqrt{-1})$, we have

$$q^2 - a = (q+\sqrt{a})(q-\sqrt{a}) = 0,$$

although we cannot conclude that either of the factors separately vanish. Or taking a more special case, the following relations are evidently satisfied $\quad (\iota_1 - i)(\iota_1 + i) = \iota_1^2 + \iota_1 i - i\iota_1 - i^2 = \iota_1^2 - i^2 = 0,$

although it would be contrary to the foundation of the calculus to conclude that $\iota_1 = \pm i$.

These considerations will materially affect our notions of the solution of equations involving higher complex numbers. Suppose, for instance, that in a system with two units, 1 and ι, we have an equation in $\xi = x + \iota b$, say of the form

$$(a + \iota b, \ \ldots \ a_n + \iota b_n)(\xi, 1)^n = 0;$$

then we know that it will have, at all events, the same roots as the corresponding equation in ordinary complex numbers, say

$$(a + ib, \ \ldots \ a_n + ib_n)(x + iy)^n = 0.$$

The latter equation, when developed, takes the form $\mathrm{U} + i\mathrm{V} = 0$; and the two equations which this involves, viz. $\mathrm{U} = 0$, $\mathrm{V} = 0$, being each of the degree n in x and y, give rise apparently to n^2 pairs of roots x_1, y_1; $\ldots\ x_{n^2}, y_{n^2}$. But of these n are real, and the remainder, when combined in the form $x + iy$, give only n distinct roots, because the remaining $n(n-1)$ turn out to be only repetitions of the first n. But when the n^2 pairs of roots are combined by means of the unit ι (different from i) in the form $x + \iota y$, there are in general no repetitions; and the equation in ξ has in fact n^2 roots.

Thus, for example, the equation $(x + \iota y)^2 = 1$

has the 4 roots $\qquad +1, \quad -1, \quad +i\iota, \quad -i\iota.$

The equation $\qquad (x + \iota y)^3 = 1,$

if we write for brevity

$$a = \frac{-1 + i\sqrt{3}}{2}, \quad a = \frac{-1 + \iota\sqrt{3}}{2},$$

has the 9 roots $1,\ a,\ a^2,\ \alpha,\ \alpha^2,\ a\alpha,\ a^2\alpha,\ \alpha a^2,\ a^2\alpha^2$;

and the equation $(x+\iota y)^4=1$

the 16 roots $\pm 1,\ \pm i,\ \pm\iota,\ \pm i\iota,\ \pm\tfrac{1}{2}(1\pm i)(1\pm\iota)$.

Similarly a quadratic equation in quaternions

$$\alpha q^2+\beta q+\gamma=0,$$

where $q=x_0+x_1\iota_1+x_2\iota_2+x_3\iota_3$, would have $2^4=16$ roots; namely those derived from the four equations in x_0, x_1, x_2, x_3, obtained by equating to zero the coefficients of ι, ι_1, ι_2, ι_3, in the developed form of $\alpha q^2+\beta q+\gamma$. A cubic would have $3^4=81$ roots; and likewise in general an equation of the degree n in quaternions would have n^4 roots.

But beside the roots above indicated, and which may perhaps be called roots proper, equations in higher complex numbers have an infinite number of roots which arise from the fact that the process of division is not a definite one. In fact if ξ be any root of the equation $\xi^2=0$, then the equation $\xi\eta=\zeta$ is satisfied simultaneously with the equation $\xi(\eta+a\xi)=\zeta$, where a is any ordinary complex number.

Similarly in quaternions the equation $q^2=a$ may in this sense have an infinite number of roots, if q be of the form $x_1\iota_1+x_2\iota_2+x_3\iota_3$; viz., it may have as the elements of its roots all those values of x_1, x_2, x_3, which satisfy the equation $x_1^2+x_2^2+x_3^2=-a$.

This is the property to which allusion was made in saying that the process of division was not a definite one, when the product of two quantities could vanish, even though neither of its products separately vanished.

It is worthy of remark that a complex system, even though unlimited (*i.e.*, when the products of its units are all different, and not in any way related to the original units), if subject to the commutative law, must satisfy the condition (4); viz., if α, β be two such complex quantities, then if $\alpha\beta=0$, and α does not vanish, β must vanish. For if α, β be linear functions of the units ι_1, ι_2, ..., and of their products and powers arranged according to their degrees, say

$$\alpha=a_1\kappa_1+a_2\kappa_2+\ldots,$$
$$\beta=b_1\lambda_1+b_2\lambda_2+\ldots;$$

then $$\alpha\beta=\Sigma a_m b_n \kappa_m\lambda_n=0.$$

And as the products are all independent, we must have

$$\Sigma a_m b_n=0.$$

And beginning with the first we have $a_1b_1=0$. But as a_1 does not vanish, the above equation gives $b_1=0$; this again gives $a_1b_2=0$, *i.e.* $b_2=0$; and so on. In other words $\beta=0$.

The *Linear Associative Algebras* of Prof. Peirce are systems of higher complex numbers, so called because the associative as well as the distributive principles in multiplication obtain throughout. The com-

mutative, as might be anticipated, does not. He defines *Pure Algebras*, which are in fact the main subject of his investigations, to be those in which every unit is connected by a definite relation with every other unit; and *Linear Algebras* to be those in which every expression may be reduced to a linear function of some or of all the units; or, in other words, where every expression may be reduced to the form $\Sigma a\iota$, where a is a quantitative coefficient, and ι a unit. His Algebras are consequently limited systems; and the products of expressions therefore do not vanish with their factors. In particular, ι_1, ι_2 being two such units, we may write

$$\begin{aligned} \iota_1\iota_2 &= a_{12}\iota_1 + b_{12}\iota_2 + \ldots\ldots, \\ \iota_2\iota_1 &= a_{21}\iota_1 + b_{21}\iota_2 + \ldots\ldots, \\ \iota_1^2 &= a_{11}\iota_1 + b_{11}\iota_2 + \ldots\ldots, \end{aligned}$$

and the problem proposed is the determination of the values of the coefficients which correspond to every variety of linear Algebra. The resulting products can be arranged in a tabular form, or multiplication table, which defines the character of the Algebra.

The following are definitions of the terms principally used. If

$\iota_1\iota_2 = 0$, ι_1 is a nil-facient, $\iota_1\iota_2 = \iota_2$, ι_1 is an idem-facient,
or, ι_2 is a nil-faciend, $\iota_1\iota_2 = \iota_1$, ι_2 is an idem-faciend,
$\iota_1^2 = 0$, ι_1 is a nil-potent, $\iota_1^2 = \iota_1$, ι_1 is an idem-potent.

This being so, it is a fundamental proposition that in every such Algebra there is at least one idem-potent or one nil-potent expression. For, if A be any expression, then in general $A^2, A^3 \ldots\ldots$ are independent of A. But since there cannot be more powers of A independent of A than there are independent units in the Algebra, it follows that there must be some least power m, which is dependent upon lower powers. And further, since the Algebra is by definition linear, the power in question, A^m, may be expressed as a linear function of these lower powers, without an independent term; viz., a relation of the form

$$(a_1, a_2, \ldots\ldots a_m)(A, 1)^{m-1} A = 0;$$

or say $BA=0$, will subsist. Multiplying this equation by $a_1 A^{m-2}$, $a_2 A^{m-3}$, $\ldots a_{m-1}$ successively, and adding, we obtain

$$B(B - a_m) = 0;$$

or, dividing throughout by a_m^2.

$$\left(\frac{B}{a_m}\right)^2 = \left(\frac{B}{a_m}\right);$$

in other words, $\frac{B}{a_m}$ is in general an idem-potent expression; unless $a_m = 0$, in which case $B^2 = 0$; *i.e.*, B is a nil-potent expression, which proves the proposition.

In the former case, $\frac{B}{a_m}$ may be taken as one of the units, and called

the basis. And by a proper selection, the remaining units may be arranged in four distinct groups with reference to the basis, as follows:

Group I., in which the units are idem-factors, say *dd*.
Group II., in which the units are idem-faciend and nil-facient, say *dn*.
Group III., in which the units are idem-facient and nil-faciend, say *nd*.
Group IV., in which the units are nil-factors, say *nn*.

The possibility of such a selection is easily shown; for if ι be the basis and A any other expression, and if $\iota A = B$, then

$$\iota B = \iota^2 A = \iota A = B,$$

and consequently $\iota (A - B) = \iota A - \iota B = B - B = 0$.

Hence A has been divided into two parts, whereof one, B, is idem-faciend, and the other A−B is nil-faciend; which was to be done. A similar process will serve to effect a sub-division of these into idem-facient and nil-facient expressions, and thus to complete the arrangement of all the units under the four heads above specified.

This having been effected, the mode in which the units of the various groups combine is given in the following multiplication table:—

	dd	*dn*	*nd*	*nn*
dd	*dd*	*dn*	0	0
dn	0	0	*dd*	*dn*
nd	*nd*	*nn*	0	0
nn	0	0	*nd*	*nn*

From this table certain general conclusions may be drawn.

(1.) That every expression which belongs to the second or third groups is nil-potent.

(2.) That all commutative products which do not vanish are restricted to the first and fourth groups.

(3.) That every continuous product which does not vanish has the same faciend name as the first facient, and the same facient name as the last faciend.

(4.) That the units of the first and fourth groups cannot be connected by multiplication, since their products vanish. They can be connected therefore only by means of the second and fourth groups. Hence, in order that such a connexion may exist, or in other words, in order that there may be a pure Algebra, it is necessary that there should be units in the second and fourth groups.

When the first group contains any units beside the basis, there is

always another expression which is either idem-potent or nil-potent. For, proceeding as before, and taking any expression A, there will be always some mth power of A which is expressible in terms of its lower powers and of the base ι; say thus,

$$(a_1, a_2, \ldots a_m, b\iota)\ (\mathrm{A}, 1)^m = 0.$$

If, then, we substitute $\mathrm{A} = \mathrm{A}_1 + hc$, and determine h by means of the ordinary algebraical equation

$$(a_1, a_2, \ldots a_m, b\iota)\ (h, 1)^m = 0,$$

we shall obtain a relation between the powers of A without an independent term. And from this, an idem-potent or a nil-potent expression B can be obtained as before.

When there is a second idem-potent unit j in the first group, the base can be so changed as to free the group from it. Thus, if ι_1 and ι_2 be two such idem-potent units, and if $\iota_1' = \iota_1 - \iota_2$ be taken as the new base, we shall obtain

$$\iota_1'^2 = (\iota_1 - \iota_2)^2 = \iota_1^2 - 2\iota_1\iota_2 + \iota_2^2 = \iota_1 - 2\iota_2 + \iota_2 = \iota_1 - \iota_2 = \iota_1',$$

so that ι_1' is idem-potent. Moreover,

$$\iota_1'\iota_2 = (\iota_1 - \iota_2)\,\iota_2 = \iota_1\iota_2 - \iota_2^2 = \iota_2 - \iota_2 = 0,$$
$$\iota_2\iota_1' = \iota_2\,(\iota_1 - \iota_2) = \iota_2\iota_1 - \iota_2^2 = \iota_2 - \iota_2 = 0;$$

that is to say, ι_2 is a nil-factor with respect to the new base; in other words, it passes into the fourth group.

From this it results that each group can be so reduced as to contain not more than one idem-potent unit. And in the groups which stand second and third, there will be only nil-potent expressions.

It may then be further shown that, in a group or Algebra which has no idem-potent expression, all the expressions are nil-potent. Such an Algebra is called a nil-potent Algebra.

When an expression is nil-potent, all its powers which do not vanish are mutually independent.

These last propositions depend upon the fact that no relation such as $(a_1, a_2, \ldots a_m)\ (\mathrm{A}, 1)^m = 0$ can subsist without giving rise to an idem-potent expression, which is contrary to the supposition.

In a nil-potent Algebra, any nil-potent expression may be taken as the base; and if that which has the greatest number of non-vanishing powers (say A) be chosen, that number is called the order of the group or Algebra. After the selection of the base, another expression independent of the base may be selected; and this, together with as many of its powers as are independent of the base, may be taken as new units; and so on, until the series is completed.

Every such expression A, whether taken as the base or not, has some least power which is nil-facient with reference to any other expression B: that power is called the facient order of B (relatively to A, if A be

not the base). And, *mutatis mutandis*, we have a definition of the facient order of B.

In every nil-potent group, the facient order of any unit which is independent of the base, can be assumed to be as low as the number of units which are independent of the base.

In a nil-potent group, of which the order is less by unity than the number of units contained in it, the unit which is independent of the base and its powers may be so selected, that its product into the base shall be equal to the highest power of the base which does not vanish; and its square shall either vanish, or shall be equal to the beforementioned power of the base. Or expressed algebraically, if ι_1, ι_2 be such units, we can make $\iota_1\iota_2 = 0$, $\iota_2\iota_1 = \iota^n = \iota_2^2$.

All the products of the expressions in the first group of an idem-potent Algebra are independent of the base.

An idem-potent Algebra can be converted into a nil-potent by taking away the idem-potent base. And, conversely, a nil-potent Algebra can be converted into an idem-potent by the addition of an idem-potent base.

By the aid of these propositions, and of some details consequent upon them, Prof. Peirce proceeds to calculate the multiplication tables of all possible varieties of pure Algebras, and to indicate the cases in which no pure Algebras are possible, for all degrees of multiplicity up to six. It would not be to the present purpose to enter into the details, for which reference must be made to the work itself; but we may take as an example a certain quadruple Algebra which, as the author states, is a form of quaternions. In this case, there is supposed to be an idem-potent base i, so that $i^2 = i$. Of the other units (j, k, l), j is supposed to belong to the second, k to the third, and l to the fourth group. This is consequently a normal case, and its multiplication table is in fact a transcript of the general multiplication table given above, viz.,

	i	j	k	l
i	i	j	0	0
j	0	0	i	j
k	k	l	0	0
l	0	0	k	l

As an illustration of the subject, I have thought it worth while to verify by the ordinary method the statement that this is a form of quaternions.

Now, generally speaking, if q, q_1, q_2 be three quaternions,

$$\begin{aligned} q &= w + ix + jy + kz, \\ q_1 &= w_1 + ix_1 + jy_1 + kz_1, \\ q_2 &= w_2 + ix_2 + jy_2 + kz_2 ; \end{aligned}$$

then the condition $q^2=0$ gives

$$w^2+x^2-y^2-z^2+2w(ix+jy+kz) = 0 ;$$

that is to say, $$x^2+y^2+z^2 = 0,\ w=0.$$

Again, the condition $q^2=q$ gives

$$w^2-x^2-y^2-z^2 = w,\ w=\tfrac{1}{2} ;$$

that is to say, $$x^2+y^2+z^2 = -\tfrac{1}{4},\ w=\tfrac{1}{2}.$$

Again, the condition $qq_2=q_1$ gives

$$\begin{aligned} & ww_1-xx_1-yy_1-zz_1 \\ & +i\ (yz_1-y_1z+wx_1+w_1x) \\ & +j\ (zx_1-z_1x+wy_1+w_1y) \\ & +k\ (xy_1-x_1y+wz_1+w_1z) \\ & = w_1+ix_1+jy_1+kz_1 ; \end{aligned}$$

whence, equating coefficients of the independent units, we obtain

$$\begin{aligned} (w-1)\,x_1-zy_1+yz_1+xw_1 &= 0, \\ zx_1+(w-1)\,y_1-xz_1+yw_1 &= 0, \\ -yx_1+xy_1+(w-1)\,z_1+zw_1 &= 0, \\ -xx_1-yy_1-zz_1+(w-1)\,w_1 &= 0. \end{aligned}$$

Or, eliminating x_1, y_1, z_1, w_1,

$$\begin{vmatrix} w-1 & -z & y & x \\ z & w-1 & -x & y \\ -y & x & w-1 & z \\ -x & -y & -z & w-1 \end{vmatrix} = 0$$

$$= \frac{1}{w-1}\{x^2+y^2+z^2+(w-1)^2\} \begin{vmatrix} w-1 & -z & y \\ z & w-1 & -x \\ -y & x & w-1 \end{vmatrix}$$

$$= \{x^2+y^2+z^2+(w-1)^2\}.$$

And a result of the same form would be deduced from the condition $q_1q=q$.

Lastly, the condition $q_1q_2=q_3$ gives

$$\begin{aligned} w_1x_2-z_1y_2+y_1z_2+x_1w_2 &= x_3, \\ z_1x_2+w_1y_2-x_1z_2+y_1w_2 &= y_3, \\ -y_1x_2+x_1y_2+w_1z_2+z_1w_2 &= z_3, \\ -x_1x_2+y_1y_2-z_1z_2+w_1w_2 &= w_3. \end{aligned}$$

Substituting q, q_1, q_2, q_3, for i, j, k, l, respectively, in the multiplication

table above written, we shall find that the following conditions must be fulfilled, viz.,

$$
\begin{aligned}
q^2 = q \quad &\text{or} \quad x^2+y^2+z^2 = -\tfrac{1}{4},\ w = \tfrac{1}{2},\\
qq_1 = q_1 \quad &,, \quad x^2+y^2+z^2+(w-1)^2 = 0,\\
qq_2 = 0 \quad &,, \quad x^2+y^2+z^2+w^2 = 0,\\
qq_3 = 0 \quad &,, \quad x^2+y^2+z^2+w^2 = 0,\\
q_1q = 0 \quad &,, \quad x^2+y^2+z^2+w^2 = 0,\\
q_1^2 = 0 \quad &,, \quad x_1^2+y_1^2+z_1^2 = 0, \quad w_1 = 0,\\
q_1q_2 = q \quad &,, \quad y_1z_2-y_2z_1 = x,\\
&\qquad\quad z_1x_2-z_2x_1 = y,\\
&\qquad\quad x_1y_2-x_2y_1 = z,\\
&\qquad\quad x_1x_2+y_1y_2+z_1z_2 = -w,\\
q_1q_3 = q_1 \quad &,, \quad x_3^2+y_3^2+z_3^2+(w_3-1)^2 = 0,\\
q_2q = q_2 \quad &,, \quad x^2+y^2+z^2+(w-1)^2 = 0,\\
q_2q_1 = q_3 \quad &,, \quad y_2z_1+y_1z_2 = x_3,\\
&\qquad\quad z_2x_1+z_1x_2 = y_3,\\
&\qquad\quad x_2y_1+x_1y_2 = z_3,\\
&\qquad\quad x_1x_2+y_1y_2+z_1z_2 = -w_3,\\
q_2^2 = 0 \quad &,, \quad x_2^2+y_2^2+z_2^2 = 0, \quad w_2 = 0,\\
q_2q_3 = 0 \quad &,, \quad x_2^2+y_2^2+z_2^2+w_2^2 = 0,\\
q_3q = 0 \quad &,, \quad x^2+y^2+z^2+w^2 = 0,\\
q_3q_1 = 0 \quad &,, \quad x_1^2+y_1^2+z_1^2+w_1^2 = 0,\\
q_3q_2 = q_2 \quad &,, \quad x_3^2+y_3^2+z_3^2+(w_3-1)^2 = 0,\\
q_3^2 = q_3 \quad &,, \quad x_3^2+y_3^2+z_3^2 = -\tfrac{1}{4}, \quad w_3 = \tfrac{1}{2}.
\end{aligned}
$$

All these conditions are comprised in the following group, viz.:

$$
\begin{aligned}
x^2+y^2+z^2 &= -\tfrac{1}{4}, \quad w = \tfrac{1}{2},\\
x_1^2+y_1^2+z_1^2 &= 0, \qquad w_1 = 0,\\
x_2^2+y_2^2+z_2^2 &= 0, \qquad w_2 = 0,\\
x_3^2+y_3^2+z_3^2 &= -\tfrac{1}{4}, \quad w_3 = \tfrac{1}{2};
\end{aligned}
$$

$$
\begin{aligned}
y_1z_2-y_2z_1 = x &= -x_3,\\
z_1x_2-z_2x_1 = y &= -y_3,\\
x_1y_2-x_2y_1 = z &= -z_3,\\
x_1x_2+y_1y_2+z_1z_2 = -w &= -w_3.
\end{aligned}
$$

In order to complete the problem, it remains only to determine x_1, y_1, z_1; x_2, y_2, z_2, in terms of x, y, z, w. With a view to this, an obvious combination of the foregoing equations gives rise to the following sets:

$$
\begin{aligned}
yz_1-y_1z-wx_1 &= 0, \qquad yz_2-y_2z+wx_2 = 0,\\
zx_1-z_1x-wy_1 &= 0, \qquad zx_2-z_2x+wy_2 = 0,\\
xy_1-x_1y-wz_1 &= 0, \qquad xy_2-x_2y+wz_2 = 0;
\end{aligned}
$$

from the first set of which we obtain, taking them two and two in a cyclic order, the following solutions:

$$\begin{aligned} x_1 : y_1 : z_1 = \; & x^2+w^2 \; : xy+wz : xz-wy \\ = \; & yx-wz : \; y^2+w^2 : yz+wz \\ = \; & zx+yw : zy-xw : \; z^2+w^2 ; \end{aligned}$$

or writing for brevity $x+y+z=s$, and adding these together, we obtain the symmetrical solution

$$\begin{aligned} x_1 : y_1 : z_1 = \; & sx+w(y-z+w) \\ & : sy+w(z-x+w) \\ & : sz+w(x-y+w), \end{aligned}$$

and similarly we should find

$$\begin{aligned} x_2 : y_2 : z_2 = \; & sx-w(y-z+w) \\ & : sy-w(z-x+w) \\ & : sz-w(x-y+w) ; \end{aligned}$$

so that finally, if we write for brevity

$$q_0 = ix+jy+kz, \quad \sigma = i+j+k,$$

the quaternions

$$\begin{aligned} q \; &= \tfrac{1}{2} + \; q_0, \\ q_1 = \; & \; sq_0 + \tfrac{1}{2}[i(y-z)+j(z-x)+k(x-y)] + \tfrac{1}{4}\sigma, \\ q_2 = \; & \; sq_0 + \tfrac{1}{2}[i(y-z)+j(z-x)+k(x-y)] - \tfrac{1}{4}\sigma, \\ q_3 &= \tfrac{1}{2} - \; q_0, \end{aligned}$$

will satisfy the multiplication table proposed; that is to say, the particular quadruple Algebra in question is a form of quaternions, which was to be proved.

Grassman and Hankel's Alternate Numbers.

I have already alluded to the principle of alternate numbers; but they appear to me to have such important applications in modern Algebra, that I venture to subjoin a few particulars as to their use.

Alternate numbers form a system of higher complex numbers, subject to the distributive and associative laws; but they are neither limited, nor linear, in the sense of those terms as used by Hankel, and by Peirce. The definition of their multiplication is very simple; viz., if $\iota_1, \iota_2, \ldots$ be the units of the system, then

$$\iota_1^2 = 0, \quad \iota_2^2 = 0, \quad \ldots \quad \iota_1\iota_2+\iota_2\iota_1 = 0, \quad \ldots$$

From this it follows at once, that if

$$\begin{aligned} \alpha &= a_1\iota_1 + a_2\iota_2 + \ldots, \\ \beta &= b_1\iota_1 + b_2\iota_2 + \ldots, \end{aligned}$$

then

$$\alpha\beta = (a_1b_2-a_2b_1)\,\iota_1\iota_2 + (a_1b_3-a_3b_1)\,\iota_1\iota_3 + \ldots + (a_2b_3-a_3b_2)\,\iota_2\iota_3 + \ldots,$$

and consequently

$$\alpha^2 = 0, \quad \beta^2 = 0, \quad \ldots \quad \alpha\beta + \beta\alpha = 0, \quad \ldots$$

In other words, the sign of a product changes with the order of any two of its consecutive factors. And generally, any linear functions of the units combine according to the same laws as the units themselves.

It will, moreover, be found, either by actual inspection of the result obtained above, or by reference to the general principles laid down in an earlier part of this paper, that a product of alternate numbers may vanish, although none of its factors vanish; and also that the process of division is not a definite one.

It further follows from the law of multiplication that if the number of factors $\alpha, \beta, \ldots$ be equal to the number of units, then

$$\alpha\beta \ldots = \begin{vmatrix} a_1 a_2 \ldots \\ b_1 b_2 \ldots \\ \ldots \end{vmatrix} \iota_1 \iota_2 \ldots ;$$

that is to say, a determinant may, by means of these alternate numbers, be expressed in the form of a product. If the number of factors exceeds that of the units, it is clear that in each term of products of units some one or more of the units must occur more than once; and consequently that each term of the product will contain the square or higher power of one of the units, and will necessarily vanish.

The ordinary theorems in determinants are by this method proved with remarkable facility. Several instances are given by Hankel in his work; and others will doubtless be found to fall in with equal readiness to the method. But in order to avoid unnecessary prolixity, I pass on to another part of the subject.

Hankel intimated in his work that he had in prospect the publication of some applications of the theory of alternate numbers to quantics; but nothing on this subject has yet appeared. In the meantime, Prof. Clifford has communicated to this Society the results of some researches of his own in the same direction. The following notes have been based upon his suggestions, and probably will be found to agree in the main with results that he will have independently obtained.

If in the quantic $(a, b, c)(x, 1)^2$ we substitute for one power of $(x, 1)$ the alternate pair (ι_1, ι_2), we obtain

$$(a, b, c)(x, 1)(\iota_1, \iota_2) = u\iota_1 + v\iota_2, \text{ suppose.}$$

Now by the properties of alternate numbers $(u\iota_1 + v\iota_2)^2 = 0$; that is, $[(a, b, c)(x, 1)(\iota_1, \iota_2)]^2 = 0$. But if $(a_1, b_1, c_1)(x, 1)^2$ be another quantic, we may form, in the same way as before, the expression

$$(a_1, b_1, c_1)(x, 1)(\iota_1, \iota_2) = u_1\iota_1 + v_1\iota_2, \text{ suppose;}$$

and then

$$\begin{aligned}&[(a, b, c)\,(x, 1)\,(\iota_1, \iota_2)]\,[(a_1, b_1, c_1)\,(x, 1)\,(\iota_1, \iota_2)]\\ &= (u\iota_1 + v\iota_2)\,(u_1\iota_1 + v_1\iota_2) = \iota_1\iota_2\,(uv_1 - u_1v)\\ &= \iota_1\iota_2 \begin{vmatrix} a\,x+b, & b\,x+c \\ a_1x+b_1, & b_1x+c_1 \end{vmatrix} = \iota_1\iota_2 \begin{pmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \end{pmatrix} (x, 1)^2;\end{aligned}$$

that is to say, the product of $(a, b, c)(x, 1)(\iota_1, \iota_2)$ into $(a_1, b_1, c_1)(x, 1)(\iota_1, \iota_2)$ is equal to the product of $\iota_1\,\iota_2$ into the covariant of the original quantics.

Suppose that we now replace the two factors of $(a, b, c)\,(x, 1)^2$ by the alternate pairs (ι_1, ι_2), (ι_3, ι_4) respectively, we obtain the form $(a, b, c)\,(\iota_1, \iota_2)\,(\iota_3, \iota_4)$; and the square of it gives

$$\begin{aligned}&[(a, b, c)\,(\iota_1, \iota_2)\,(\iota_3, \iota_4)]^2\\ &= [a\iota_1\iota_2 + b\,(\iota_1\iota_4 + \iota_2\iota_3) + c\iota_3\iota_4)]^2\\ &= -2\iota_1\iota_2\iota_3\iota_4\,(ac - b^2);\end{aligned}$$

that is to say, the square of the form in question is equal to $-2\iota_1\iota_2\iota_3\iota_4$ into the invariant of the original quantic.

Again, from the cubic $(a, b, c, d)\,(x, 1)^3$ we may form the expression $(a, b, \ldots)\,(x, 1)^2\,(\iota_1, \iota_2)$, the square of which of course vanishes. But taking a second cubic $(a_1, b_1, \ldots)\,(x, 1)^3$, and forming the corresponding expression $(a_1, b_1, \ldots)\,(x, 1)^2\,(\iota_1, \iota_2)$, we may obtain from the product

$$[(a, b, \ldots)\,(x, 1)^2\,(\iota_1, \iota_2)]\,[(a_1, b_1, \ldots)\,(x, 1)^2\,(\iota_1, \iota_2)]$$

the covariant, quartic in the variable, and quadratic in the coefficients, of the two cubics.

Similarly, the square of the second derivative of the cubic will give the quadratic covariant, as follows:

$$\begin{aligned}&[(a, b, \ldots)\,(x, 1)\,(\iota_1, \iota_2)\,(\iota_3, \iota_4)]^2\\ &= -2\iota_1\iota_2\iota_3\iota_4\,\{(ax+b)\,(cx+d) - (bx+c)^2\}.\end{aligned}$$

If for brevity we write down only the suffixes of the units, *e. g.*, if we write $(1, 2)$ for (ι_1, ι_2), we may say that the quadricovariant is given by

$$[(a, b, \ldots)\,(x, 1)\,(1, 2)\,(3, 4)]^2;$$

and similarly it will be found that the cubicovariant is given by the formula

$$\begin{aligned}&[(a, b, \ldots)\,(x, 1)^2\,(1, 2)]\\ &\times [(a, b, \ldots)\,(x, 1)\,(3, 4)\,(5, 6)]\\ &\times [(a, b, \ldots)\,(1, 2)\,(3, 4)\,(5, 6)].\end{aligned}$$

The quartic invariant is obtained by the following considerations:— The invariant of the quadricovariant is (by the same process as that used in the case of the quadratic) given by the formula

$$\left[\begin{matrix}(a, b, c, d)\,(1, 2)\,(3, 4)\,(5, 6)\\ \times(a, b, c, d)\,(1, 2)\,(3, 4)\,(7, 8)\end{matrix}\right]^2;$$

viz., discarding the factor $\iota_1\iota_2\iota_3\iota_4$, this is

$$= [2(ac-b^2)\,\iota_5\iota_7+(ad-bc)\,(\iota_5\iota_8+\iota_6\iota_7)+2(bd-c^2)\,\iota_6\iota_8]^2,$$

the developed form of which is equal to $-4\iota_5\iota_6\iota_7\iota_8$ into the known invariant.

These few remarks will perhaps suffice, on the present occasion, to direct attention to the connexion of alternate numbers with other methods which have been proposed for forming invariants and covariants of quantics. I must conclude this fragmentary communication with two more applications of the method of alternate numbers.

The first relates to Jacobi's theorem in functional determinants. If $y=f(x)$, let

$$y=\iota_1y_1+\iota_2y_2+\ldots,$$
$$x=\iota_1x_1+\iota_2x_2+\ldots,$$

then

$$\frac{dy}{dx}=\iota_1\frac{dy_1}{dx}+\iota_2\frac{dy_2}{dx}+\ldots$$

$$= \iota_1\left(\frac{dy_1}{dx_1}\frac{dx_1}{dx}+\frac{dy_1}{dx_2}\frac{dx_2}{dx}+\ldots\right) = \left(\iota_1\frac{dy_1}{dx_1}+\iota_2\frac{dy_2}{dx_1}+\ldots\right)\frac{dx_1}{dx}$$
$$+\iota_2\left(\frac{dy_2}{dx_1}\frac{dx_1}{dx}+\frac{dy_2}{dx_2}\frac{dx_2}{dx}+\ldots\right) \quad +\left(\iota_1\frac{dy_1}{dx_2}+\iota_2\frac{dy_2}{dx_2}+\ldots\right)\frac{dx_2}{dx}$$
$$+\ldots\ldots \qquad +\ldots\ldots$$

But, by the properties of alternate numbers,

$$\left(\frac{dy}{dx}\right)^n=\iota_1\iota_2\ldots\iota_n\frac{dx_1}{dx}\frac{dx_2}{dx}\ldots\frac{dx_n}{dx}\left|\begin{matrix}\frac{dy_1}{dx_1}, & \frac{dy_2}{dx_1}, & \ldots\\ \frac{dy_1}{dx_2}, & \frac{dy_2}{dx_2}, & \ldots\\ \cdot & \cdot & \ldots\end{matrix}\right|,$$

also

$$\frac{dx_1}{dx}=\frac{1}{\iota_1},\quad \frac{dx_2}{dx}=\frac{1}{\iota_2},\quad \ldots\ldots;$$

whence

$$\left(\frac{dy}{dx}\right)^n=\left|\begin{matrix}\frac{dy_1}{dx_1}, & \frac{dy_2}{dx_1}, & \ldots\\ \frac{dy_2}{dx_1}, & \frac{dy_2}{dx_2}, & \ldots\\ \cdot & \cdot & \ldots\end{matrix}\right|;$$

similarly

$$\left(\frac{dx}{dy}\right)^n=\left|\begin{matrix}\frac{dx_1}{dy_1}, & \frac{dx_2}{dy_1}, & \ldots\\ \frac{dx_1}{dy_2}, & \frac{dx_2}{dy_2}, & \ldots\\ \cdot & \cdot & \ldots\end{matrix}\right|;$$

whence finally $$\left|\begin{matrix} \frac{dy_1}{dx_1}, & \frac{dy_2}{dx_1}, & \dots \\ \frac{dy_1}{dx_2}, & \frac{dy_2}{dx_2}, & \dots \\ . & . & \dots \end{matrix}\right| \left|\begin{matrix} \frac{dx_1}{dy_1}, & \frac{dx_2}{dy_1}, & \dots \\ \frac{dx_1}{dy_2}, & \frac{dx_2}{dy_2}, & \dots \\ . & . & \dots \end{matrix}\right| = 1,$$

which is Jacobi's Theorem.

The second note relates to the connexion between the differential coefficients of y with respect to x, and the partial differential coefficients of y with respect to $x_1, x_2, \dots$ in such a system as that written above. Now, generally speaking, when $x, x_1, x_2, \dots$; $y, y_1, y_2, \dots$ are related as above, we have

$$\frac{dy}{dx} = \frac{dy}{dx_1}\frac{dx_1}{dx} + \frac{dy}{dx_2}\frac{dx_2}{dx} + \dots$$

$$= \frac{1}{\iota_1}\frac{dy}{dx_1} + \frac{1}{\iota_2}\frac{dy}{dx_2} + \dots ;$$

or, putting $\iota_1\iota_2 \dots = \mathrm{I}$,

$$\mathrm{I}\frac{dy}{dx} = \iota_2\iota_3 \dots \frac{dy}{dx_1} \pm \iota_3\iota_4 \dots \frac{dy}{dx_2} + \dots,$$

whence, multiplying both sides of the equation by $\iota_1, \iota_2, \dots$, we obtain the disjunctive equation

$$\mathrm{I}\frac{dy}{dx}(\iota_1, \iota_2, \dots) = \mathrm{I}\left(\frac{dy}{dx_1}, \frac{dy}{dx_2}, \dots\right),$$

by which formula any of the partial differential coefficients of y in respect of $x_1, x_2, \dots$ may be expressed in terms of the differential coefficients of y in respect of x.

A similar process will apply to second, and to higher differential coefficients; and by means of it the relations may be established which are comprised in the following formula:

$$\frac{d^2y}{dx^2}(\iota_1, \iota_2, \dots)^2 = \left(\frac{d}{dx_1}, \frac{d}{dx_2}, \dots\right)^2 y.$$

As a last instance, it may be shown that a partial differential equation of the first order, such as

$$\mathrm{X}_1 dx_1 + \mathrm{X}_2 dx_2 + \dots = 0,$$

may be transformed into the product of complex numbers thus. Let

$$x = \iota_1 x_1 + \iota_2 x_2 + \dots,$$
$$\mathrm{X} = \iota_2\iota_3 \dots \mathrm{X}_1 \pm \iota_3\iota_4 \dots \mathrm{X}_2 + \dots,$$

the upper or lower sign being taken according as the number of terms is odd or even. Then, by the properties of the units $\iota_1, \iota_2, \dots$, we obtain

$$(\iota_2\iota_3 \dots \mathrm{X}_1 \pm \iota_3\iota_4 \dots \mathrm{X}_2 + \dots)(\iota_1 dx_1 + \iota_2 dx_2 + \dots)$$
$$= \iota_1\iota_2 \dots (\mathrm{X}_1 dx_1 + \mathrm{X}_2 dx_2 + \dots),$$

and consequently the proposed equation takes the form

$$X dx = 0.$$

Partial differential equations of higher orders also admit of transformation to a similar form. For, from what was shown before, the partial differential coefficients of the dependent variable y with respect to the independent variables $x_1, x_2, \ldots$ may be expressed in terms of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ And if we then put

$$y = p_0, \quad \frac{dy}{dx} = p_1, \quad \frac{d^2y}{dx^2} = p_2, \quad \ldots,$$

the proposed equation will become

$$f(p_0, p_1, \ldots) = 0,$$

together with $\quad dp_0 = p_1 dx, \quad dp_1 = p_2 dx, \quad \ldots;$

and if, by means of the equation $f = 0$, we eliminate one of the variables, it will remain only to integrate the last system. If in these we equate to zero the coefficients of the various combinations of the units, we shall have a series of equations which may be reduced to the form

$X dx = 0$, say $\quad X_1 dx = 0, \quad X_2 dx = 0, \quad \ldots,$

which, by a former process, may be comprised under the general expression $\quad X dx = 0.$

But this will not, in general, be free from the units $\iota_1, \iota_2, \ldots$; and consequently methods of integration applicable to the reduced form of equations of the first order will not be applicable to this case.

Sur l'Intégration des Fonctions Circulaïres. *Par* M. Hermite.

[*Read November* 14*th*, 1872.]

En désignant par $f(\sin x, \cos x)$ une fonction rationnelle de $\sin x$ et $\cos x$, le procédé enseigné dans les éléments pour obtenir l'intégrale $\int f(\sin x, \cos x)\, dx$ consiste à changer de variable en posant $\tan \frac{1}{2}x = t$, ce qui ramène en effet à opérer sur une simple fonction rationnelle de t. Toutefois à l'égard des expressions de la forme $\sin^m x$, ou $\sin^m x \cos^n x$, on suit une autre marche en employant soit l'intégration par parties, soit la transformation en fonction linéaire des sinus et cosinus des multiples de la variable. Cette diversité de procédés rend, il faut le reconnaître, les éléments du calcul intégral moins intéressants que ceux de la géométrie et de l'algèbre, et c'est ce qui m'a porté à suivre une autre marche, en déduisant de l'étude que je vais exposer de la

fonction transcendante $f(\sin x, \cos x)$ la méthode qui en donne l'intégrale sous forme finie explicite.

I. Je partirai de la transformation en une fonction rationnelle de la quantité $f(\sin x, \cos x)$, qu'on obtient en posant : $e^{x\sqrt{-1}} = z$; de là résulte en effet :

$$\sin x = \frac{z^2-1}{2z\sqrt{-1}}, \quad \cos x = \frac{z^2+1}{2z};$$

de sorte qu'on peut faire :

$$f(\sin x, \cos x) = \frac{F_1(z)}{F(z)},$$

$F(z)$ et $F_1(z)$ désignant des polynômes entiers en z. Cela posé, je vais montrer que de la décomposition en fonctions simples de la fonction rationnelle $\frac{F_1(z)}{F(z)}$ résulte une décomposition en éléments simples qu'on donnera semblablement et d'une manière immédiate l'intégration. Considérant dans ce but la quantité $\frac{1}{(z-a)^n}$, qui est le type des fonctions simples, je pose $a = e^{\alpha\sqrt{-1}}$, ce qui sera toujours possible en excceptant le cas de $a=0$, et je remarque qu'on aura :

$$\frac{1}{z-a} = \frac{1}{e^{x\sqrt{-1}} - e^{\alpha\sqrt{-1}}} = -\frac{1+\sqrt{-1}\cot\frac{x-\alpha}{2}}{2e^{\alpha\sqrt{-1}}}.$$

De là résulte une première transformation du groupe des fractions partielles :

$$\frac{P}{z-a} + \frac{P_1}{(z-a)^2} + \ldots\ldots + \frac{P_n}{(z-a)^{n+1}}$$

en un polynôme entier et du degré $n+1$ en $\cot\frac{x-\alpha}{2}$, mais nous pouvons faire :

$$\cot^2 x = -1 - \frac{d\cot x}{dx}, \quad \cot^3 x = -\cot x + \frac{1}{2}\frac{d^2\cot x}{dx^2}, \quad \text{etc.},$$

et la relation identique :

$$\cot^{k+1} x = -\cot^{k-1} x - \frac{1}{k}\frac{d\cot^k x}{dx}$$

montre que de proche en proche on exprimera linéairement $\cot^n x$ au moyen des dérivées successives de $\cot x$, jusqu'à celle d'ordre $n-1$. Nous parvenons donc à ce nouveau résultat savoir :

$$\frac{P}{z-a} + \frac{P_1}{(z-a)^2} + \ldots\ldots + \frac{P_n}{(z-a)^{n+1}}$$

$$= C + A\cot\frac{x-\alpha}{2} + A_1\frac{d\cot\frac{x-\alpha}{2}}{dx} + \ldots\ldots + A_n\frac{d^n\cot\frac{x-\alpha}{2}}{dx^n}$$

les constantes C, A, A_1, ... A_n dépendant linéairement des divers numérateurs P, P_1, ... P_n. Ce point établi, je mettrai en évidence, si elles existent, les racines nulles du polynôme $F(z)$, en faisant :

$$F(z) = z^{m+1}(z-a)^{n+1}(z-b)^{p+1}\ldots\ldots(z-l)^{s+1},$$

et je modifierai la formule générale de décomposition en fractions simples, en réunissant à la partie entière du quotient $\frac{F_1(z)}{F(z)}$ les fractions partielles en $\frac{1}{z}$, $\frac{1}{z^2}$, ... $\frac{1}{z^{m+1}}$, de manière à avoir :

$$\begin{aligned}\frac{F_1(z)}{F(z)} = F(z) &+ \frac{P}{z-a} + \frac{P_1}{(z-a)^2} + \ldots\ldots + \frac{P_n}{(z-a)^{n+1}}\\ &+ \frac{Q}{z-b} + \frac{Q_1}{(z-b)^2} + \ldots\ldots + \frac{Q_p}{(z-b)^{p+1}}\\ &+ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\\ &+ \frac{S}{z-l} + \frac{S_1}{(z-l)^2} + \ldots\ldots + \frac{S_s}{(z-l)^{s+1}}\end{aligned}$$

ou $F(z)$ sera par conséquent de la forme $\Sigma a_k z^k$, avec des puissances entières mais positives ou négatives de z. Maintenant nous conclurons de cette formule élémentaire en revenant à la valeur $z = e^{x\sqrt{-1}}$ l'expression suivante de la fonction $f(\sin x, \cos x)$. La quantité $F(z)$ d'abord devenant :

$$\Sigma a_k e^{kx\sqrt{-1}} = \Sigma a_k (\cos kx + \sqrt{-1}\sin kx)$$

nous donne une première partie que je désignerai par $\Pi(x)$ et qui en sera considérée comme la partie entière. Les fractions partielles donnent ensuite une seconde partie $\Phi(x)$, qui en posant :

$$a = e^{\alpha\sqrt{-1}}, \quad b = e^{\beta\sqrt{-1}}, \quad \ldots\ldots \quad l = e^{\lambda\sqrt{-1}}$$

aura la forme suivante :

$$\begin{aligned}\Phi(x) = C + A\cot\frac{x-\alpha}{2} &+ A_1\frac{d\cot\frac{x-\alpha}{2}}{dx} + \ldots\ldots + A_n\frac{d^n\cot\frac{x-\alpha}{2}}{dx^n}\\ + B\cot\frac{x-\beta}{2} &+ B_1\frac{d\cot\frac{x-\beta}{2}}{dx} + \ldots\ldots + B_p\frac{d^p\cot\frac{x-\beta}{2}}{dx^p}\\ \ldots\ldots \quad &\ldots\ldots \quad \ldots\ldots\\ + L\cot\frac{x-\lambda}{2} &+ L_1\frac{d\cot\frac{x-\lambda}{2}}{dx} + \ldots\ldots + L_s\frac{d^s\cot\frac{x-\lambda}{2}}{dx^s}.\end{aligned}$$

La détermination des coefficients A, B, ... L ; A_1, B_1, ... etc. rendra plus complète encore l'analogie de la formule que nous venons d'obtenir :

$$f(\sin x, \cos x) = \Pi(x) + \Phi(x),$$

avec celle de la décomposition des fractions rationnelles en fractions simples.

II. Je ferai dans ce but, en ayant en vue le groupe des coefficients $A, A_1, \ldots A_n$, $x = a+h$, et je développerai les deux membres suivant les puissances croissantes de h. Or les séries provenant ainsi de la partie entière, et de $\cot\frac{x-\beta}{2}$, ... $\cot\frac{x-\lambda}{2}$, ne contiendront que des puissances entières et positives de h, tandis que la quantité $\cot\frac{x-a}{2}$ et les dérivées donneront un nombre fini et limité de puissances négatives. Nous avons en effet :

$$\cot\frac{x-a}{2} = \cot\frac{h}{2} = \frac{2}{h} - \frac{h}{6} - \frac{h^3}{360} - \ldots,$$

et comme la dérivée de h prise par rapport à x est l'unité, on déduira de cette relation, si on se dispense d'écrire la partie entière :

$$\frac{d^n \cot\frac{x-a}{2}}{dx^n} = (-1)^n 1.2 \ldots n \frac{2}{h^{n+1}},$$

le développement du second membre $\Pi(x)+\Phi(x)$ se composant ainsi des termes

$$2\left[\frac{A}{h} - \frac{A_1}{h^2} + \frac{1.2\,A_2}{h^3} - \ldots\ldots + (-1)^n \frac{1.2 \ldots n\,A_n}{h^{n+1}}\right]$$

et d'une série infinie de puissances positives de h, nous obtiendrons $A_1, A_2, \ldots A_n$, en formant la portion du développement du premier membre, qui est composée des seules puissances négatives de h. Supposant à cet effet :

$$f[\sin(a+h), \cos(a+h)] = \frac{\mathfrak{A}}{h} - \frac{\mathfrak{A}_1}{h^2} + \frac{1.2\,\mathfrak{A}_2}{h^3} - \ldots + (-1)^n \frac{1.2 \ldots n\,\mathfrak{A}_n}{h^{n+1}},$$

on aura immédiatement :

$$2A_1 = \mathfrak{A}_1, \quad 2A_2 = \mathfrak{A}_2, \quad \ldots\ldots \quad 2A_n = \mathfrak{A}_n,$$

et j'ajoute que si l'on multiplie membre à membre l'égalité précédente avec celle-ci que donne le théorême de Taylor :

$$\cot\frac{x-a-h}{2} = \cot\frac{x-a}{2} - \frac{h}{1}\frac{d\cot\frac{x-a}{2}}{dx} + \frac{h^2}{1.2}\frac{d^2\cot\frac{x-a}{2}}{dx^2} - \ldots\ldots$$

$$\ldots\ldots + (-1)^n \frac{h^n}{1.2 \ldots n}\frac{d^n\cot\frac{x-a}{2}}{dx^n} + \ldots\ldots$$

on trouvera pour le coëfficient divisé par deux, du terme en $\frac{1}{h}$, précisément :

$$A\cot\frac{x-a}{2} + A_1\frac{d\cot\frac{x-a}{2}}{dx} + \ldots\ldots + A_n\frac{d^n\cot\frac{x-a}{2}}{dx^n}.$$

Le groupe total des éléments simples, se rapportant à la quantité $x=a$ qui rend infinie la fonction proposée, est ainsi le demi résidu correspondant à $h=0$ de l'expression :

$$f[\sin(a+h), \cos(a+h)] \cot\frac{x-a-h}{2},$$

résultat analogue comme on voit à un théorème connu de Lagrange.

III. Après avoir jusqu'ici suivi pas à pas la théorie de la décomposition des fractions rationnelles en fractions simples, nous allons introduire une considération nouvelle qui a son origine dans la propriété caractéristique de la fonction $f(\sin x, \cos x)$ d'être périodique. Je remarque que, d'après la relation : $\cot\frac{x}{2} = \cot x + \operatorname{cosec} x$, la fonction $\Phi(x)$ l'exprime en termes de deux formes, à savoir : $\frac{d^n \cot x}{dx^n}$ et $\frac{d^n \operatorname{cosec} x}{dx^n}$, les premiers ayant pour période π, et les autres se reproduisant en signe contraire lorsqu'on change x en $x+\pi$. Or à l'égard de

$$\Pi(x) = \Sigma a_k (\cos kx + \sqrt{-1}\sin kx),$$

si l'on désigne par $\theta(x)$ l'ensemble des termes où k est pair, et par $\eta(x)$ l'ensemble des termes où k est impair, on aura de même :

$$\theta(x+\pi) = \theta(x), \quad \eta(x+\pi) = -\eta(x).$$

De là résulte la décomposition de la fonction proposée en deux parties $\Theta(x)$, $\mathrm{H}(x)$, de sorte qu'on aura :

$$f(\sin x, \cos x) = \Theta(x) + \mathrm{H}(x),$$

avec les conditions :

$$\Theta(x+\pi) = \Theta(x), \quad \mathrm{H}(x+\pi) = -\mathrm{H}(x),$$

les expressions des nouvelles fonctions introduites étant :

$$\begin{aligned}\Theta(x) = \theta(x) &+ \mathrm{A}\cot(x-a) + \mathrm{A}_1\frac{d\cot(x-a)}{dx} + \ldots + \mathrm{A}_n\frac{d^n\cot(x-a)}{dx^n}\\ &+ \mathrm{B}\cot(x-\beta) + \mathrm{B}_1\frac{d\cot(x-\beta)}{dx} + \ldots + \mathrm{B}_p\frac{d^p\cot(x-\beta)}{dx^p}\\ &\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ &+ \mathrm{L}\cot(x-\lambda) + \mathrm{L}_1\frac{d\cot(x-\lambda)}{dx} + \ldots + \mathrm{L}_s\frac{d^s\cot(x-\lambda)}{dx^s},\end{aligned}$$

et

$$\begin{aligned}\mathrm{H}(x) = \eta(x) &+ \mathrm{A}\operatorname{cosec}(x-a) + \mathrm{A}_1\frac{d\operatorname{cosec}(x-a)}{dx} + \ldots + \mathrm{A}_n\frac{d^n\operatorname{cosec}(x-a)}{dx^n}\\ &+ \mathrm{B}\operatorname{cosec}(x-\beta) + \mathrm{B}_1\frac{d\operatorname{cosec}(x-\beta)}{dx} + \ldots + \mathrm{B}_r\frac{d^r\operatorname{cosec}(x-\beta)}{dx^r}\\ &\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ &+ \mathrm{L}\operatorname{cosec}(x-\lambda) + \mathrm{L}_1\frac{d\operatorname{cosec}(x-\lambda)}{dx} + \ldots + \mathrm{L}_s\frac{d^s\operatorname{cosec}(x-\lambda)}{dx^s}.\end{aligned}$$

Nous voyons donc apparaître deux éléments simples distincts, $\cot x$ et $\operatorname{cosec} x$, ou $\frac{1}{\sin x}$, appartenant en propre aux fonctions dont la périodicité est celle de $\Theta(x)$ ou $\Pi(x)$, au lieu de $\cot\frac{x}{2}$, qui dans le cas général a le rôle de la quantité $\frac{1}{x}$ à l'égard des fonctions rationnelles. C'est par les applications qu'on reconnaîtra surtout l'utilité de ces distinctions, et pour commencer par un cas bien facile, j'envisagerai d'abord la fonction $\frac{1}{\cos\alpha-\cos x}$. En premier lieu j'observe que si l'on introduit le variable $z=e^{x\sqrt{-1}}$, il vient :

$$\frac{1}{\cos\alpha-\cos x}=\frac{2z}{2z\cos\alpha-1-z^2};$$

or les racines du dénominateur sont évidemment les quantités $e^{\alpha\sqrt{-1}}$, $e^{-\alpha\sqrt{-1}}$; le numérateur est seulement du premier degré, ainsi la partie entière $\Pi(x)$ n'existe point, et nous aurons :

$$\frac{1}{\cos\alpha-\cos x}=\mathrm{C}+\mathrm{A}\cot\frac{x-\alpha}{2}+\mathrm{B}\cot\frac{x+\alpha}{2}.$$

Calculant maintenant les résidus pour $x=\alpha$ et $x=-\alpha$, j'obtiens les quantités $\frac{1}{\sin\alpha}$, $-\frac{1}{\sin\alpha}$, et par suite, en divisant par deux, les valeurs $\mathrm{A}=\frac{1}{2\sin\alpha}$, $\mathrm{B}=-\frac{1}{2\sin\alpha}$; de sorte qu'il vient :

$$\frac{1}{\cos\alpha-\cos x}=\mathrm{C}+\frac{1}{2\sin\alpha}\left[\cot\frac{x-\alpha}{2}-\cot\frac{x+\alpha}{2}\right].$$

On trouve d'ailleurs sans peine que $\mathrm{C}=0$, mais voici pour des cas moins faciles une détermination directe et immédiate de cette constante.

Supposons en général : $f(\sin x,\cos x)=\frac{\mathrm{F}_1(z)}{\mathrm{F}(z)}$, $\mathrm{F}(z)$ ne contenant point le facteur z et étant de degré au moins égal à celui du numérateur ; la partie désignée par $\Phi(x)$ existera seule dans l'expression de la fonction qui sera ainsi :

$$\begin{aligned}f(\sin x,\cos x)=\mathrm{C}&+\mathrm{A}\cot\frac{x-\alpha}{2}+\mathrm{A}_1\frac{d\cot\frac{x-\alpha}{2}}{dx}+\ldots\ldots\\&+\mathrm{B}\cot\frac{x-\beta}{2}+\mathrm{B}_1\frac{d\cot\frac{x-\beta}{2}}{dx}+\ldots\ldots\\&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\&+\mathrm{L}\cot\frac{x-\lambda}{2}+\mathrm{L}_1\frac{d\cot\frac{x-\lambda}{2}}{dx}+\ldots\ldots\end{aligned}$$

Or je dis qu'en nommant G et H les valeurs de $\frac{F_1(z)}{F(z)}$ pour z nul et infini, on aura : $C = \frac{1}{2}(G+H)$. En effet, la relation

$$\cot\frac{x-a}{2} = \sqrt{-1}\,\frac{e^{(x-a)\sqrt{-1}}+1}{e^{(x-a)\sqrt{-1}}-1} = \sqrt{-1}\,\frac{ze^{-a\sqrt{-1}}+1}{ze^{-a\sqrt{-1}}-1}$$

fait voir qu'en supposant z nul et infini, toutes les quantités $\cot\frac{x-a}{2}$ se réduisent à $-\sqrt{-1}$ et $+\sqrt{-1}$; elle montre aussi que leurs dérivées des divers ordres s'évanouissent, nous avons par conséquent :

$$G = C-(A+B+\ldots\ldots+L)\sqrt{-1},$$
$$H = C+(A+B+\ldots\ldots+L)\sqrt{-1},$$

d'où : $$A+B+\ldots\ldots+L = \frac{G-H}{2}\sqrt{-1}, \quad C = \frac{G+H}{2}.$$

Dans l'exemple considéré tout à l'heure, on trouve sur le champ $G=0$, $H=0$, de sorte que C est nul comme nous l'avons dit. Ces deux relations que nous venons d'obtenir, appliquées à la fonction $\Phi(x)\cot\frac{x-t}{2}$ donneraient, comme il est aisé de voir, la décomposition de $\Phi(t)$ en éléments simples, mais sans m'arrêter à cette remarque, je vais immédiatement donner plusieurs exemples de cette décomposition.

1°. L'expression $\frac{\cos mx}{\cos x-\cos a}$, où m est un nombre entier, devient en faisant $e^{x\sqrt{-1}} = z$, $\frac{z^{2m}+1}{z^{m-1}(1-2z\cos a+z^2)}$, et contient par conséquent une partie entière qui s'obtient ainsi. Partant de ces deux identités :

$$\frac{\sin a}{1-2z\cos a+z^2} = \sin a + z\sin 2a + \ldots\ldots + z^{m-2}\sin(m-1)a + z^{m-1}\frac{\sin ma - z\sin(m-1)a}{1-2z\cos a+z^2}$$

$$\frac{\sin a}{1-2z\cos a+z^2} = \frac{\sin a}{z^2} + \frac{\sin 2a}{z^3} + \ldots\ldots \frac{\sin ma}{z^{m+1}} + \frac{1}{z^{m+1}}\frac{z\sin(m+1)a - \sin ma}{1-2z\cos a+z^2},$$

je les ajoute après avoir divisé la première par z^{m-1} et multiplié la seconde par z^{m+1}, et j'obtiens :

$$\frac{(z^{2m}+1)\sin a}{z^{m-1}(1-2z\cos a+z^2)} = (z^{m-1}+z^{1-m})\sin a + (z^{m-2}+z^{2-m})\sin 2a + \ldots + (z+z^{-1})\sin(m-1)a + \sin ma + \frac{z[\sin(m+1)a - \sin(m-1)a]}{1-2z\cos a+z^2}.$$

Si maintenant nous remplaçons z par l'exponentielle $e^{a\sqrt{-1}}$, nous aurons,

en divisant par $\sin a$:

$$\frac{\cos mx}{\cos x - \cos a} = \Pi(x) + \frac{\cos ma}{\cos x - \cos a},$$

en faisant :

$$\Pi(x) = \frac{1}{\sin a}[2 \sin a \cos (m-1)x + 2 \sin 2a \cos (m-2)x + \ldots\ldots + 2 \sin (m-1)a \cos x + \sin ma].$$

2°. La fonction $\frac{\sin mx}{\sin nx}$, où m et n sont entiers, prend la forme $z^{n-m}\frac{z^{2m}-1}{z^{2n}-1}$ et aura de même une partie entière, si l'on suppose $m > n$. J'emploie pour l'obtenir l'identité :

$$z^{n-m}\frac{z^{2m}-1}{z^{2n}-1} = z^{m-n} + z^{n-m} + z^{n-3m} + z^{3n-m} + \ldots\ldots + z^{m-(2k-1)n} + z^{(2k-1)n-m} + \frac{z^{(2k+1)m-n} - z^{n-(2k-1)m}}{z^{2n}-1},$$

et je prends pour k le plus grand entier contenu dans $\frac{m-n}{2n}$, de sorte qu'on ait $k = \frac{m-n}{2n} + \epsilon$, ϵ étant positif et moindre que l'unité. Il en résulte que

$$(2k+1)m - n = 2\epsilon n \quad \text{et} \quad n - (2k-1)m = 2(1-\epsilon)n ;$$

ainsi dans la fraction du second membre le numérateur est de degré inférieur au degré du dénominateur. L'identité employé se vérifie d'ailleurs sur le champ, car elle se transforme, en remplaçant z par $e^{x\sqrt{-1}}$, dans l'équation connue :

$$\frac{\sin mx}{\sin nx} = 2 \cos (m-n)x + 2 \cos (m-3n)x + \ldots\ldots + 2 \cos [m-(2k-1)n]x + \frac{\sin (2kn-m)x}{\sin nx}.$$

Nous obtenons ainsi :

$$\Pi(x) = 2 \cos (m-n)x + 2 \cos (m-3n)x + \ldots + 2 \cos [m-(2k-1)n]x$$

et $\Phi(x) = \frac{\sin (2kn-m)x}{\sin nx}$, ou simplement $\Phi(x) = \frac{\sin mx}{\sin nx}$,

m étant supposé actuellement inférieur à n en valeur absolue. Ceci établi, les racines de l'équation $z^{2n} - 1 = 0$ sont données par la formule $z = e^{\frac{k\pi}{n}\sqrt{-1}}$ pour $k = 0, 1, 2, \ldots 2n-1$, et si l'on fait $a = \frac{k\pi}{n}$, le résidu de la fonction $\frac{\sin mx}{\sin nx}$ correspondant à la valeur $x = a$ sera :

$$\frac{\sin ma}{n \cos na} = \frac{(-1)^k \sin ma}{n},$$

et nous obtenons par conséquent :

$$\frac{\sin mx}{\sin nx} = \frac{1}{2n} \Sigma (-1)^k \sin ma \cot \frac{x-a}{2}.$$

Mais ayant $\Phi(x+\pi) = (-1)^{m+n}\Phi(x)$, la fonction appartiendra au type $\Theta(x)$ ou $H(x)$, suivant que $m+n$ sera pair ou impair ; de sorte qu'il vient pour le premier cas :

$$\frac{\sin mx}{\sin nx} = \frac{1}{2n}\Sigma(-1)^k \sin ma \cot(x-a),$$

et pour le second :

$$\frac{\sin mx}{\sin nx} = \frac{1}{2n}\Sigma\frac{(-1)^k \sin ma}{\sin(x-a)}.$$

Or les termes des deux sommes qui correspondent aux valeurs k et $k+n$ sont égaux, on peut donc en doublant se borner à prendre $k = 1, 2, \ldots n-1$, le résidu relatif à $k=0$ étant nul.

3°. L'expression $\cot(x-a)\cot(x-\beta)\ldots\ldots\cot(x-\kappa)\cot(x-\lambda)$, en désignant par n le nombre des facteurs, et faisant : $a=e^{a\sqrt{-1}}$, $b=e^{\beta\sqrt{-1}}$, $\ldots$ $l=e^{\lambda\sqrt{-1}}$, aura pour transformée en z :

$$(\sqrt{-1})^n\frac{(z^2+a^2)(z^2+b^2)\ldots\ldots(z^2+l^2)}{(z^2-a^2)(z^2-b^2)\ldots\ldots(z^2-l^2)}.$$

On voit que le numérateur et le dénominateur sont du même degré, ainsi il n'existe pas de partie entière et nous avons seulement à calculer $\Phi(x)$. Or les $2n$ racines du dénominateur sont d'une part : $e^{a\sqrt{-1}}$, $e^{\beta\sqrt{-1}}$, $\ldots\ldots$ $e^{\lambda\sqrt{-1}}$, et en outre ces mêmes quantités changées de signe, c'est à dire : $e^{(a+\pi)\sqrt{-1}}$, $e^{(\beta+\pi)\sqrt{-1}}$, $\ldots\ldots$ $e^{(\lambda+\pi)\sqrt{-1}}$; d'ailleurs $\Phi(x+\pi) = \Phi(x)$, ainsi la fonction proposée appartient au type $\Theta(x)$, et ses éléments simples où figurent les arguments a et $a+\pi$, β et $\beta+\pi$, etc., se réduisent à ceux-ci : $\cot(x-a)$, $\cot(x-\beta)$, $\ldots$ $\cot(x-\lambda)$. Nous aurons en conséquence :

$$\Phi(x) = C + A\cot(x-a) + B\cot(x-\beta) + \ldots\ldots + L\cot(x-\lambda),$$

A, B, ... L étant les résidus de $\Phi(x)$ pour $x = a, \beta, \ldots \lambda$; c'est à dire :

$$A = \cot(a-\beta)\cot(a-\gamma)\ldots\ldots\cot(a-\lambda),$$
$$B = \cot(\beta-a)\cot(\beta-\gamma)\ldots\ldots\cot(\beta-\lambda),$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$L = \cot(\lambda-a)\cot(\lambda-\beta)\ldots\ldots\cot(\lambda-\kappa).$$

Enfin la constante C s'obtient par l'équation $C = \frac{1}{2}(G+H)$, au moyen des valeurs, $G = (-\sqrt{-1})^n$, $H = (\sqrt{-1})^n$, qu'on tire de la transformée en z pour z nul et infini, ce qui donne simplement : $C = \cos\frac{n\pi}{2}$.

4°. On traitera de même enfin l'expression plus générale :

$$\frac{F(\sin x, \cos x)}{\sin(x-a)\sin(x-\beta)\ldots\sin(x-\lambda)},$$

où le numérateur est un polynôme entier en $\sin x$ et $\cos x$: en supposant qu'il soit homogène et de degré $n-1$, on sera amené à la relation suivante :

$$\frac{F(\sin x, \cos x)}{\sin(x-\alpha)\sin(x-\beta)\ldots\sin(x-\lambda)}$$

$$= \frac{F(\sin\alpha, \cos\alpha)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)\ldots\sin(\alpha-\lambda)} \cdot \frac{1}{\sin(x-\alpha)}$$

$$+ \frac{F(\sin\beta, \cos\beta)}{\sin(\beta-\alpha)\sin(\beta-\gamma)\ldots\sin(\beta-\lambda)} \cdot \frac{1}{\sin(x-\beta)}$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$+ \frac{F(\sin\lambda, \cos\lambda)}{\sin(\lambda-\alpha)\sin(\lambda-\beta)\ldots\sin(\lambda-\varkappa)} \cdot \frac{1}{\sin(x-\lambda)},$$

Nous en déduisons, en chassant le dénominateur :

$$F(\sin x, \cos x) = \frac{\sin(x-\beta)\sin(x-\gamma)\ldots\sin(x-\lambda)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)\ldots\sin(\alpha-\lambda)} F(\sin\alpha, \cos\alpha)$$

$$+ \frac{\sin(x-\alpha)\sin(x-\gamma)\ldots\sin(x-\lambda)}{\sin(\beta-\alpha)\sin(\beta-\gamma)\ldots\sin(\beta-\lambda)} F(\sin\beta, \cos\beta)$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$+ \frac{\sin(x-\alpha)\sin(x-\beta)\ldots\sin(x-\varkappa)}{\sin(\lambda-\alpha)\sin(\lambda-\beta)\ldots\sin(\lambda-\varkappa)} F(\sin\lambda, \cos\lambda),$$

résultat qui se rapporte à la théorie de l'interpolation, comme donnant l'expression de la fonction $F(\sin x, \cos x)$ où entrent n coefficients arbitraires, au moyen des n valeurs qu'elle prend pour $x = \alpha, \beta, \ldots \lambda$.

IV. C'est pour obtenir l'intégrale de la fonction $f(\sin x, \cos x)$ qu'a été établie la formule de décomposition en éléments simples, et on voit en effet que d'une part $\int \Pi(x)\,dx$ se calcule sur le champ, et de l'autre en ce qui concerne $\int \Phi(x)\,dx$, il suffit d'employer la relation qui s'établit immédiatement :

$$\int \cot\frac{x-\alpha}{2}\,dx = 2\log\sin\frac{x-\alpha}{2}.$$

À l'égard des quantités : $\int \Theta(x)\,dx$, $\int H(x)\,dx$, on se servira de celles-ci :

$$\int \cot(x-\alpha)\,dx = \log\sin(x-\alpha) \quad \text{et} \quad \int \operatorname{cosec}(x-\alpha)\,dx = \log\operatorname{tang}\frac{x-\alpha}{2}.$$

Quant aux expressions qui figurent en dehors des termes logarithmiques, à savoir :

$$A_1\cot(x-\alpha) + A_2\frac{d\cot(x-\alpha)}{dx} + \ldots + A_n\frac{d^{n-1}\cot(x-\alpha)}{dx^n},$$

et $$A_1\operatorname{cosec}(x-\alpha) + A_2\frac{d\operatorname{cosec}(x-\alpha)}{dx} + \ldots + A_n\frac{d^{n-1}\operatorname{cosec}(x-\alpha)}{dx^{n-1}},$$

on prouve aisément que la première étant un polynôme du n^e degré en $\cot(x-a)$, la seconde est le produit d'un polynôme semblable en $\cot(x-a)$, du degré $n+1$, multiplié par le facteur $\cos(x-a)$. Ces quantités sous ces formes nouvelles sont parfois d'une détermination plus facile, et j'en donnerai un exemple en considérant l'intégrale $\int \cot^{n+1} x\,dx$, l'exposant n étant entier et positif. D'après la méthode générale, on fera :

$$\cot^{n+1} x = C + A\cot x + A_1 \frac{d\cot x}{dx} + \ldots\ldots + A_n \frac{d^n \cot x}{dx^n},$$

et les coefficients s'obtiendront, soit au moyen des relations successives :

$$\cot^2 x = -1 - \frac{d\cot x}{dx}, \quad \cot^3 x = -\cot x + \frac{1}{2}\frac{d^2\cot x}{dx^2}, \quad \text{etc.},$$

soit en formant la puissance $n+1$ et les dérivées de la série : $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{49} - \ldots$. Or la substitution $\cot x = t$, qui est indiquée par la forme connue d'avance de l'intégrale, en donne facilement la valeur, puisqu'elle devient ainsi : $-\int \frac{t^{n+1}\,dt}{1+t^2}$, et on obtient, si n est impair :

$$\int \cot^{n+1} x\,dx = -\frac{\cot^n x}{n} + \frac{\cot^{n-2} x}{n-2} - \ldots - (-1)^{\frac{n-1}{2}} \cot x + (-1)^{\frac{n-1}{2}} x,$$

et en supposant n pair :

$$\int \cot^{n+1} x\,dx = -\frac{\cot^n x}{n} + \frac{\cot^{n-2} x}{n-2} - \ldots + (-1)^{\frac{n}{2}} \frac{\cot^2 x}{2} + (-1)^{\frac{n}{2}} \log\sin x.$$

Rapprochant ces résultats de l'expression donnée par la formule générale, savoir :

$$\int \cot^{n+1} x\,dx = Cx + A\log\sin x + A_1\cot x + \ldots + A_n \frac{d^{n-1}\cot x}{dx^{n-1}}$$

nous en tirons cette conséquence que l'on a : $A=0$ ou $A=(-1)^{\frac{n}{2}}$, selon que n est impair ou pair. En terminant je remarquerai que cette seule connaissance du résidu A relatif à la valeur $x=0$ de la fonction $\cot^{n+1} x$, suffit pour la détermination complète de la série :

$$\cot x = \frac{a}{x} + b + cx + dx^2 + \ldots\ldots$$

Et d'abord de ce que le terme en $\frac{1}{x}$ manque dans le carré, la quatrième puissance, et toutes les puissances paires, on conclut de proche en proche les conditions : $b=0$, $d=0$, ... c'est à dire que le développement ne contient que des puissances impaires de la variable, et a la forme :

$\cot x = \frac{a}{x} + \beta x + \gamma x^3 + \ldots$ De ce que la coefficient du même terme est $+1$, -1, $+1$, etc., dans la première, la troisième, la cinquième puissance, etc., on tire aisément les égalités : $a=1$, $3a^2\beta = -1$, $9a^4\gamma + 10a^3\beta^2 = 1$, d'où $\beta = -\frac{1}{3}$, $\gamma = -\frac{1}{45}$, etc. C'est donc à un nouveau point de vue une détermination des nombres de Bernouilli.

On the Mechanical Description of a Cubic Curve.

By Prof. Cayley, F.R.S.

[*Read 14th November*, 1872.]

If the coordinates x, y of a point on a curve are rational functions of $\sin\phi$, $\cos\phi$, $\sqrt{1-k^2\sin^2\phi}$, the curve has the deficiency 1, and conversely in any curve of deficiency 1 the coordinates x, y can be thus expressed in terms of the parameter ϕ. Hence writing $\sin\theta = k\sin\phi$, the coordinates will be rational functions of $\sin\phi$, $\cos\phi$, $\cos\theta$, or say of $\sin\phi$, $\cos\phi$, $\sin\theta$, $\cos\theta$; and for the mechanical representation of the relation $k\sin\phi = \sin\theta$, we require only a rod OA rotating about the fixed point O, and connected with it by a pin at A, a rod AB, the other extremity of which, B, moves in a fixed line Ox. The curve most readily obtained by such an arrangement is that described by a point C rigidly connected with the rod AB; this is however a quartic curve (with two dps., since its deficiency is $=1$). I first considered the cubic

curve $$xy - 1 = \sqrt{(1-x^2)(1-k^2x^2)},$$

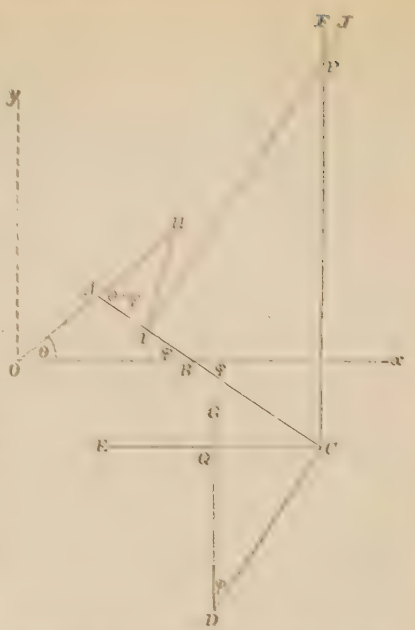
or say $$xy - 1 = -\sqrt{(1-x^2)(1-k^2x^2)};$$

writing herein $x = \sin\phi$, and as before $k\sin\phi = \sin\theta$, we have then $y\sin\phi = 1 - \cos\theta\cos\phi$; which values may be written

$$x = \sin\phi,$$
$$y = \frac{1-\cos(\theta+\phi)}{\sin\phi} - \sin\theta.$$

I found, however, that this was *not* the cubic curve most easily constructed; and I ultimately devised a mechanical arrangement consisting of—

1. Rod OH, and connected with it by a pin at H, rod HI.*
2. Square ACD, and connected with it by a pin at D, rod DG.
3. Square ECF; the two squares being connected by a pin at C.
4. Rod IJ.



The rod OH rotates about a pin at O; taking HA = HI, there is a pin at A connecting a fixed point of this rod with the extremity A of the square ACD: the fixed point B of this square moves along the line Ox. There is a pin at I connecting the extremities of the rods HI, IJ; and this slides along the leg AC of the square ACD, the rod IJ being always at right angles thereto: finally the legs of the square ECF are always parallel to Ox, Oy, and the rod DG at right angles to EC. I have omitted from the description the parallel-motion rods or other arrangements necessary for giving these fixed directions to the rod IJ, the square ECF, and the rod DG. It will be seen that the angles AOB, ABO are variable angles connected by an equation of the form above referred to; and that the lines IJ, CF determine by their intersection the point P; and the lines CE, DG determine by their intersection the point Q; the curve about to be considered is that determined by the relative motion of P in regard to Q; or say the curve the coordinates of a point of which are

$$x = \mathrm{QC}, \quad y = \mathrm{CP}.$$

I write

$$\angle \mathrm{AOB} = \theta, \quad \angle \mathrm{ABO} = \phi,$$

$$\mathrm{OA} = a, \quad \mathrm{AB} = b, \quad \mathrm{AC} = c, \quad \mathrm{CD} = d,$$

$$\mathrm{AH} = \mathrm{HI} = \tfrac{1}{2}h.$$

We then have $a \sin\theta = b \sin\phi$; and moreover the length AI being $= h\cos(\theta+\phi)$, and therefore $\mathrm{IC} = c - h\cos(\theta+\phi)$, we have

$$y = \mathrm{CP} = \frac{c - h\cos(\theta+\phi)}{\sin\phi},$$

$$x = \mathrm{QC} = d\sin\phi;$$

* There was a mechanical convenience in this, but observe that producing OH to meet IP in I′, the single straight rod OHI′ might have been made use of.

whence also $$xy = d\{c - h\cos(\theta+\phi)\};$$

or we have $$xy = d\left\{c - h\sqrt{\left(1-\frac{x^2}{d^2}\right)}\sqrt{\left(1-\frac{b^2}{a^2}\frac{x^2}{d^2}\right)} + h\frac{b}{a}\frac{x^2}{d^2}\right\},$$

that is $$x\left(y - \frac{bh}{ad}x\right) - cd = -dh\sqrt{\left(1-\frac{x^2}{d^2}\right)}\sqrt{\left(1-\frac{b^2}{a^2}\frac{x^2}{d^2}\right)};$$

or rationalising and reducing, this is

$$x^2y^2 - \frac{2bh}{ad}x^2y - 2cdxy + \left\{2\frac{b}{a}ch + h^2\left(1+\frac{b^2}{a^2}\right)\right\}x^2 + d^2(c^2-h^2) = 0,$$

a quartic curve with two dps.

In the particular case $a=b$, the relation between θ, ϕ is simply $\theta=\phi$; the curve should become unicursal.

Writing in the equation $\frac{b}{a} = 1$, the equation takes the form

$$\left\{x\left(y - \frac{2h}{d}x\right) - d(c-h)\right\}\{xy - d(c+h)\} = 0;$$

the second factor is extraneous, and the curve is the hyperbola

$$x\left(y - \frac{2h}{d}x\right) - d(c-h) = 0,$$

as at once appears from the foregoing irrational form of the equation.

In the particular case $h=c$, the equation contains the factor x, and omitting this it becomes

$$x\left(y^2 - \frac{2bc}{ad}x^2\right) - 2cdy + c\left(1+\frac{b}{a}\right)^2 x = 0;$$

viz., we have here a cubic curve with three real asymptotes meeting in a point which is also the centre of the curve.

If simultaneously $a=b$ and $h=c$, then the equation is

$$x\left(y - \frac{2c}{d}x\right)(xy - 2cd) = 0,$$

the actual locus being in this case the line $y - \frac{2c}{d}x = 0$.

Writing $h=c$, and for greater convenience $h=c=d=1$; also to fix the ideas supposing $b<a$, and writing $\frac{b}{a} = k$, $=\sin\lambda$, then we have

$$\sin\theta = k\sin\phi,$$
$$x = \sin\phi,$$
$$y = \frac{1-\cos(\theta+\phi)}{\sin\phi};$$

that is $$xy = 1 - \sqrt{1-x^2}\sqrt{1-k^2x^2} + kx^2,$$

giving the rationalised equation

$$x(y^2 - 2kx^2) - 2y + 4x = 0;$$

the angle ϕ may be anything whatever, but θ varies between the limits

$\pm\lambda$, the simultaneous values of these angles and of the coordinates being

$\phi=0$	$\theta=0$	$x=0$	$y=0$
$\phi=90°$	$\theta=\lambda$	$x=1$	$y=1+\sin\lambda$
$\phi=180°$	$\theta=0$	$x=0$	$y=\pm\infty$
$\phi=270°$	$\theta=-\lambda$	$x=-1$	$y=-(1+\sin\lambda)$
$\phi=360°$	$\theta=0$	$x=0$	$y=0$;

and it thus appears that the mechanism gives the continuous branch which belongs to the asymptote $x=0$ of the cubic curve; the other two branches belong to $x=\sin\phi$, $y=\dfrac{1+\cos(\theta+\phi)}{\sin\phi}$, which would require a slight alteration in the arrangement of the mechanism.

I remark that if AH, HI had been unequal, then writing $\angle \mathrm{HIA}=\chi$, this would be connected with $\theta+\phi$ by an equation of the form

$$\sin(\theta+\phi)=m\sin\chi,$$

and the coordinates x, y would be rational functions of the sines and cosines of θ, ϕ, χ; the deficiency is in this case >1.

On a Theorem relating to Polyhedra, analogous to Mr. Cotterill's Theorem on Plane Polygons. By Prof. Clifford.

[*Read November 14th*, 1872.]

Mr. Cotterill's theorem, presented last year to the Society, is as follows: For every plane polygon of n vertices there is a curve of class $n-3$ touching all the diagonals; the number of diagonals is such as to exactly determine this curve and no more; and when the curve touches the line at infinity, the area of the polygon is zero.

The proof of this depends essentially upon the fact that if we join the vertices of the polygon to any point in its plane, the area of the polygon is equal to the sum of the triangles so formed, taken of course with their proper signs according to the rule of Möbius.

The analogous theorem in space should therefore apply in the first instance to those solids whose volume can be expressed as the sum of tetrahedra, having one vertex at an arbitrary point of space, and the other three at three vertices of the figure; that is to say, it should apply to solids having *triangular faces*.

For such solids I find accordingly that the analogy is very complete and exact. It is convenient to define a plane containing three vertices but not being a face, as a diagonal plane; and a line joining two vertices but not being an edge, as a diagonal line. This being so, the theorems which I shall prove are the following:

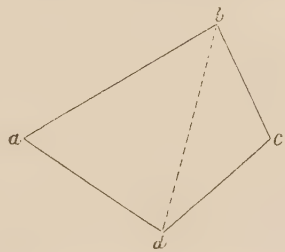
For every polyhedron of n summits having only triangular faces (△-faced *n*-acron, CAYLEY) *there is a surface of class* $n-4$ *touching all the diagonal planes.*

This surface contains all the diagonal lines.

The diagonal planes and lines are so situated, however, that the conditions of touching the planes and containing the lines are precisely sufficient to determine a surface of class $n-4$.

When this surface touches the plane at infinity, the volume of the solid is zero.

To apply these propositions to polyhedra having other than triangular faces, we must consider such polygonal faces as *singularities.* Each of them, in fact, may by a small deformation of the polyhedron be resolved into a certain number of triangles; and we may thus regard a quadrangular face, for example, as the special case of two adjacent trianglular faces being in one plane. Thus the quadrangular face *abcd* may be regarded as produced by coplanarity of the triangles *abd*, *cbd*. The effect of this is also to unite together the two diagonal planes *abc*,



adc, and to make the diagonal line *ac* lie in the face. Thus the surface of class $n-4$ must touch the face *abcd;* but it does not in general contain the lines *ac*, *bd*. It touches the face at their point of intersection. And, in general, it is not necessary to consider the diagonals of a polygonal face as diagonals of the polyhedron, and they do not in fact lie upon the surface d_{n-4}. But *a polygonal face with m vertices is a multiple tangent plane of order* $m-3$, *and the curve of contact is Mr. Cotterill's curve appertaining to the polygon.*

It is interesting to consider from this point of view the correlative propositions. Just as we have regarded a solid with a given number of summits, or *polyacron*, as having normally or in the most general case only triangular faces, while polygonal faces present themselves as singularities, and polyacra possessing them as degenerate forms; so we must regard a *polyhedron*, or solid with a given number of faces, as having normally or in general only three-edged summits (tripleural summits, CAYLEY), while summits having a greater number of edges will present themselves as singularities, and polyhedra possessing them as

degenerate. Every solid with plane faces, except the tetrahedron, must have singularities of one kind or the other; just as only loci of the second order are general at the same time of their order and of their class.

The proof of these results is as follows. Let $a, b, c, \ldots l, m, n$ be the summits of a Δ-faced polyacron, and p any point in space; let also $X=0$ be the equation to the plane at infinity, and the result of substituting in X the coordinates of any point, as a, be denoted by aX. Now if fgh is a face, and the summits f, g, h, looked at from p, go round the face clockwise, then the expression $\frac{(pfgh)}{pX.fX.gX.hX}$ represents the volume of the tetrahedron $pfgh$ according to the rule of Möbius. [Here $(pfgh)$ means the determinant formed with the coordinates of p, f, g, h.] Hence, if V be the volume of the whole solid, we have

$$\Sigma \frac{(pfgh)}{pX.fX.gX.hX} = V,$$

the summation being extended over all the faces, and the summits of each so mentioned that every edge occurs twice in two different orders; that is to say, if we have mentioned $(pfgh)$, we must not mention $(pfgk)$, but $(pgfk)$ or $(pfkg)$ or $(pkgf)$. To render this equation homogeneous in all the quantities mentioned, I call to mind that the volume of a tetrahedron is not given absolutely by the formula $\frac{(pfgh)}{pX.fX.gX.hX}$, but only to a factor *près*, depending on the unit of volume employed. If we take as this unit of volume the volume of the fundamental tetrahedron, whose vertices may be denoted by 1, 2, 3, 4, then our equation becomes

$$\Sigma \frac{(pfgh)}{pX.fX.gX.hX} = V.\frac{(1234)}{1X.2X.3X.4X} \quad \ldots\ldots\ldots\ldots (1).$$

Here V is a ratio, depending on the positions of the points $a, b, c, \ldots$ relatively to the plane X, but absolutely independent of the position of p. If, then, we make the equation integral, by multiplying both sides into $pX.1X.2X.3X.4X.\Pi.fX$, we see that the expression

$$\Sigma (pfgh) \frac{\Pi.fX}{fX.gX.hX}$$

must be divisible by pX; because its equivalent on the other side is so divisible, and the equation is an identity so far as p is concerned. The result of the division is of the order $n-4$ in X; or, which is the same thing, *if* X *be regarded as a variable plane, the equation*

$$\Sigma (pfgh) \frac{\Pi.fX}{pX.fX.gX.hX} = 0 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

represents a surface of class $n-4$.

Two things are now clear from our previous equation and from the form of this one.

1°. If the equation is satisfied when X is the plane at infinity, then $V=0$; or, *if the surface* (2) *touch the plane at infinity, the volume of the solid is zero.*

2°. The equation (2) is satisfied if $lX=0$, $mX=0$, $nX=0$, where l, m, n are any three vertices not in the same face. Therefore *the surface* (2) *touches all the diagonal planes.*

The investigation, so far, is a mere reproduction of that of Mr. Cotterill, with the addition of an extra letter to apply it to three dimensions instead of two. I shall take the liberty of calling the surface thus arrived at the *index-surface* of the polyacron, and shall denote it by the symbol v_{n-4}.

The index-surface contains all the diagonal lines. For let ab be a diagonal line, and c any other summit of the solid; then abc is a diagonal plane. For if it were a face, ab would be an edge, contrary to the supposition. Consequently $n-2$ diagonal planes can be drawn through every diagonal line; now all these are touched by the index-surface. But if more than $n-4$ tangent planes can be drawn through a straight line to a surface of class $n-4$, the line must lie altogether in the surface.

Through an edge of the solid, on the other hand, two faces and $n-4$ diagonal planes can be drawn; which latter are, of course, the tangent planes from it to the surface.

There are certain diagonal planes which it is convenient to consider separately. They are those which contain three edges of the solid; and I shall call them *single planes*. A diagonal plane may, of course, contain three diagonal lines, or two, or one, or none; but if it contains any diagonal line, the condition of touching it is already involved in the condition of containing that line. So that the facts we know about the index-surface may be summed up in saying that it passes through all the diagonal lines and touches all the single planes.

I now go on to prove that in general these conditions are precisely sufficient to determine a surface of class $n-4$. In order to do this, it will be necessary to make use of the researches of Prof. Cayley upon the Δ-faced polyacra, contained in the 1st volume of the 3rd series of the "Manchester Memoirs," p. 248; particularly of the following passage:—

"An n-acron has n summits, $3n-6$ edges, $2n-4$ faces; and it is easy to see that there are the following three cases only, viz.:

1. The polyacron has at least one tripleural summit.
2. The polyacron, having no tripleural summit, has at least one tetrapleural summit.
3. The polyacron, having no tripleural or tetrapleural summit, has at least twelve pentipleural summits.

In fact, if the polyacron has c tripleural summits, d tetrapleural summits, e pentipleural summits, and so on, then we have

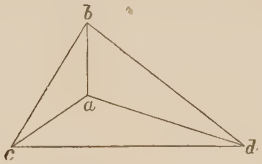
$$n = c+d+e+f+g+h+\&c.,$$
$$6n-12 = 3c+4d+5e+6f+7g+8h+\&c.;$$

and therefore $$12 = 3c+2d+e+0f-g-2h-\&c.,$$

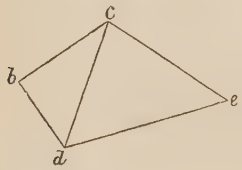
or $$3c+2d+e = 12+g+2h+\&c.;$$

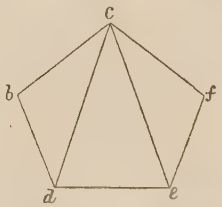
whence, if $c=0$ and $d=0$, then $e=12$ at least."

Upon this theorem Prof. Cayley founds a method of deriving all polyacra with $n+1$ summits from those with n summits. If we remove from a polyacron a tripleural summit, as a in the figure, we may derive



from it a new polyacron with one summit less by regarding the diagonal plane bcd as a face of the new solid. Conversely, we may add one summit to any polyacron by crowning any one of its faces with a tripleural summit, and then regarding this face as a diagonal plane. This process is called by Prof. Cayley the First Process. In a similar manner, the skew quadrilateral $bced$ formed by two adjacent faces

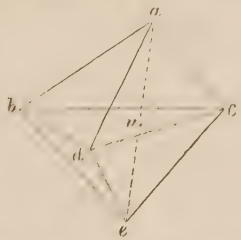




may be crowned by a tetrapleural summit a, with the edges ab, ac, ae, ad, the faces bcd, cde becoming diagonal planes of the new solid; this is called the Second Process. Again, the skew pentagon $befed$ formed by three adjacent faces may be crowned by a pentipleural summit a, with the edges ab, ac, af, ae, ad, the faces bcd, cde, cef becoming diagonal planes; this is called the Third Process. And it appears from the theorem quoted above, that every $(n+1)$-acron can be made out of an n-acron by one or other of these processes, according as it belongs to the first, second, or third case of the theorem.

I shall now show, then, that if the conditions of containing the diagonal lines and touching the single planes are precisely sufficient to determine the index-surface of an n-acron, then the same thing will be

true for any $(n+1)$-acron derived from it by either of these processes. This will prove that the theorem is true for all Δ-faced polyacra, provided we can show that it is true for all pentacra. Now there is only one pentacron, the figure formed of two tetrahedra with a common face, *abcde*. This figure has the diagonal line *ae* and the single plane



bcd; and the index-surface is the point *v*, which is precisely determined as the intersection of these.

In determining the number of conditions involved in passing through a system of lines, we must remember that every intersection of two lines diminishes the number by one, except where three or more lines are in one plane. We have only to deal with the case of three lines in one plane; the number of conditions is then reduced by two for the three intersections.

First Process.—Let D be the number of diagonal lines of the n-acron; then when we pass to the $(n+1)$-acron, the following is the increase in the number of conditions:

The D-lines are on a surface of class $n-3$ instead of $n-4$; this makes an increase	$+D$
There are $n-3$ new diagonals joining the new summit a to all the old summits except b, c, d	$+(n-2)(n-3)$
One or other of these, however, meets each of the old ones at least once; which is all that need be counted, because if any old diagonal meets two new ones a triangle is formed	$-D$
The new diagonals all meet in a point, counting as $\frac{1}{2}(n-3)(n-4)$ intersections	$-\frac{1}{2}(n-3)(n-4)$
There is a new single plane *bcd*	$+1$
The total increase is therefore	$\frac{1}{2}(n-1)(n-2)$,

which is the difference between the number of conditions required to determine a surface of class $n-3$ and the number required for class $n-4$.

Second Process.—Let D be the number of old diagonal lines less *be*; then we have the following increase:

The D lines on surface of higher class	$+D$
There are $n-4$ new diagonals	$+(n-2)(n-4)$
One or other of these, however, meets each of the D lines at least once; and, as before, this is all that need be counted	$-D$
The new diagonals all meet in a point, counting as $\frac{1}{2}(n-4)(n-5)$ intersections	$-\frac{1}{2}(n-4)(n-5)$
The edge cd becomes a diagonal line	$+(n-2)$
If, however, we join this edge to the $n-4$ summits different from $bcde$, there is a reduction 1 in the case of each; for either the plane was a single plane, or it contained one or two diagonals	$-(n-4)$
The diagonal be is on surface of higher class	$+1$
Total, as before	$\frac{1}{2}(n-1)(n-2)$.

Third Process.—Let D be the number of old diagonal lines, less df, fb, be; then we have the following increase:—

The D lines on surface of higher class	$+D$
There are $n-5$ new diagonals	$+(n-2)(n-5)$
Intersections of these with D lines	$-D$
New diagonals meet in a point	$-\frac{1}{2}(n-5)(n-6)$
The edges cd, ce become diagonal lines	$+2(n-2)$
If we join these to the $n-5$ summits different from $bcdef$, there is a reduction 1 for each plane	$-2(n-5)$
The diagonals df, fb, be on surface of higher class ...	$+3$
Their intersections with cd, ce, and of these with one another	-3
Total as before	$\frac{1}{2}(n-1)(n-2)$.

Passing now to the consideration of polygonal faces, I remark first that, by direct application of Mr. Cotterill's theorem, we have an expression for the volume of the pyramid standing on a plane polygon. For let p be the vertex of the pyramid, q any point in the plane of the polygon; then we have

$$\Sigma' \frac{(pqab)}{pX \cdot qX \cdot aX \cdot bX} = U \frac{(1234)}{1X \cdot 2X \cdot 3X \cdot 4X},$$

in which U is the ratio of the volume of the pyramid to that of the fundamental tetrahedron, and Σ' refers to a summation going round the m sides of the polygon in order. From this it appears that the expression

$$\frac{1}{qX} \cdot \Sigma' \{(pqab) \cdot \Pi' \cdot cX\},$$

(in which $\Pi'.cX$ means a product involving all the vertices of the polygon except a and b) is integral, independent of q and of the order $m-3$ in X. If we equate it to zero, we in fact obtain the equation of Mr. Cotterill's curve belonging to the polygon.

Now if this polygon form a face P of a polyacron, we obtain, as before, the following expression for the volume of the solid:—

$$\Sigma'.\frac{(pqab)}{pX.qX.aX.bX}+\Sigma.\frac{(pfgh)}{pX.fX.gX.hX}=V.\frac{(1234)}{1X.2X.3X.4X}.$$

Thus the equation of the index-surface may be written

$$\frac{\Pi.fX}{pX.qX}.\Sigma'\{(pqab).\Pi'.cX\}+\frac{1}{pX}.\Sigma\{(pfgh)\Pi.aX\}=0.$$

Here $\Pi.aX$ must in every case contain $m-2$ factors at least belonging to points on the plane P; or it vanishes in the order $m-2$ when the coordinates of P are substituted in it. The term Σ' vanishes as we have seen in the order $m-3$ in the same case. Thus P is a multiple tangent plane of order $m-3$, and the curve of contact is determined by the term Σ'; that is to say, it is Mr. Cotterill's curve belonging to the polygon.

December 12th, 1872.

Dr. HIRST, F.R.S., President, in the Chair.

Mr. R. Pendlebury, B.A., Fellow of St. John's College, Cambridge, was elected a Member; and the following gentlemen were proposed for election:—Mr. G. B. Finch, M.A., Queen's College, Cambridge, Barrister-at-law; Mr. T. O. Harding, Scholar of Trinity College, Cambridge; and Mr. John Macleod, University of Glasgow, Instructor in Mathematics in the Royal Military Academy, Woolwich.

Prof. Cayley read his paper "On the Mechanical Description of certain Quartic Curves by a Modified Oval Chuck," and exhibited the action of the instrument. Mr. Roberts spoke on the subject of the paper. Prof. Cayley then gave an account of his communication "On Geodesic Lines, especially those of a Quadric Surface." Mr. J. J. Walker made a few remarks on the breaking up of the Anharmonic Ratio Sextic. Mr. J. W. L. Glaisher next gave a description, and worked out part, of his paper "On a Deduction from Von Staudt's Property of Bernoulli's Numbers." Prof. Clifford read a paper, "Geometry on an Ellipsoid." The Secretary (Mr. Tucker) communicated two notes, one from Prof. P. G. Tait on a determinant, and the other from the Hon. J. W. Strutt on a remarkable echo.

The following presents were received :—

"The Theory of Screws; a Geometrical Study of the Kinematics, Equilibrium, and Small Oscillations of a Rigid Body. By R. Stawell Ball." (Trans. of Royal Irish Academy, vol xxv., read November 13, 1871.) From the author.

"Journal of the Institute of Actuaries and Assurance Magazine." No. lxxxix., October, 1872.

"Constitution and Laws of the Institute of Actuaries of Great Britain and Ireland, and List of Members, 1871."

"Nautical Almanac for 1876," presented by authority of the Lords Commissioners of the Admiralty.

"Monatsbericht," Juli, 1872.

"Vergleichende Betrachtungen über neuere geometrische Forschungen, von Dr. Felix Klein." Erlangen, 1872. From the author.

"Mémoires de la Société des Sciences physiques et naturelles," tome viii., 4^{me} cahier, 1872.

"Extrait du Bulletin des Sciences mathématiques et astronomiques." (Notice sur la vie de Jean-Auguste Grunert.) Paris.

"Vierteljahrsschrift der naturforschenden Gesellschaft in Zürich, redigirt von Dr. Rudolf Wolf; sechszehnten Jahrgang, viertes Heft; siebzehnten, erstes Heft." Zurich, 1871.

"Royal Society's Catalogue of Scientific Papers," vol. vi. TKA—ZYL. 1872.

"Proceedings of Royal Society," vol. xx., Nos. 137, 138.

"Catalogue of the Mathematical and Scientific Library of the late Charles Babbage." From the compiler (R. Tucker, M.A.)

"War Department Weather Map, Signal Service, U.S. Army." Washington, Friday, November 22nd, 1872, at 7.35 a.m., 4.35 p.m., 11 p.m. (3 maps.)

On the Mechanical Description of certain Quartic Curves by a Modified Oval Chuck. *By* Prof. Cayley.

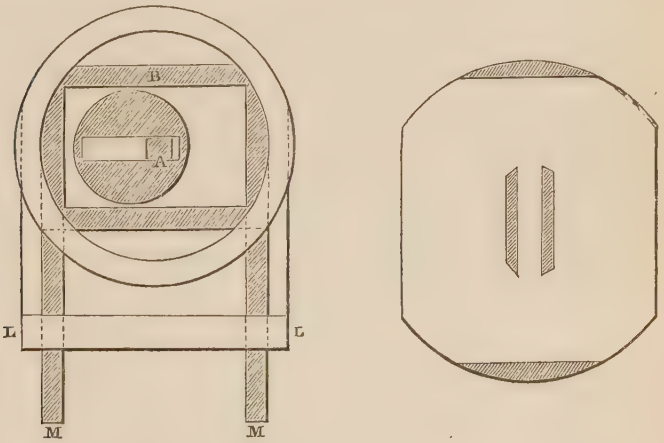
[*Read December 12th*, 1872.]

The geometrical principle of the oval chuck is the well-known one that if a plane move in such manner that two lines Ox, Oy, fixed in the plane and moveable with it, pass through two fixed points A, B, respectively, then any fixed point P traces out on the plane an ellipse. The point A is on the (geometrical) axis of the mandril; there is connected with the head a guide-ring moving horizontally; the point B is the centre of the guide-ring, this being a ring connected with the head.

moveable horizontally at right angles to the axis in such wise that the distance AB of the two centres is adjustable to any given value; the fixed point P is the tool, which practically is held on the level of the axis, that is, at a point in the line AB. The guide-ring remains fixed during the motion of the lathe.

It occurred to me that a chuck applicable to ornamental turning might be constructed by giving to the guide-ring a reciprocating motion synchronous with the rotation of the mandril; viz., for this purpose it is only necessary to affix to the axis of the mandril an excentric, working in a frame attached to the guide-ring so as to move the centre B of the guide-ring backwards and forwards along the line AB: the curve is thus that described by the fixed point P upon a plane moving in such manner that the lines Ox, Oy pass always through the points A, B respectively; the former of these being a fixed point, the latter of them a point moving according to a determinate law backwards and forwards along a fixed line through A.

The plan is carried out in a drawing apparatus which I have had constructed in wood, the axis being here vertical instead of horizontal, and the details of course different from what they would be for a lathe.

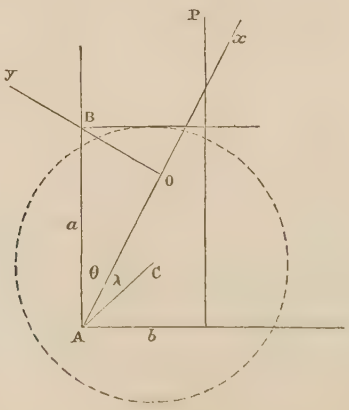


The apparatus consists of a piece of inch-board, about 10 inches long by 7 inches broad, pierced with a circular hole of 1 inch diameter for a vertical axis: the edges of the board serve as guides for the frame L, which carries the guide-ring, and resting on the board we have the frame M, itself guided by the frame L: the two frames move independently of each other, but they can be clamped together; the axis has upon it a square nut, the sides of which work in the slot of an excentric, the throw of this being adjustable by means of a screw passing through

the nut and axis, and there is above the excentric a square nut shown in the figure. This is capable of rotation round the axis, so that two of its sides may be placed either parallel with or inclined to the sides of the slot; but I fix it with two sides parallel to those of the slot by means of a screw run into the axis. The upper surfaces of the last-mentioned nut and of the guide-ring are flush with each other; and we then have a table or bed having, on its under-surface, guides which work on the outer edges of the guide-ring and on two edges of the nut. It will be observed that the bed may be placed in two different positions, viz., the guides may work on either pair of edges of the nut, those which are parallel to the sides of the slot, or those which are at right angles to it.

Supposing the bed placed as above upon the guide-ring and nut, then if the frames L and M are disconnected, and the former of them is fixed, the frame M will, on rotation of the axis, be carried backwards and forwards by the excentric, but this will in no wise affect the motion of the bed; the arrangement is then equivalent to the oval chuck, and a pencil fixed above the bed in any given position will trace out upon it an ellipse. If, however, the frame L, instead of being fixed, is clamped to the frame M, then the two frames, and therefore the guide-ring, are carried backwards and forwards by the excentric, and the curve traced out by the pencil is no longer an ellipse; it is, as I proceed to show, a special form of trinodal quartic; viz., there is a tacnode (=two nodes) at infinity, and a third node, which may be a crunode, cusp, or acnode. In the last mentioned case, the acnode or conjugate point is, as usual, not exhibited by the mechanical description, and the curve has no visible singularity.

Let the coordinates of the fixed point P, referred to axes through A, the first of them perpendicular to, and the second coincident with, AB, be b, c; let the distance AB be $=a$; and let θ denote the angle BAO:



then, if x, y are the coordinates of P referred to the origin O and axes Ox, Oy, we have

$$\begin{aligned} x+a\cos\theta &= \quad b\sin\theta+c\cos\theta, \\ y &= -b\cos\theta+c\sin\theta, \end{aligned}$$

which, if a be constant, gives a quadric equation, or the curve is an ellipse; and, in particular, if $b=0$, that is, if the point P is on the line AB, then we have

$$x=(c-a)\cos\theta, \quad y=c\sin\theta,$$

or the curve is

$$\frac{x^2}{(c-a)^2}+\frac{y^2}{c^2}=1.$$

But if a is a given function of θ, then the equation is still found by eliminating θ between the two equations for x and y. In particular, if the distance AB is given as the perpendicular upon the tangent of a circle, as shown in the figure, then if k be the radius AC of this circle, and λ the inclination of AC to Ax (k and λ being taken to be constants),

we have

$$a=k\cos(\theta+\lambda),$$

and the equations are

$$\begin{aligned} x &= \quad b\sin\theta+\{c-k\cos(\theta+\lambda)\}\cos\theta, \\ y &= -b\cos\theta+c\sin\theta. \end{aligned}$$

The elimination is nearly the same as if b were $=0$; viz., we may determine γ, α in such wise that

$$\begin{aligned} b\sin\theta+c\cos\theta &= \gamma\cos(\theta+\alpha), \quad &&= \gamma\cos\phi \text{ suppose}, \\ -b\cos\theta+c\sin\theta &= \gamma\sin(\theta+\alpha), \quad &&= \gamma\sin\phi; \end{aligned}$$

and then

$$\begin{aligned} x &= \gamma\cos\phi-k\cos(\phi+\lambda-\alpha)\cos(\phi-\alpha), \\ y &= \gamma\sin\phi. \end{aligned}$$

I will, for greater simplicity, at once write $b=0$: the equations thus are

$$\begin{aligned} x &= (c-k\cos\theta\cos\lambda+k\sin\theta\sin\lambda)\cos\theta, \\ y &= c\sin\theta; \end{aligned}$$

or say the first is

$$x=(c+k\sin\theta\sin\lambda)\cos\theta-k\cos^2\theta\cos\lambda;$$

whence we have

$$x+\frac{k\cos\lambda}{c^2}(c^2-y^2)=\left(c+\frac{ky}{c}\sin\lambda\right)\frac{1}{c}\sqrt{c^2-y^2},$$

that is,

$$c^2x+k\cos\lambda\,(c^2-y^2)=(c^2+ky\sin\lambda)\sqrt{c^2-y^2},$$

an equation which will assume a more simple form if either $\lambda=0$ or $\lambda=90°$; that is, if in the apparatus the nut-sides which guide the bed are either at right angles, or parallel, to the sides of the slot.

Taking the general case, and writing for convenience $\cos\theta = \xi$, $\sin\theta = \eta$, the curve is given by equations of the form

$$x = (\xi, \eta, 1)^2,$$
$$y = (\xi, \eta, 1)^2,$$
$$\xi^2+\eta^2 = 1;$$

viz., the elimination of ξ, η from these equations leads to the equation of the curve. The points of the curve have thus a $(1, 1)$ correspondence with those of the circle $\xi^2+\eta^2 = 1$; or, the circle being unicursal, the curve is also unicursal. Moreover, considering the intersections of the curve with an arbitrary line $ax+by+c = 0$, the points of intersection correspond to the points of intersection of the circle by the quadric $a(\xi, \eta, 1)^2+b(\xi, \eta, 1)^2+c = 0$; viz., there are four points of intersection, or the curve is a quartic, and hence it is a binodal quartic. But it is a binodal quartic of a special form: to show this more clearly, I introduce for homogeneity the coordinates z, ζ, so that the foregoing equations become

$$x : y : z = (\xi, \eta, \zeta)^2 : (\xi, \eta, \zeta)^2 : \zeta^2,$$
$$\xi^2+\eta^2-\zeta^2 = 0;$$

the curve corresponding to these equations is, as just seen, a binodal quartic. But in the case in hand the form is the more special one,

$$x : y : z = (\xi, \eta, \zeta)^2 : \xi\zeta : \zeta^2,$$
$$\xi^2+\eta^2-\zeta^2 = 0.$$

The intersections by the arbitrary line $ax+by+cz = 0$ are the points corresponding to the intersections of the circle $\xi^2+\eta^2-\zeta^2 = 0$ by the quadric $a(\xi, \eta, \zeta)^2+b\xi\zeta+c\zeta^2 = 0$, giving four intersections. But the intersections by the line $by+cz = 0$ (that is, by any line through the point $y=0$, $z=0$) are obtained from the equation $b\xi\zeta+c\zeta^2 = 0$; viz., this breaks up into $b\xi+c\zeta = 0$, $\zeta=0$, and the last factor combined with the equation of the circle gives $\zeta=0$, $\xi^2+\eta^2=0$, the two circular points at infinity, corresponding each to the point $y=0$, $z=0$: the other factor gives points corresponding to two variable points on the curve; that is, a line through the point $y=0$, $z=0$ meets the curve in this point twice and in two other points. Again, making $b=0$, or taking the line to be the line at infinity $z=0$, the equations then are $\zeta^2=0$, $\xi^2+\eta^2 = 0$; viz., we then have the circular points at infinity each twice, corresponding to the point $y=0$, $z=0$ four times, and no other point; that is, the line $z=0$ meets the curve in the point $y=0$, $z=0$ four times. We thus see that the curve has at $y=0$, $z=0$, that is, at infinity on the line $y=0$, a tacnode (counting as two nodes), the tangent at this point being the line at infinity $z=0$. The curve being trinodal, has of course one other node.

On Geodesic Lines, in particular those of a Quadric Surface.

By Prof. Cayley, F.R.S.

[*Read December 12th*, 1872.]

The present Memoir contains an investigation of the differential equation (of the second order) of the geodesic lines on a surface, the coordinates of a point on the surface being regarded as given functions of two parameters p, q, and researches in connection therewith; a deduction of Jacobi's differential equation of the first order in the case of a quadric surface, the parameters p, q being those which determine the two sets of curves of curvature; formulæ where the parameters are those which determine the two right lines through the surface; and a discussion of the forms of the geodesic lines in the two cases of an ellipsoid and a skew hyperboloid respectively.

Preliminary Formulæ.

1. I call to mind the fundamental formulæ in the Memoir by Gauss, *Disquisitiones generales circa superficies curvas*, Comm. Gott. recent. t. vi., 1827, (reprinted as an Appendix in Liouville's edition of Monge,) together with some that I have added to them. The coordinates x, y, z of a point on a surface are regarded as given functions of two parameters p, q, these expressions of x, y, z in effect determining the equation of the surface, and we have

$$dx+\tfrac{1}{2}d^2x = a\,dp+a'\,dq+\tfrac{1}{2}(a\,dp^2+2a'\,dp\,dq+a''\,dq^2),$$
$$dy+\tfrac{1}{2}d^2y = b\,dp+b'\,dq+\tfrac{1}{2}(\beta\,dp^2+2\beta'\,dp\,dq+\beta''\,dq^2),$$
$$dz+\tfrac{1}{2}d^2z = c\,dp+c'\,dq+\tfrac{1}{2}(\gamma\,dp^2+2\gamma'\,dp\,dq+\gamma''\,dq^2),$$
$$\mathrm{A},\ \mathrm{B},\ \mathrm{C} = bc'-b'c,\ ca'-c'a,\ ab'-a'b;$$

whence differential equation of surface is

$$\mathrm{A}dx+\mathrm{B}dy+\mathrm{C}dz = 0.$$

Also $$\mathrm{E},\ \mathrm{F},\ \mathrm{G} = a^2+b^2+c^2,\ aa'+bb'+cc',\ a'^2+b'^2+c'^2;$$

so that element of length on the surface is given by

$$dx^2+dy^2+dz^2 = \mathrm{E}dp^2+2\mathrm{F}dp\,dq+\mathrm{G}dq^2;$$

or as I write it,
$$= (\mathrm{E},\ \mathrm{F},\ \mathrm{G}\,\text{⨟}\,dp,\ dq)^2;$$

and moreover
$$\mathrm{V}^2 = \mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2 = \mathrm{EG}-\mathrm{F}^2.$$

The equation $(\mathrm{E},\ \mathrm{F},\ \mathrm{G}\,\text{⨟}\,dp,\ dq)^2 = 0$ determines at each point on the surface two directions (necessarily imaginary) which are called the "circular" directions. Passing on the surface from point to point along the circular directions, we obtain two series of curves (always imaginary) which are the "circular" curves; the equation $(\mathrm{E},\mathrm{F},\mathrm{G}\,\text{⨟}\,dp,dq)^2 = 0$ is the differential equation of these curves; and if we have $\mathrm{E}=0$, $\mathrm{G}=0$, then this becomes $dp\,dq=0$; viz., we have in this case $p=$const. and $q=$const. as the equations of the two sets of circular curves re-

spectively. It is clear *à priori*, and will be shown analytically in the sequel, that the circular curves are geodesic lines.

I write also

$$\mathrm{E}', \mathrm{F}', \mathrm{G}' = \mathrm{A}\alpha+\mathrm{B}\beta+\mathrm{C}\gamma,\ \mathrm{A}\alpha'+\mathrm{B}\beta'+\mathrm{C}\gamma',\ \mathrm{A}\alpha''+\mathrm{B}\beta''+\mathrm{C}\gamma'',$$

or what is the same thing, $\mathrm{E}', \mathrm{F}', \mathrm{G}'$ represent the determinants

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha, & \beta, & \gamma \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha', & \beta', & \gamma' \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}, \text{ respectively,}$$

(these last symbols do not occur in Gauss).

2. Radius of curvature of normal section corresponding to direction $dp : dq$ is given by $\dfrac{\rho}{\mathrm{V}} = \dfrac{(\mathrm{E}, \mathrm{F}, \mathrm{G}\,\mathrm{X}\,dp, dq)^2}{(\mathrm{E}', \mathrm{F}', \mathrm{G}'\,\mathrm{X}\,dp, dq)^2}$;

whence it appears that the directions of the inflexional or chief tangents (the *Haupt-tangenten*) are determined by the equation

$$(\mathrm{E}', \mathrm{F}', \mathrm{G}'\,\mathrm{X}\,dp, dq)^2 = 0.$$

The directions in question are imaginary on a surface such as the ellipsoid where the curvatures are in the same direction, but on a concavo-convex surface they are real; and in particular on the hyperboloid they coincide with the directions of the generating lines. We may on any surface pass from point to point along the chief directions; we have thus on the surface two sets of curves which are the chief curves; the differential equation of these is $(\mathrm{E}', \mathrm{F}', \mathrm{G}'\,\mathrm{X}\,dp, dq)^2 = 0$; and in particular if $\mathrm{E}'=0$, $\mathrm{G}'=0$, then this becomes $dp\,dq = 0$, or we have $p=$const., $q=$const. for the equations of the two sets of chief curves respectively. On the hyperboloid the chief curves are the two sets of generating lines. The chief curves are not in general geodesic lines, but on the hyperboloid, *quà* straight lines, they are, it is clear, geodesic lines.

3. The directions of the curves of curvature, or the principal tangents, and the corresponding values of the radius of curvature are determined by

$$\frac{\rho}{\mathrm{V}} = \frac{\mathrm{E}dp+\mathrm{F}dq}{\mathrm{E}'dp+\mathrm{F}'dq} = \frac{\mathrm{F}dp+\mathrm{G}dq}{\mathrm{F}'dp+\mathrm{G}'dq};$$

or what is the same thing, these directions are determined by the equation

$$\begin{vmatrix} dq^2, & -dq\,dp, & dp^2 \\ \mathrm{E}, & \mathrm{F}, & \mathrm{G} \\ \mathrm{E}', & \mathrm{F}', & \mathrm{G}' \end{vmatrix} = 0.$$

The same equations may be written

$$-dq : dp = \frac{\rho\mathrm{E}'-\mathrm{V}\mathrm{E}}{\rho\mathrm{F}'-\mathrm{V}\mathrm{F}} = \frac{\rho\mathrm{F}'-\mathrm{V}\mathrm{F}}{\rho\mathrm{G}'-\mathrm{V}\mathrm{G}};$$

that is, the principal radii of curvature are determined by the equation

$$\rho^2(\mathrm{E}'\mathrm{G}'-\mathrm{F}'^2)-\rho\mathrm{V}(\mathrm{E}\mathrm{G}'+\mathrm{E}'\mathrm{G}-2\mathrm{F}\mathrm{F}')+\mathrm{V}^2(\mathrm{E}\mathrm{G}-\mathrm{F}^2) = 0.$$

(last term is $=V^4$, but it is better to retain the original form): and then, ρ being either root, the last preceding equations give the direction of the curve of curvature corresponding to the given value of the radius of curvature.

If $p=\text{const.}$, $q=\text{const.}$ are the equations of the two systems of curves of curvature respectively, then the quadric equation in (dp, dq) must become $dpdq=0$; this will be so if $F=0$, $F'=0$; and we thus have these equations, viz., written at full length they are

$$d_p x\, d_q x + d_p y\, d_q y + d_p z\, d_q z = 0,$$

$$\begin{vmatrix} d_p x, & d_p y, & d_p z \\ d_q x, & d_q y, & d_q z \\ d_p d_q x, & d_p d_q y, & d_p d_q z \end{vmatrix} = 0,$$

as the conditions in order that $p=\text{const.}$, $q=\text{const.}$ may be the two systems of curves of curvature. The former of these equations merely expresses that the two sets of curves always intersect at right angles.

General Theory of the Geodesic Lines on a Surface.

4. I now proceed to investigate the theory of geodesic lines on a surface, the surface being determined as above by means of given expressions of the coordinates x, y, z in terms of the parameters p, q.

The differential equation obtained by Gauss for the geodesic lines is in a form not symmetrical in regard to the two variables; viz., his equation is

$$\frac{dE}{dq}\,dp^2 + 2\,\frac{dF}{dp}\,dp\,dq + \frac{dG}{dp}\,dp^2 = 2ds\,d\,\frac{E dp + F dq}{ds},$$

where, as above, $ds^2 = (E, F, G \mathbin{\text{\textbigtimes}} dp, dq)^2$.

If we herein consider p, q as functions of a parameter θ, and write

for shortness $d_\theta p, d_\theta q, d_\theta^2 p$, &c. $= p', q', p''$, &c.,

also $\Omega = (E, F, G \mathbin{\text{\textbigtimes}} p', q')^2$,

and $d_p E = E_1$, $d_q E = E_2$, &c.,

then the equation is

$$(E_1, F_1, G_1)(p', q')^2 - 2\sqrt{\Omega}\left(\frac{Ep' + Fq'}{\sqrt{\Omega}}\right)' = 0.$$

We have $$\left(\frac{Ep' + Fq'}{\sqrt{\Omega}}\right)' = \frac{1}{\Omega\sqrt{\Omega}}(M+N),$$

where N is the part containing p'', q'', which I will first calculate; viz., we have

$$\begin{aligned} N &= \Omega(Ep'' + Fq'') - (Ep' + Fq')\,\tfrac{1}{2}\Omega' \\ &= \Omega(Ep'' + Fq'') - (Ep' + Fq')\{(Ep' + Fq')\,p'' + (Fp' + Gq')\,q''\} \\ &= p''\{E\Omega - (Ep' + Fq')^2\} + q''\{F\Omega - (Ep' + Fq')(Fp' + Gq')\}; \end{aligned}$$

or substituting for Ω its value, this is

$$= p''(EG-F^2)\,q'^2 - q''(EG-F^2)\,p'q', \quad = -q'(EG-F^2)\,(p'q''-p''q');$$

wherefore

$$\left(\frac{Ep'+Fq'}{\sqrt{\Omega}}\right)' = \frac{1}{\Omega\sqrt{\Omega}}\{M - q'(EG-F^2)\,(p'q''-p''q')\},$$

and the equation becomes

$$\Omega\,(E_1p'^2+2F_1p'q'+G_1q'^2) - 2M + 2q'(EG-F^2)\,(p'q''-p''q') = 0;$$

whence we foresee that the whole equation must divide by q'.

5. We have

$$\begin{aligned} M = {} & (p'dE+q'dF)\,\Omega - (Ep'+Fq')\,(\tfrac{1}{2}p'^2dE + p'q'dF + \tfrac{1}{2}q'^2dG) \\ = {} & dE\,\{p'\Omega - \tfrac{1}{2}p'^2\,(Ep'+Fq')\} \\ & + dF\,\{q'\Omega - p'q'\,(Ep'+Fq')\} \\ & + dG\,\{\quad - \tfrac{1}{2}q'^2\,(Ep'+Fq')\}, \end{aligned}$$

or say

$$\begin{aligned} 2M = {} & dE\,[\;\; p'^2\,(p'E+q'F) + 2p'q'\,(p'F+q'G)] \\ & + dF\,[\qquad\qquad\quad 2q'^2\,(p'F+q'G)] \\ & + dG\,[-q'^2\,(p'E+q'F)\qquad\qquad\quad] \\ = {} & (E_1p'+E_2q')\,[\;\; p'^2\,(Ep'+Fq') + 2p'q'\,(Fp'+Gq')] \\ & + (F_1p'+F_2q')\,[\qquad\qquad\quad 2q'^2\,(Fp'+Gq')] \\ & + (G_1p'+G_2q')\,[-q'^2\,(Ep'+Fq')\qquad\qquad\quad]. \end{aligned}$$

The term in p'^4 is $EE_1p'^4$, which is also the term in p'^4 of $\Omega\,(E_1p'^2+2F_1p'q'+G_1q'^2)$; whence $\Omega\,(E_1p'^2+2F_1p'q'+G_1q'^2)-2M$ divides by q'.

Proceeding to the reduction:

Term in E_1 is

$$E_1\,.\,p'^2\Omega - p'^3\,(Ep'+Fq') - 2p'^2\,q'\,(Fp'+Gq') = E_1\,.\,-p'^2q'\,(Fp'+Gq');$$

term in F_1 is

$$F_1\,.\,2p'q'\Omega - 2p'q'^2\,(Fp'+Gq') = F_1\,.\,2p'^2q'\,(Ep'+Fq');$$

term in G_1 is

$$G_1\,.\,q'^2\Omega + p'q'^2\,(Ep'+Fq') = G_1\,\{2p'q'^2\,(Ep'+Fq') + q'^3\,(Fp'+Gq')\}.$$

6. The remaining terms in E_2, F_2, G_2 require no reduction, and the result is

$$\begin{aligned} & E_1\,\{-p'^2\,(Fp'+Gq')\} - E_2\,\{\;p'^2\,(Ep'+Fq') + 2p'q'\,(Fp'+Gq')\} \\ & + F_1\,\{\;2p'^2\,(Ep'+Fq')\} - F_2\,\{2q'^2\,(Fp'+Gq')\} \\ & + G_1\,\{2p'q'\,(Ep'+Fq') + q'^2\,(Fp'+Gq')\} - G_2\,\{-q'^2\,(Ep'+Fq')\} \\ & \qquad + 2\,(EG-F^2)\,(p'q''-p''q') = 0, \end{aligned}$$

or what is the same thing,

$$\begin{aligned}&(\mathrm{E}p'+\mathrm{F}q')\,\{(2\mathrm{F}_1-\mathrm{E}_2)\,p'^2+2\mathrm{G}_1\,p'q'+\mathrm{G}_2q'^2\}\\-&(\mathrm{F}p'+\mathrm{G}q')\,\{\mathrm{E}_1\,p'^2+2\mathrm{E}_2\,p'q'+(2\mathrm{F}_2-\mathrm{G}_1)\,q'^2\}\\+&2\,(\mathrm{EG}-\mathrm{F}^2)\,(p'q''-p''q')=0,\end{aligned}$$

which is the required differential equation of the second order: the independent variable has been taken to be the arbitrary quantity θ; but taking $\theta=p$, or q, say $\theta=p$, we have $p'=1$, $p''=0$, and the equation then contains (besides p and q) only q'' and q', that is, d_p^2q and d_pq, and is therefore a differential relation between p and q.

7. Instead of starting, as above, from the equation given by Gauss, we may use the geometrical property that at each point of a geodesic line the osculating plane passes through the normal of the surface.

Considering, as above, p, q as functions of a variable θ, then, θ becoming $\theta+d\theta$, the new values of p, q are

$$p+p'd\theta+\tfrac{1}{2}p''d\theta^2,\quad q+q'd\theta+\tfrac{1}{2}q''d\theta^2;$$

and that of x is

$$x+\alpha\,(p'd\theta+\tfrac{1}{2}p''d\theta^2)+\alpha'(q'd\theta+\tfrac{1}{2}q''d\theta^2)+\tfrac{1}{2}\,(\alpha p'^2+2\alpha'p'q'+\alpha''q'^2)\,d\theta^2;$$

or calling this $x+x'd\theta+\frac{1}{2}x''d\theta^2$, we have

$$\begin{aligned}&x'=\alpha p'+\alpha'q',\quad x''=\alpha p''+\alpha'q''+\alpha p'^2+2\alpha'p'q'+\alpha''q'^2,\\\text{and so}\quad&y'=\beta p'+\beta'q',\quad y''=\beta p''+\beta'q''+\beta p'^2+2\beta'p'q'+\beta''q'^2,\\&z'=\gamma p'+\gamma'q',\quad z''=\gamma p''+\gamma'q''+\gamma p'^2+2\gamma'p'q'+\gamma''q'^2.\end{aligned}$$

The condition in question is expressed by the equation

$$\begin{vmatrix}\mathrm{A}, & \mathrm{B}, & \mathrm{C}\\ x', & y', & z'\\ x'', & y'', & z''\end{vmatrix}=0,$$

that is,

$$\begin{vmatrix}\mathrm{A}, & \mathrm{B}, & \mathrm{C}\\ \alpha p'+\alpha'q', & \beta p'+\beta'q', & \gamma p'+\gamma'q'\\ \alpha p''+\alpha'q'', & \beta p''+\beta'q'', & \gamma p''+\gamma'q''\end{vmatrix}$$

$$+\begin{vmatrix}\mathrm{A}, & \mathrm{B}, & \mathrm{C}\\ \alpha p'+\alpha'q', & \beta p'+\beta'q', & \gamma p'+\gamma'q'\\ \alpha p'^2+2\alpha'p'q'+\alpha''q'^2, & \beta p'^2+2\beta'p'q'+\beta''q'^2, & \gamma p'^2+\gamma'p'q'+\gamma''q'^2\end{vmatrix}=0$$

8. The first determinant is the sum of three terms such as $\mathrm{A}\,(bc'-b'c)\,(p'q''-p''q')$; viz., this is $\mathrm{A}^2(p'q''-p''q')$, or the determinant is $(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2)\,(p'q''-p''q')$, $=(\mathrm{EG}-\mathrm{F}^2)\,(p'q''-p''q')$.

The second determinant is the sum of three terms such as $(\alpha p'^2+2\alpha'p'q'+\alpha''q'^2)\,[\mathrm{B}\,(cp'+c'q')-\mathrm{C}\,(bp'+b'q')]$, where the factor in [ ] is $p'\,[c(c\alpha'-c'\alpha)-b(ab'-a'b)]+q'\,[c'(c\alpha'-c'\alpha)-b'(ab'-a'b)]$,

which is $= p'(a'E-aF)+q'(a'F-aG)$. We have thus the second determinant, and the equation becomes

$$\begin{aligned}&(EG-F^2)\,(p'q''-p''q')\\&\quad+(ap'^2+2a'p'q'+a''q'^2)\,\{p'(a'E-aF)+q'(a'F-aG)\}\\&\quad+(\beta p'^2+2\beta'p'q'+\beta''q'^2)\,\{p'(b'E-bF)+q'(b'F-bG)\}\\&\quad+(\gamma p'^2+2\gamma'p'q'+\gamma''q'^2)\,\{p'(c'E-cF)+q'(c'F-cG)\}=0,\end{aligned}$$

an equation of the same form as that previously obtained, and which can be without difficulty identified therewith.

The Circular Curves are Geodesics.

9. I proceed to show that the circular curves are geodesics; viz., that an integral of the geodesic equation is

$$(E, F, G \mathbin{\text{\textbigtimes}} p', q')^2 = 0.$$

Starting from this equation, we have

$$\begin{aligned}&2\,\{(Ep'+Fq')\,p''+(Fp'+Gq')\,q''\}\\&+(E_1p'+E_2q')\,p'^2+2\,(F_1p'+F_2q')\,p'q'+(G_1p'+G_2q')\,q'^2=0.\end{aligned}$$

Now the equation, writing therein $Ep'+Fq'=\lambda q'$, gives $Fp'+Gq'=-\lambda p'$: these equations may be written

$$\begin{aligned}Ep'+(F-\lambda)\,q'&=0,\\(F+\lambda)\,p'+G\quad q'&=0,\end{aligned}$$

the value of λ being therefore $-\lambda^2=EG-F^2$. The result just obtained thus becomes

$$\begin{aligned}&2\lambda\,(p''q'-p'q'')\\&\quad+[(E_1p'+E_2q')\,p'+(F_1p'+F_2q')\,q']\,.\,-\frac{1}{\lambda}\,(Fp'+Gq')\\&\quad+[(F_1p'+F_2q')\,p'+(G_1p'+G_2q')\,q']\,.\quad\frac{1}{\lambda}\,(Ep'+Fq'),\end{aligned}$$

that is,

$$\begin{aligned}&2\,(EG-F^2)\,(p'q''-p''q')\\&\quad-(Fp'+Gq')\,[E_1p'^2+(E_2+F_1)\,p'q'+F_2q'^2]\\&\quad+(Ep'+Fq')\,[F_1p'^2+(F_2+G_1)\,p'q'+G_2q'^2]=0;\end{aligned}$$

or what is the same thing, adding hereto the zero value

$$\Delta\,(E, F, G \mathbin{\text{\textbigtimes}} p', q')^2, = (Fp'+Gq')\,\Delta q'+(Ep'+Fq')\,\Delta p',$$

where Δ is arbitrary, the equation is

$$\begin{aligned}&2\,(EG-F^2)\,(p'q''-p''q')\\&\quad+(Ep'+Fq')\,[F_1p'^2+(F_2+G_1)\,p'q'+G_2q'^2+\Delta p']\\&\quad-(Fp'+Gq')\,[E_1p'^2+(E_2+F_1)\,p'q'+F_2q'^2-\Delta q']=0;\end{aligned}$$

viz., taking $\Delta=(F_1-E_2)\,p'+(G_1-F_2)\,q'$, this agrees with the geodesic equation.

The foregoing integral, $(E, F, G \mathbin{\text{\textbigtimes}} p', q')^2 = 0$, is, I believe, a particular, not a singular, solution of the geodesic equation.

The Chief Lines are not in general Geodesics.

10. That the chief lines are not in general geodesics appears most readily as follows:

To find the condition in order that $p = \text{const.}$ may be a geodesic, we write in the geodesic equation $p' = 0$, $p'' = 0$: the equation thus becomes

$$Fq' \cdot G_2 q'^2 - Gq'(2F_2 - G_1) q'^2 = 0;$$

that is, we have

$$FG_2 - 2F_2 G + GG_1 = 0$$

as the condition in order that $p = \text{const.}$ may be a geodesic: the condition that it may be a chief curve is $G' = 0$, which is a different condition.

We have of course, in like manner,

$$2EF_1 - E_1 F - EE_2 = 0$$

as the condition in order that $q = \text{const.}$ may be a geodesic; and $E' = 0$ as the condition that this may be a chief curve. If $p = \text{const.}$, $q = \text{const.}$ are each of them at once a geodesic and a chief curve, then the four equations must all be satisfied, viz., we must have

$$FG_2 - 2F_2 G + GG_1 = 0, \quad 2EF_1 - E_1 F - EE_2 = 0,$$
$$G' = 0, \quad E' = 0.$$

Special Form of the Geodesic Equation.

11. In the case where the curves $p = \text{const.}$, $q = \text{const.}$ intersect at right angles (and in particular when these are the curves of curvature), we have $F = 0$; whence also $F_1 = 0$, $F_2 = 0$; and the geodesic equation assumes the more simple form,

$$\begin{aligned} &Ep'(-E_2 p'^2 + 2G_1 p'q' + G_2 q'^2) \\ &-Gq'(\ E_1 p'^2 + 2E_2 p'q' - G_1 q'^2) \\ &+2EG(p'q'' - p''q') = 0. \end{aligned}$$

Geodesics on a Quadric Surface.

12. In the case of a quadric surface $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$, writing for shortness $\alpha, \beta, \gamma = b-c, c-a, a-b$, we may express the coordinates x, y, z in terms of two parameters p, q as follows:

$$\begin{aligned} -\beta\gamma x^2 &= a(a+p)(a+q), \\ -\gamma\alpha y^2 &= b(b+p)(b+q), \\ -\alpha\beta z^2 &= c(c+p)(c+q), \end{aligned}$$

where, in fact, $p = \text{const.}$, $q = \text{const.}$ are the equations of the two sets of curves of curvature respectively. Writing moreover

$$\mathrm{P} = \frac{p}{(a+p)(b+p)(c+p)}, \quad \mathrm{Q} = \frac{q}{(a+q)(b+q)(c+q)},$$

we have $ds^2 = \frac{1}{4}(p-q)(\mathrm{P}dp^2 - \mathrm{Q}dq^2)$,

that is, $\mathrm{E}, \mathrm{F}, \mathrm{G} = \frac{1}{4}(p-q)\mathrm{P},\ 0,\ \frac{1}{4}(q-p)\mathrm{Q}$;

and the geodesic equation becomes

$$\begin{aligned}
&\mathrm{P}p'\{\mathrm{P}p''^2 - 2\mathrm{Q}p'q' + (\mathrm{Q} + \overline{q-p}\,\mathrm{Q}')\,q''^2\}\\
&+\mathrm{Q}q'\{(\mathrm{P} + \overline{p-q}\,\mathrm{P}')\,p''^2 - 2\mathrm{P}p'q' + \mathrm{Q}q''^2\}\\
&-2(p-q)\,\mathrm{PQ}(p'q'' - p''q') = 0,
\end{aligned}$$

where P′, Q′ stand for $d_p\mathrm{P}$ and $d_q\mathrm{Q}$ respectively; viz., this is

$$\begin{aligned}
&p'^3 \,.\quad \mathrm{P}^2\\
&+p'^2q' \,.\, -\mathrm{PQ} + (p-q)\,\mathrm{P}'\mathrm{Q}\\
&+p'\,q'^2 \,.\, -\mathrm{PQ} + (q-p)\,\mathrm{PQ}'\\
&+\quad q'^3 \,.\quad \mathrm{Q}^2\\
&-2(p-q)\,\mathrm{PQ}(p'q'' - p''q') = 0.
\end{aligned}$$

13. This has a first integral,

$$p'\sqrt{\left(\frac{\mathrm{P}}{\theta+p}\right)} + q'\sqrt{\left(\frac{\mathrm{Q}}{\theta+q}\right)} = 0,$$

where θ is the constant of integration; or say this is

$$\theta(p'^2\mathrm{P} - q'^2\mathrm{Q}) + p'^2q\mathrm{P} - q'^2p\mathrm{Q} = 0;$$

viz., differentiating logarithmically, this gives

$$\begin{aligned}
&\frac{2p'p''\mathrm{P} - 2q'q''\mathrm{Q} + p'^3\mathrm{P}' - q'^3\mathrm{Q}'}{p'^2\mathrm{P} - q'^2\mathrm{Q}}\\
&= \frac{2p'p''q\mathrm{P} - 2q'q''p\mathrm{Q} + p'^2(qp'\mathrm{P}' + q'\mathrm{P}) - q'^2(pq'\mathrm{Q}' + p'\mathrm{Q})}{p'^2q\mathrm{P} - q'^2p\mathrm{Q}},
\end{aligned}$$

which, multiplying out the denominators, is in fact the foregoing geodesic equation. To verify, consider first the part involving p'', q'': this is

$$(2p'p''\mathrm{P} - 2q'q''\mathrm{Q})(p'^2q\mathrm{P} - q'^2p\mathrm{Q}) - (2p'p''q\mathrm{P} - 2q'q''p\mathrm{Q})(p'^2\mathrm{P} - q'^2\mathrm{Q}),$$

which is $= 2p'p''\mathrm{P}\,.\,q'^2\mathrm{Q}\,(q-p) - 2q'q''\mathrm{Q}\,.\,p'^2\mathrm{P}\,(q-p)$,

that is, $= 2(q-p)\,\mathrm{PQ}p'q'(p''q' - p'q'')$,

or say $= 2(p-q)\,\mathrm{PQ}p'q'(p'q'' - p''q')$.

We have next the part

$$\begin{aligned}
&(p'^3\mathrm{P}' - q'^3\mathrm{Q}')(p'^2q\mathrm{P} - q'^2p\mathrm{Q})\\
&\qquad - \{p'^2(qp'\mathrm{P}' + q'\mathrm{P}) + q'^2(pq'\mathrm{Q}' + p'\mathrm{Q})\}\,(p'^2\mathrm{P} - q'^2\mathrm{Q}),
\end{aligned}$$

which is readily found to be

$$= -p'q'\{p'^3\mathrm{P}^2 + p'^2q'(-\mathrm{PQ} + \overline{p-q}\,\mathrm{P}'\mathrm{Q}) + p'q'^2(-\mathrm{PQ} + \overline{q-p}\,\mathrm{PQ}') + q'^3\mathrm{Q}^2\}$$

and the equation is thus verified.

14. We have consequently

$$dp\sqrt{\left(\frac{p}{(a+p)(b+p)(c+p)(\theta+p)}\right)}+dq\sqrt{\left(\frac{q}{(a+q)(b+q)(c+q)(\theta+q)}\right)}=0,$$

involving the arbitrary constant θ as the differential equation of the first order of the geodesic lines on the quadric surface $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$: the geodesic lines in question all touch the curve of curvature determined by the parameter θ, that is, the curve which is the intersection of the surface by the confocal surface $\frac{x^2}{a+\theta}+\frac{y^2}{b+\theta}+\frac{z^2}{c+\theta}=1$.

15. In the particular case $\theta=\infty$, the equation becomes

$$Pdp^2-Qdq^2=0,$$

that is, $$\frac{p\,dp^2}{(a+p)(b+p)(c+p)}-\frac{q\,dq^2}{(a+q)(b+q)(c+q)}=0,$$

which is the differential equation of the circular curves on the surface.

16. The signification of the case $\theta=0$ is not at first sight so obvious. Supposing that θ is first indefinitely small, and writing the equation in the form $$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1-\theta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+\&c.=0,$$

we have the series of geodesics touching the (imaginary) curve of curvature, the intersection of the surface by the imaginary cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$. *These are, in fact, the right lines on the surface:* I apprehend that the intersection in question is not a proper envelope, but is the locus of nodes of the geodesics, viz., each geodesic is to be considered as a pair of lines belonging to the two sets: I do not, however, quite understand this.

17. I say that the geodesics in question are the right lines on the surface; viz., writing in the differential equation $\theta=0$, it is to be shown that the differential equation of the right lines is

$$\frac{dp}{\sqrt{(a+p)(b+p)(c+p)}}+\frac{dq}{\sqrt{(a+q)(b+q)(c+q)}}=0,$$

or what is the same thing, that the integral of this equation represents the right lines on the surface.

Writing the equation of the surface in the form $\frac{x^2}{a}+\frac{y^2}{b}=1-\frac{z^2}{c}$, we have at once

$$\frac{x}{\sqrt{a}}+\frac{iy}{\sqrt{b}}=\sigma\left(1+\frac{z}{\sqrt{c}}\right),$$

$$\frac{x}{\sqrt{a}}-\frac{iy}{\sqrt{b}}=\frac{1}{\sigma}\left(1-\frac{z}{\sqrt{c}}\right),$$

(σ an arbitrary parameter) as the equations of a right line on the surface; viz., considering x, y, z as denoting the foregoing functions of p and q, these two equations are forms of a single relation between p, q, σ, which relation expresses that the point (p, q) is situate on the right line determined by the parameter σ. We may from this integral equation deduce without difficulty the foregoing differential equation;

viz., we have

$$\frac{dx}{\sqrt{a}} + \frac{idy}{\sqrt{b}} = \sigma \frac{dz}{\sqrt{c}},$$
$$\frac{dx}{\sqrt{a}} - \frac{idy}{\sqrt{b}} = -\frac{1}{\sigma} \frac{dz}{\sqrt{c}},$$

or multiplying these equations,

$$\frac{dx^2}{a} + \frac{dy^2}{b} + \frac{dz^2}{c} = 0;$$

and substituting herein for dx, dy, dz their values in terms of dp and dq, we find the required equation

$$\frac{dp^2}{(a+p)(b+p)(c+p)} - \frac{dq^2}{(a+q)(b+q)(c+q)} = 0.$$

18. I return to the integral equation involving σ: we have to rationalise this equation, that is, obtain from it an equation containing x^2, y^2, z^2, and then substituting for these their values in terms of p, q, we have the required relation between p, q, σ. We at once obtain

$$\left\{2\left(\frac{x^2}{a} - \frac{y^2}{b}\right) - \left(\sigma^2 + \frac{1}{\sigma^2}\right)\left(1 + \frac{z^2}{c}\right)\right\}^2 - 4\left(\sigma^2 - \frac{1}{\sigma^2}\right)^2 \frac{z^2}{c} = 0;$$

or if for greater convenience we introduce in place of σ a new parameter ϕ, determined by the equation $\sigma^2 + \frac{1}{\sigma^2} = \frac{2\phi}{\gamma}$, the equation is

$$\left\{\gamma\left(\frac{x^2}{a} - \frac{y^2}{b}\right) - \phi\left(1 + \frac{z^2}{c}\right)\right\}^2 - 4(\phi^2 - \gamma^2)\frac{z^2}{c} = 0.$$

Writing for shortness $p+q = \mathrm{X}$, $pq = \mathrm{Y}$, we have

$$-\beta\gamma\frac{x^2}{a} = a^2 + a\mathrm{X} + \mathrm{Y}, \quad -\gamma\alpha\frac{y^2}{b} = b^2 + b\mathrm{X} + \mathrm{Y}, \quad -\alpha\beta\frac{z^2}{c} = c^2 + c\mathrm{X} + \mathrm{Y};$$

and substituting these values, the equation becomes

$$\{\beta(b^2 + b\mathrm{X} + \mathrm{Y}) - \alpha(a^2 + a\mathrm{X} + \mathrm{Y}) - \phi\gamma(\alpha\beta - c^2 - c\mathrm{X} - \mathrm{Y})\}^2$$
$$+ 4\alpha\beta(\phi^2 - \gamma^2)(c^2 + c\mathrm{X} + \mathrm{Y}) = 0,$$

or what is the same thing,

$$\{\beta b^2 - \alpha a^2 - \phi\gamma(\alpha\beta - c^2) + \mathrm{X}(\beta b - \alpha a + \phi\gamma c) + \mathrm{Y}(\beta - \alpha + \phi\gamma)\}^2$$
$$+ 4\alpha\beta(\phi^2 - \gamma^2)(c^2 + c\mathrm{X} + \mathrm{Y}) = 0.$$

19. This is an equation, quadric as regards p, and also as regards q, viz., it is of the form

$$\begin{aligned}&(a+2hp+gp^2)\\+&2q\,(h+2bp+fp^2)\\+&\;q^2(g+2fp+cp^2)=0,\end{aligned}$$

and it leads to a differential equation

$$\frac{dp}{\sqrt{P}}+\frac{dq}{\sqrt{Q}}=0,$$

where $P=(h+2bp+fp^2)^2-(a+2hp+gp^2)(g+2fp+cp^2)$,
$Q=(h+2bq+fq^2)^2-(a+2hq+gq^2)(g+2fq+cq^2)$;

and upon effecting the calculation, it is found that we have

$$P=-8a^2\beta^2(\phi^2-\gamma^2)(a+b-2c-\phi)(a+p)(b+p)(c+p),$$
$$Q=-8a^2\beta^2(\phi^2-\gamma^2)(a+b-2c-\phi)(a+q)(b+q)(c+q),$$

viz., P, Q are the same multiples of $(a+p)(b+p)(c+p)$, $(a+q)(b+q)(c+q)$ respectively; so that, omitting the common factor, or taking P, Q to represent the last-mentioned functions respectively, we have

$$\frac{dp}{\sqrt{P}}+\frac{dq}{\sqrt{Q}}=0;$$

and since the parameter ϕ has disappeared, we see that the original equation involving ϕ is the general integral of this differential equation; viz., that the differential equation belongs to the right lines on the surface.

20. The form of the integral equation may be simplified by introducing instead of ϕ a new parameter K, connected with it by the equation $K=\dfrac{\beta b-\alpha a+\phi c}{\beta-\alpha+\phi}$, or $\phi=\dfrac{(\beta-\alpha)K-\beta b+\alpha a}{c-K}$,

viz., we thence deduce

$$\begin{aligned}\beta-\alpha+\phi&=-2\alpha\beta &&(\div),\\ \beta b-\alpha a+\phi c&=-2a\beta K &&(\div),\\ \beta b^2-\alpha a^2-\phi(a\beta-c^2)&=\;2a\beta(ac+bc-ab+2cK) &&(\div),\\ \phi+\gamma&=\;2\beta(K-b) &&(\div),\\ \phi-\gamma&=-2a(K-a) &&(\div),\end{aligned}$$

where the sign $(\div)$ is used to signify that the functions preceding it have to be divided by a denominator which in fact is $=c-K$. The equation thus becomes

$$(ac+bc-ab-2cK-KX-Y)^2-4(K-a)(K-b)(c^2+cX+Y)=0;$$

and if we moreover write

$$\nu,\ \mu,\ \lambda\ =\ abc,\ bc+ca+ab,\ a+b+c,$$

and instead of K introduce the parameter C, $=\lambda-K$, the equation becomes

$$\{-\mu-2c^2+2cC-(\lambda-C)X-Y\}^2+(b+c-C)(c+a-C)(c^2+cX+Y) = 0;$$

or expanding and reducing, this is

$$\begin{aligned}&\{Y+(\lambda-C)X\}^2\\&\quad+Y(-2\mu+4\lambda C-4C^2)\\&\quad+X(2\mu\lambda-4\nu-2\mu C)\\&\quad+\mu^2-4\nu C=0,\end{aligned}$$

or say

$$\begin{aligned}&\mu^2-4\nu C\\&+(2\mu\lambda-4\nu-2\mu C)(p+q)\\&+(-2\mu+4\lambda C-4C^2)pq\\&+(\lambda-C)^2(p^2+q^2+2pq)\\&+2(\lambda-C)pq(p+q)\\&+\quad p^2q^2\qquad\quad=0,\end{aligned}$$

viz., this, containing the constant C, is the general integral of the differential equation

$$\frac{dp}{\sqrt{P}}+\frac{dq}{\sqrt{Q}}=0,$$

where

$$\begin{aligned}P&=(a+p)(b+p)(c+p), &&=\nu+\mu p+\lambda p^2+p^3,\\Q&=(a+q)(b+q)(c+q), &&=\nu+\mu q+\lambda q^2+q^3.\end{aligned}$$

21. The constant C is connected with the parameter σ, which originally served to determine the right line, by the equation

$$\sigma+\frac{1}{\sigma}=\frac{2}{\gamma}\frac{(\beta-a)(\lambda-C)-\beta b+\alpha a}{c-(\lambda-C)},$$

or what is the same thing,

$$\sigma+\frac{1}{\sigma}=\frac{2}{b-c}\frac{2c^2-a^2-b^2-C(2c-a-b)}{C-a-b}.$$

Reverting to the equation between p, q, ϕ, I remark that if ϕ be therein considered as variable, we have the differential equation

$$\sqrt{Q}\,dp+\sqrt{P}\,dq+\sqrt{\Phi}\,d\phi=0,$$

where P, Q have the foregoing values

$$P=-8a^2\beta^2(\phi^2-\gamma^2)(a+b-c-\phi)(a+p)(b+p)(c+p),\quad Q=\&c.;$$

and where, if the integral equation be written in the form

$$L+2M\phi+N\phi^2=0,$$

then we have $\Phi=M^2-NL$, viz., we thus find

$$\Phi=16a^2\beta^2(a+p)(b+p)(c+p)(a+q)(b+q)(c+q).$$

22. Changing the notation, and writing

$$\begin{aligned} \mathrm{P} &= (a+p)(b+p)(c+p), \\ \mathrm{Q} &= (a+q)(b+q)(c+q), \\ \Phi &= (\phi^2-\gamma^2)(a+b-2c-\phi), \end{aligned}$$

the equation is $$\frac{dp}{\sqrt{\mathrm{P}}} + \frac{dq}{\sqrt{\mathrm{Q}}} + \frac{\sqrt{2}\, d\phi}{\sqrt{\Phi}} = 0;$$

or if, instead of ϕ, we introduce the original parameter σ, then, observing that $\dfrac{2d\sigma}{\sigma} = \dfrac{d\phi}{\sqrt{\phi^2-\gamma^2}}$, we at once find

$$\frac{dp}{\sqrt{\mathrm{P}}} + \frac{dq}{\sqrt{\mathrm{Q}}} + \frac{4d\sigma}{\sqrt{\Sigma}} = 0,$$

where $$\Sigma = \gamma(1+\sigma^4) - 2(a+b-2c)\,\sigma^2,$$

or what is the same thing,

$$\Sigma = a\,(\sigma^2-1)^2 - b\,(\sigma^2+1)^2 + c\,.\,4\sigma^2;$$

viz., passing from a point (p, q) on the line σ to a consecutive point $(p+dp, q+dq)$ on the line $\sigma+d\sigma$, the above is the relation between the variations $dp, dq, d\sigma$. If τ be the parameter of the other line through the same point, then we have in like manner, say

$$\frac{dp}{\sqrt{\mathrm{P}}} - \frac{dq}{\sqrt{\mathrm{Q}}} + \frac{4d\tau}{\sqrt{\mathrm{T}}} = 0,$$

(viz., one of the radicals $\sqrt{\mathrm{P}}$, $\sqrt{\mathrm{Q}}$ must present itself with a reversed sign): and we thus have dp, dq each expressed in terms of $d\sigma, d\tau$; viz., we have the increments dp, dq when a point passes from (σ, τ) to $(\sigma+d\sigma, \tau+d\tau)$. These results will be presently obtained in a more simple manner.

Formulæ where the position of a Point on the Surface is determined by means of the two Lines through the Point.

23. We may determine the position of a point by means of the parameters σ, τ of the two lines through the point. The equations of these are

$$\frac{x}{\sqrt{a}} + \frac{iy}{\sqrt{b}} = \sigma\left(1+\frac{z}{\sqrt{c}}\right), \quad \frac{x}{\sqrt{a}} + \frac{iy}{\sqrt{b}} = \tau\left(1-\frac{z}{\sqrt{c}}\right),$$

$$\frac{x}{\sqrt{a}} - \frac{iy}{\sqrt{c}} = \frac{1}{\sigma}\left(1-\frac{z}{\sqrt{c}}\right), \quad \frac{x}{\sqrt{a}} - \frac{iy}{\sqrt{b}} = \frac{1}{\tau}\left(1+\frac{z}{\sqrt{c}}\right),$$

and from these equations we deduce

$$\frac{x}{\sqrt{a}} = \frac{\sigma\tau+1}{\tau+\sigma}, \quad \frac{iy}{\sqrt{b}} = \frac{\sigma\tau-1}{\tau+\sigma}, \quad \frac{z}{\sqrt{c}} = \frac{\tau-\sigma}{\tau+\sigma}.$$

We have thence

$$\frac{dx}{\sqrt{a}} = (\tau^2-1)\,d\sigma+(\sigma^2-1)\,d\tau \quad (\div),$$

$$\frac{idy}{\sqrt{b}} = (\tau^2+1)\,d\sigma+(\sigma^2+1)\,d\tau \quad (\div),$$

$$\frac{dz}{\sqrt{c}} = \quad -2\tau\,d\sigma+ \qquad 2\sigma\,d\tau \quad (\div),$$

where denom. $=(\tau+\sigma)^2$: regarding σ, τ as the parameters in place of p, q, these show the values of the first differential coefficients a, a'; b, b'; c, c'. We deduce

$$A = -2i\sqrt{bc}\,(\sigma\tau+1) \div, \qquad B = -2i\sqrt{ca}\,(\sigma\tau-1) \div,$$
$$C = -2i\sqrt{ba}\,(\tau-\sigma) \div,$$

where denom. $=(\tau+\sigma)^3$. We have, moreover,

$$\frac{d^2x}{\sqrt{a}} = -2(\tau^2-1)\,d\sigma^2+2(\sigma\tau+1)\,2d\sigma\,d\tau-2(\sigma^2-1)\,d\tau^2 \quad (\div),$$

$$\frac{2d^2y}{\sqrt{b}} = -2(\tau^2+1)\,d\sigma^2+2(\sigma\tau-1)\,2d\sigma\,d\tau-2(\sigma^2+1)\,d\tau^2 \quad (\div),$$

$$\frac{d^2z}{\sqrt{c}} = + \qquad 4\tau\,d\sigma^2+2(\tau-\sigma)\,2d\sigma\,d\tau+ \qquad 4\sigma\,d\tau^2 \quad (\div),$$

where denom. $=(\tau+\sigma)^3$: giving α, α', α''; β, β', β''; γ, γ', γ''. We deduce as the numerators of $E'(=A\alpha+B\beta+C\gamma)$ and $G'(=A\alpha''+B\beta''+C\gamma'')$,

$$4i\sqrt{abc}\,\{(\sigma\tau+1)(\tau^2-1)-(\sigma\tau-1)(\tau^2+1)-2\tau(\tau-\sigma)\}, = 0,$$

and $$4i\sqrt{abc}\,\{(\sigma\tau+1)(\sigma^2-1)-(\sigma\tau-1)(\sigma^2+1)+2\sigma(\tau-\sigma)\}, = 0;$$

that is, $E'=0$ and $G'=0$; or the differential equation of the chief lines is $d\sigma\,d\tau=0$, which is right. The value of $F'(=A\alpha'+B\beta'+C\gamma')$ is hardly required, but it is readily found to

$$= 4i\sqrt{abc}\,\{-(\sigma\tau+1)^2+(\sigma\tau-1)^2-(\tau-\sigma)\}^2 \div (\tau+\sigma)^6,$$

or since the term in $\{\ \}$ is $=-(\tau+\sigma)^2$, we have

$$F' = \frac{-4i\sqrt{abc}}{(\tau+\sigma)^4}.$$

24. The values of E, F, G $(ds^2 = Ed\sigma^2+2Fd\sigma\,d\tau+Gd\tau^2)$ are

$$E = a(\tau^2-1)^2 \qquad -b(\tau^2+1)^2 \qquad +c.4\tau^2 \quad (\div),$$
$$F = a(\tau^2-1)(\sigma^2-1)-b(\tau^2+1)(\sigma^2+1)-c.4\tau\sigma \quad (\div),$$
$$G = a(\sigma^2-1)^2 \qquad -b(\sigma^2+1)^2 \qquad +c.4\sigma^2 \quad (\div),$$

where denom. $=(\tau+\sigma)^4$.

We have, it is clear, $(E_1 = d_\sigma E,\ E_2 = d_\tau E,$ &c.)

$$E_1 = -\frac{4}{\sigma+\tau}E, \quad G_2 = -\frac{4}{\sigma+\tau}G.$$

Hence the condition $FG_2 - 2F_2G + GG_1 = 0$, in order that $\sigma = \text{const.}$ may be a geodesic, reduces itself to

$$-\frac{4}{\sigma+\tau}F - 2F_2 + G_1 = 0;$$

and similarly, the condition $-2EF_1 + E_1F + EE_2 = 0$, in order that $\tau = \text{const.}$ may be a geodesic, reduces itself to

$$-\frac{4}{\sigma+\tau}F - 2F_1 + E_2 = 0.$$

We have at once

$$E_2 = 4a(\tau^2-1)(\tau\sigma+1) \quad -4b(\tau^2+1)(\tau\sigma-1) \quad +4c.2\tau(\sigma-\tau) \ (\div),$$
$$G_1 = 4a(\sigma^2-1)(\tau\sigma+1) \quad -4b(\sigma^2+1)(\tau\sigma-1) \quad -4c.2\sigma(\sigma-\tau) \ (\div),$$
$$F_1 = 2a(\tau^2-1)(\tau\sigma-\sigma^2+2) - 2b(\tau^2+1)(\tau\sigma-\sigma^2-2) - 2c.2\tau(\tau-3\sigma) \ (\div),$$
$$F_2 = 2a(\sigma^2-1)(\tau\sigma-\tau^2+2) - 2b(\sigma^2+1)(\tau\sigma-\tau^2-2) - 2c.2\sigma(\sigma-3\tau) \ (\div),$$

where denom. $= (\sigma+\tau)^5$; and substituting these values, the conditions are verified: we thus again see *à posteriori* that the right lines $\sigma = \text{const.}$ and $\tau = \text{const.}$ are geodesics.

25. The last-mentioned values of E, G are $E = T \div (\tau+\sigma)^4$, $G = \Sigma \div (\tau+\sigma)^4$; and writing for a moment

$$A = a(\tau^2-1)(\sigma^2-1) - b(\tau^2+1)(\sigma^2+1) - c.4\sigma\tau,$$

we have $F = A \div (\tau+\sigma)^4$, the value of ds^2 is thus

$$= Td\sigma^2 + 2Ad\sigma d\tau + \Sigma d\tau^2 \div (\tau+\sigma)^4,$$

which should be $= \frac{1}{4}(p-q)\left(p\frac{dp^2}{P} - q\frac{dq^2}{Q}\right)$,

where, as before, $P, Q = (a+p)(b+p)(c+p)$, $(a+q)(b+q)(c+q)$, respectively. We have already found

$$\frac{dp}{\sqrt{P}} + \frac{dq}{\sqrt{Q}} + \frac{4d\sigma}{\sqrt{\Sigma}} = 0,$$
$$\frac{dp}{\sqrt{P}} - \frac{dq}{\sqrt{Q}} + \frac{4d\tau}{\sqrt{T}} = 0;$$

or what is the same thing,

$$\frac{dp}{\sqrt{P}} = -2\left(\frac{d\sigma}{\sqrt{\Sigma}} + \frac{d\tau}{\sqrt{T}}\right),$$
$$\frac{dq}{\sqrt{Q}} = -2\left(\frac{d\sigma}{\sqrt{\Sigma}} - \frac{d\tau}{\sqrt{T}}\right);$$

and we ought therefore to have identically

$$(p-q)\left\{p\left(\frac{d\sigma}{\sqrt{\Sigma}} + \frac{d\tau}{\sqrt{T}}\right)^2 - q\left(\frac{d\sigma}{\sqrt{\Sigma}} - \frac{d\tau}{\sqrt{T}}\right)^2\right\}$$
$$= Td\sigma^2 + 2Ad\sigma d\tau + \Sigma d\tau^2 \div (\tau+\sigma)^4;$$

that is,
$$(p-q)^2 = \mathrm{T}\Sigma \div (\tau+\sigma)^4,$$
$$p^2-q^2 = \mathrm{A}\sqrt{\mathrm{T}\Sigma} \div (\tau+\sigma)^4;$$

or what is the same thing,
$$(p-q)^2 = \mathrm{T}\Sigma \div (\tau+\sigma)^4,$$
$$p+q \quad = \mathrm{A} \div (\tau+\sigma)^2,$$

which are easily verified.

26. In fact, the equation
$$\frac{x^2}{a+u} + \frac{y^2}{b+u} + \frac{z^2}{c+u} = 1$$

gives for u a quadric equation, the roots of which are $u=p$, $u=q$; that is, we have
$$p+q = \quad x^2+y^2+z^2-a-b-c,$$
$$pq \quad = -(b+c)\,x^2-(c+a)\,y^2-(a+b)\,z^2+bc+ca+ab;$$

and substituting herein for x^2, y^2, z^2 their values in terms of σ, τ, we find
$$p+q = a\,(\sigma^2-1)\,(\tau^2-1)-b\,(\sigma^2+1)\,(\tau^2+1)-4c\tau\sigma \div (\tau+\sigma)^2,$$
$$pq \quad = bc\,(\sigma\tau+1)^2-ca\,(\sigma\tau-1)^2+ab\,(\sigma-\tau)^2 \div (\tau+\sigma)^4,$$

the first of which is, in fact, $p+q = \mathrm{A} \div (\tau+\sigma)^2$. And from the two equations, forming the combination $(p+q)^2-4pq$, we at once obtain the other equation $(p-q)^2 = \Sigma\mathrm{T} \div (\tau+\sigma)^4$.

27. The most ready way of obtaining the relations between the differentials of p, q, σ, τ, is from the foregoing expressions of $p+q$, pq. Writing for a moment $p+q = \mathrm{A} \div (\sigma+\tau)^2$, $pq = \mathrm{B} \div (\sigma+\tau)^2$, we have $p^2\,(a+\tau)^2-\mathrm{A}p+\mathrm{B} = 0$, $q^2\,(\sigma+\tau)^2-\mathrm{A}q+\mathrm{B} = 0$; viz., the first of these equations is
$$p^2\,(\sigma+\tau)^2-p\,[a\,(\sigma^2-1)\,(\tau^2-1)-b\,(\sigma^2+1)\,(\tau^2+1)-4c\sigma\tau]$$
$$+bc\,(\sigma\tau+1)^2-ca\,(\sigma\tau-1)^2+ab\,(\sigma-\tau)^2 = 0,$$

which is quadric in p, σ, τ. The negative discriminants in regard to these variables respectively are $\Sigma\mathrm{T}$, $4\mathrm{PT}$, $4\mathrm{P}\Sigma$ respectively, and we have thus the equation
$$\frac{dp}{\sqrt{\mathrm{P}}} + 2\left(\frac{d\sigma}{\sqrt{\Sigma}} + \frac{d\tau}{\sqrt{\mathrm{T}}}\right) = 0;$$

and the like equation for dq.

28. In the first integral of the geodesic lines, introducing σ, τ instead of p, q, the equation becomes
$$\sqrt{\left(\frac{p}{p+\theta}\right)}\left(\frac{d\sigma}{\sqrt{\Sigma}} + \frac{d\tau}{\sqrt{\mathrm{T}}}\right) - \sqrt{\left(\frac{q}{q+\theta}\right)}\left(\frac{d\sigma}{\sqrt{\Sigma}} - \frac{d\tau}{\sqrt{\mathrm{T}}}\right) = 0;$$

or what is the same thing,

$$p(q+\theta)\left(\frac{d\sigma}{\sqrt{\Sigma}}+\frac{d\tau}{\sqrt{\Gamma}}\right)^2-q(p+\theta)\left(\frac{d\sigma}{\sqrt{\Sigma}}-\frac{d\tau}{\sqrt{\Gamma}}\right)^2=0;$$

that is, $(p-q)\theta\left(\frac{d\sigma^2}{\Sigma}+\frac{d\tau^2}{\Gamma}\right)+2[2pq+\theta(p+q)]\frac{d\sigma\,d\tau}{\sqrt{\Sigma\Gamma}}=0;$

or substituting herein for $p-q$, pq, $p+q$ the values $\sqrt{\Sigma\Gamma}$, B, A, each divided by $(\sigma+\tau)^2$, this is

$$\theta(\Gamma d\sigma^2+\Sigma d\tau^2)+2(2\mathrm{B}+\theta\mathrm{A})\,d\sigma\,d\tau=0,$$

or say

$$\theta(\Gamma d\sigma^2+2\mathrm{A}d\sigma\,d\tau+\Sigma d\tau^2)+4\mathrm{B}d\sigma\,d\tau=0;$$

viz., writing herein $\theta=0$, the equation is $d\sigma\,d\tau=0$, giving the right lines on the surface; and writing $\theta=\infty$, it is $\Gamma d\sigma^2+2\mathrm{A}d\sigma\,d\tau+\Sigma d\tau^2=0$, giving the circular lines.

29. The equation $ds^2=\Gamma d\sigma^2+2\mathrm{A}d\sigma\,d\tau+\Sigma d\tau^2\div(\tau+\sigma)^4$ shows that the right lines σ, $\sigma+d\sigma$, τ, $\tau+d\tau$ form on the surface an indefinitely small parallelogram, the sides whereof are $\sqrt{\Gamma}\,d\sigma\div(\tau+\sigma)^2$ and $\sqrt{\Sigma}\,d\tau\div(\tau+\sigma)^2$, viz., the ratio of the coefficients of $d\sigma$, $d\tau$ is of the form function $\sigma\div$ function τ; and it thus appears that it is possible to draw on the surface the two sets of right lines, the lines of each set being at such intervals that the surface is divided into parallelograms, the sides of which have to each other any given ratio (the angles being variable); viz., if this ratio be as $m:1$, then, to determine the relation between σ, τ, we must have $\sqrt{\Gamma}\,d\sigma=\pm m\sqrt{\Sigma}\,d\tau$, or what is the same thing, $\frac{d\sigma}{\sqrt{\Sigma}}=\pm m\frac{d\tau}{\sqrt{\Gamma}}$. In particular, if $m=1$, the parallelograms will be rhombs; and we must then have $\frac{d\sigma}{\sqrt{\Sigma}}=\pm\frac{d\tau}{\sqrt{\Gamma}}$;

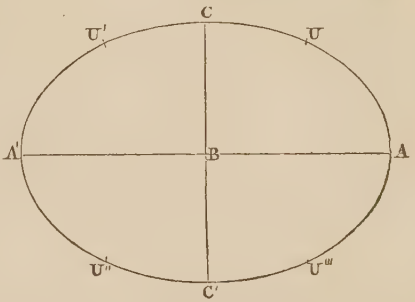
viz., this being in terms of σ, τ, the differential equation of the curves of curvature, it appears that the two sets of lines may be taken so as to divide the surface into indefinitely small rhombs, such that, drawing the diagonals of these, we have the two sets of curves of curvature.

The Ellipsoid and the Skew Hyperboloid.

30. I have thus far considered a quadric surface in general, the various theorems being applicable as well to the ellipsoid and the hyperboloid of two sheets as to the skew hyperboloid, the right lines being of course imaginary for the first-mentioned surfaces; but I will now consider the ellipsoid and the skew hyperboloid separately.

31. First the ellipsoid. We have here a, b, c all positive, and I assume as usual $a>b>c$. The principal sections are all ellipses, viz., $\frac{x^2}{a}+\frac{y^2}{b}=1$ is the major-mean, or say the minor section, $\frac{y^2}{b}+\frac{z^2}{c}=1$

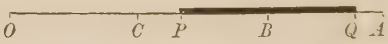
the minor-mean, or say the major section, and $\frac{x^2}{a}+\frac{z^2}{c}=1$ the mean, or umbilicar section. The elliptic coordinates p, q enter into the equations symmetrically, but we distinguish them by taking p to extend from $-c$ to $-b$, and q to extend from $-b$ to $-a$. Thus $p=$ const. denotes the curves of curvature of the one kind; viz., $p=-c$ denotes the major-mean section $\frac{x^2}{a}+\frac{y^2}{b}=1$, $p=-b$ the portions UU′ and U″U‴ of the umbilicar section; and $q=$ const. denotes the curves of curvature of the other kind, viz., $q=-b$ the remaining portions U′U″ and U‴U of the umbilicar section, $q=-a$ the minor-mean section $\frac{y^2}{b}+\frac{z^2}{c}=1$; say $p=$const. the major-mean curves, and $q=$ const. the minor-mean curves.



32. Hence, in order that the equation

$$dp\sqrt{\left(\frac{p}{(a+p)(b+p)(c+p)(\theta+p)}\right)} \pm dq\sqrt{\left(\frac{q}{(a+q)(b+q)(c+q)(\theta+q)}\right)}=0$$

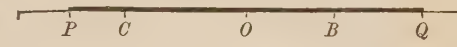
of the geodesic lines may be real [observing that we have $a+p$, $b+p=+$, $c+p$, $p=-$, and $a+q=+$, $b+q$, $c+q$, $q=-$, consequently $p\div(a+p)(b+p)(c+p)=+$, but $q\div(a+q)(b+q)(c+q)=-$], we must have $\theta+p$, $\theta+q$ of opposite signs, that is, $\theta+p=+$ and $\theta+q=-$; or θ included between the limits a, c. Or what is the same thing, $-\theta$ is included between the limits $-c$, $-b$, say $-\theta$ has a p-value; or else between the limits $-b$, $-a$, say $-\theta$ has a q-value. This is conveniently shown in the annexed diagram of the values of $-p$, $-q$, $-\theta$. Hence on the ellipsoid we have two kinds of geodesic lines, each of them touching a real curve of curvature; viz., those which touch a major-mean curve and those which touch a minor-mean curve: the transition case, answering to the value $\theta=b$, is that of the geodesic lines which pass through an umbilicus. I have considered the theory more in detail in my memoir "On the Geodesic Lines of an Ellipsoid," Mem. R. Ast. Soc., t. xxx., pp. 31—53, 1872.



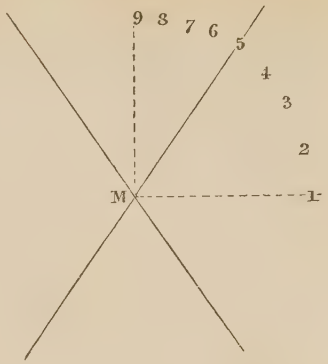
33. Next, for the skew hyperboloid, we have a and $b=+$, $c=-$, and

I assume for convenience $a>b$. Attending to the signs, we still have therefore $a>b>c$. The principal sections are one of them an ellipse, and the other two hyperbolas, viz., the minor section is the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$, the major section is the hyperbola $\frac{y^2}{b}+\frac{z^2}{c}=1$, and the mean section is the hyperbola $\frac{x^2}{a}+\frac{z^2}{c}=1$: there are no umbilici. The elliptic coordinates enter symmetrically; but, as before, we distinguish them, viz., we take p to extend from $-c$ (a positive value) to infinity, and q from $-b$ to $-a$. Thus $p=$ const. denotes the curves of curvature of the one kind, viz., $p=-c$ the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$, and every other value of p an oval curve surrounding the hyperboloid; and $q=$ const. the curves of curvature of the other kind, viz., $q=-c$ the major hyperbola $\frac{y^2}{b}+\frac{z^2}{c}=1$, $q=-b$ the mean hyperbola $\frac{x^2}{a}+\frac{z^2}{c}=1$, and each intermediate value gives a curve of curvature of a hyperbolic form: we may say that $p=$ const. determines the oval curves of curvature, and $q=$ const. the hyperbolic curves of curvature.

34. In the equation of the geodesic lines we have $a+p$, $b+p$, $c+p$, p all positive; but $a+q=+$, $b+q$, $c+q$, q all $=-$; hence $p\div(a+p)(b+p)(c+p)=+$, but $q\div(a+q)(b+q)(c+q)=-$; therefore $\theta+p$ and $\theta+q$ must be of opposite signs, or we must have $\theta+p=+$ and $\theta+q=-$; or what is the same thing, θ may have any value from $-p$ to $-q$, or say $-\theta$ any value from p to q; that is, the value of $-\theta$ may be positive and greater than $-c$, positive and less than $-c$, negative and less than $-b$, negative and between $-b$ and $-a$; viz., in the first case $-\theta$ has a p-value, and in the fourth case it has a q-value, but in the second and third cases it has neither a p- nor a q-value. This is better seen from the diagram. It follows that we have, on the hyperboloid, geodesic lines of four different kinds: those which touch a real curve of curvature, oval or hyperbolic, and those which touch no real curve of curvature, but for which $-\theta$ has a positive value from 0 to $-c$, or a negative value from 0 to $-b$. And there are the transitional cases $-\theta=-c$, where the geodesic touches the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$; $\theta=0$, where the geodesic becomes a right line; and $-\theta=-b$, where the geodesic touches the mean hyperbola $\frac{x^2}{a}+\frac{z^2}{c}=1$.



35. To explain this more in detail, consider the geodesics which start from a point M of the hyperboloid. To fix the ideas, consider the axis of z as vertical, and take the point M in the positive octant of the hyperboloid; and let M1 represent the direction of the oval curve of curvature, M9 that of the hyperbolic curve of curvature, M5 that of one of the right lines.



The geodesic of initial direction M1 touches at M the oval curve of curvature M1, and lies wholly above this curve; it makes an infinity of convolutions round the upper part of the hyperboloid, cutting all the oval curves of curvature for which p has a (positive) value greater than p_1 (if p_1 is the value of p corresponding to the oval curve through M), and ascending to infinity: or considering the curve as described in the opposite sense, it descends from infinity to touch the oval curve through M, after which again ascends to infinity.

Next, if the initial direction is M2, we have a geodesic of the same kind, only descending below M to touch a certain oval curve having for its parameter p_2 ($p_2 > -c < p_1$).

We come next to a critical direction M3, for which the geodesic descends below M to touch the oval curve of parameter $p_3 = -c$, that is, the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$. But it is to be observed that, whatever the initial point M may be, the geodesic makes below M an infinity of convolutions round the hyperboloid, so that it does not in fact ever actually touch the ellipse, but has this ellipse for an asymptote. That this is so appears from the consideration that the ellipse, *quâ* plane curve of curvature, is a geodesic; so that, starting from a point of the ellipse in the direction of the ellipse, the geodesic coincides with the ellipse, or, besides the ellipse itself, there is not any geodesic which touches the ellipse.

Next, if the initial direction be M4, the geodesic does not here touch any oval curve; it descends through M below the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$, lying in the upper and lower portions of the hyperboloid, and making round it an infinity of convolutions.

36. We come, then, to the initial direction M5, which is that of the right line; the geodesic here coincides with the right line.

In the cases which follow, the geodesic lies in the upper and lower portions of the hyperboloid, cutting all the oval curves of curvature.

Initial direction M6: the geodesic does not touch any hyperbolic curve of curvature, but makes round the hyperboloid an infinity of convolutions.

Initial direction M7: the geodesic touches at opposite infinities the mean hyperbola $\frac{x^2}{a} + \frac{z^2}{c} = 1$, it lies wholly in front of the plane $y=0$ of this hyperbola.

Initial direction M8: the geodesic touches a hyperbolic curve of curvature parameter q_8 where q_8 (negative) is between $-b$ and q_9 the parameter of the hyperbolic curve of curvature through M; viz., it cuts all the hyperbolic curves the parameters of which are between $-b$ and q_8, but does not cut the remaining curves the parameters of which extend from q_8 to $-a$.

Lastly, initial direction is M9, that of the hyperbolic curve of curvature through M; the geodesic touches this curve, cutting all the hyperbolic curves the parameters of which are between $-b$ and q_9, but not any of those the parameters of which are between q_9 and $-a$.

37. If in the differential equation of the geodesic line we consider p, q as the elliptic coordinates of a given point M of the curve, the equation for a given value of θ determines the direction of the curve; or conversely, if the direction be given, the equation determines the value of the parameter θ. Writing

$$\frac{dp\sqrt{p}}{\sqrt{(a+p)(b+p)(c+p)}} = \mathrm{P}, \quad \frac{dq\sqrt{q}}{\sqrt{(a+q)(b+q)(c+q)}} = \mathrm{Q},$$

then P, Q are proportional to the rectangular coordinates of a consecutive point M′, measured from M in the directions of the hyperbolic and oval curves of curvature respectively; and the differential equation of the geodesic lines gives $\frac{\mathrm{P}}{\sqrt{p+\theta}} \pm \frac{\mathrm{Q}}{\sqrt{q+\theta}} = 0$;

viz., if ϕ be the inclination of the geodesic to the hyperbolic curve of curvature, then $\mathrm{Q} = \mathrm{P}\tan\phi$, or we have $\frac{1}{p+\theta} = \frac{\tan^2\phi}{q+\theta}$, that is, $p\tan^2\phi - q = \theta(1-\tan^2\phi)$; hence, if for the right line $\phi = \lambda$, then $p\tan^2\lambda - q = 0$; and therefore $\theta = \frac{p(\tan^2\phi - \tan^2\lambda)}{1-\tan^2\phi}$; viz., $\phi = 0$, $\theta = -q$, $= \infty$, $\theta = -p$, as it should be.

On a Deduction from Von Staudt's Property of Bernoulli's Numbers. By J. W. L. Glaisher, B.A., F.R.A.S., Fellow of Trinity College, Cambridge.

[*Read December 12th*, 1872.]

Some time since I was engaged on a quadruple calculation of the value of Euler's constant to a great many decimal places by making $x=1000$, 500, 200, and 100, in the formula

$$\gamma = 1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{x}-\log x-\frac{1}{2x}+\frac{B_1}{2x^2}-\frac{B_2}{4x^4}+\frac{B_3}{6x^6}-\cdots;$$

and for this purpose the numerators of the first 29 Bernoulli's numbers were divided out by their denominators till the periods of the resulting decimals were found; the same was also done (only not in both cases as far as B_{20}) for the numbers divided and multiplied respectively by the power of 2 assigned by the above formula, viz., for $\frac{B_n}{2^{2n}}$ and $2^{2n}B_n$; and I then remarked that the circulating period of B_n, $\frac{B_n}{2^{2n}}$, and $2^{2n}B_n$ was identical in all cases, thus affording an important verification of the accuracy with which the divisions had been performed.*

Thus, for example,

$$B_{18}=\frac{2631527155305347737 3}{1919190}$$
$$=13711655205088\cdot\dot{3}32772159087948561\dot{6},$$

$$\frac{B_{18}}{2^{36}}=199\cdot530844185040518236333270740020565749\dot{1}32772159087948561\dot{6},$$

$$2^{36}B_{18}=942257770878120992761412\cdot\dot{8}32772159087948561\dot{6};$$

so that the period $\dot{3}2772159087948561\dot{6}$ is the same in all three.

At a meeting of the Cambridge Philosophical Society on May 27, 1871, during a discussion on Bernoulli's numbers, I enunciated the above result as having been found to be practically true for the first 29 numbers; but it is only recently that it has occurred to me that it is an almost immediate consequence of a property of Bernoulli's numbers given by Von Staudt in Crelle's Journal, t. xxi., p. 372 (1840).

The property in question is that if the n^{th} Bernoulli's number be reduced to its lowest terms, and if α, β ... be all the factors of $2n$ (including 1 and $2n$), then those of the numbers $\alpha+1$, $\beta+1$, ... which are prime, are the factors of the denominator of B_n; thus, for $n=18$, the factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36; and of 2, 3, 4, 5, 7, 10,

* In one or two instances this agreement did not at first take place; but, on examination of the work, one of the divisions was found to be inaccurate in every such case.

13, 19, 37 the numbers 4 and 10 are rejected, as they are not prime: so that the denominator of B_{18} is $= 2.3.5.7.13.19.37 = 1919190$.

It follows from this property at once that $10(5^{2n}-1)B_n$ is always integral; for $\alpha, \beta, \gamma, \ldots$ being the factors of $2n$, $5^{2n}-1$ is divisible by $5^{\alpha}-1$, $5^{\beta}-1$, &c.; and when $\alpha+1$ is a prime number, $5^{\alpha}-1$ is divisible by $\alpha+1$, by Fermat's Theorem, so that all the factors in the denominator of B_n are factors also of $5^{2n}-1$. There is only one case of exception, viz., when $\alpha+1=5$, for then $\alpha+1$ is not prime to the quantity raised to the power, as is requisite for the truth of Fermat's Theorem; but the factor 10 (introduced to meet this case) is divisible by 5; thus, $10(5^{2n}-1)B_n$ is always integral, and the Theorem in question follows at once, for this quantity

$$= 10\left\{10^{2n}.\frac{B_n}{2^{2n}} - B_n\right\},$$

which could not be integral unless B_n and $\frac{B_n}{2^{2n}}$ had the same period.

It is evident from the above reasoning that the theorem is also true for B_n and $\frac{B_n}{5^{2n}}$, or, which is the same thing, for B_n and $2^{2n}B_n$; and generally it is true for B_n and $2^{\pm 2mn}B_n$. It also appears that the theorem holds good for B_n and $2^{\pm mn}B_n$ (m an odd integer), except when $2n+1$ is prime.

The above results are not of nearly so much interest as Von Staudt's property, from which they are derived; as, while the latter, depending only on the partition of numbers and the properties of primes, is independent of the radix of the scale of numbers, the former is only true for radix 10. It is easy to see that for any other radix r (factors $a, b, c, \ldots$), the results are generally true for $\frac{B_n}{a^{2n}}$, $\frac{B_n}{b^{2n}}$, &c.; but if r be prime, there is no theorem.

Von Staudt's property (with another giving the numerator of the fractional part of the Bernoulli's number) was given by him, as before stated, in "Crelle," t. xxi. It was published, without proof, by Clausen in the "Astron. Nach.," t. xvii. c. 351, as a specimen of a book he intended to write; and Von Staudt, in consequence, published his proof in the above paper, with the remark that he had communicated the theorem to Gauss several years previously. Other proofs are given by Prof. Schläfli and Mr. Jeffery in vol. vi. of the "Quarterly Journal of Mathematics"; and another is contained in a paper recently communicated to the Cambridge Philosophical Society by Prof. Adams.*

* The portion of Von Staudt's which is required in the proof of the deduction contained in this paper is also demonstrated in the "Messenger of Mathematics," New Series, Vol. ii., p. 191 (April, 1873).—May 5, 1873.

The principles used in this note can also be applied to another form of Bernoulli's numbers, as follows:

$$B_n = \frac{2\Pi(2n)}{(2\pi)^{2n}}\left\{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\cdots\right\}$$

$$= \frac{2\Pi(2n)}{(2\pi)^{2n}}\,\frac{2^{2n}.3^{2n}.5^{2n}\ldots}{(2^{2n}-1)(3^{2n}-1)(5^{2n}-1)\ldots} \quad \ldots\ldots\ldots\ldots (1),$$

since $$1+\frac{1}{2^m}+\frac{1}{3^m}+\cdots = \frac{1}{\left(1-\frac{1}{2^m}\right)\left(1-\frac{1}{3^m}\right)\left(1-\frac{1}{5^m}\right)\cdots}$$

by a known formula, 2, 3, 5, &c. being the series of prime numbers; $\Pi(2n)$ is written for $1.2\ldots 2n$.

By taking $n=1$, we have

$$\frac{\pi^2}{6} = \frac{2^2.3^2.5^2\ldots}{(2^2-1)(3^2-1)(5^2-1)\ldots};$$

therefore, raising both sides of this equation to the n^{th} power, and substituting for $(2\pi)^{2n}$ in (1), we find

$$B_n = \frac{2\Pi(2n)}{(24)^n}\,\frac{(2^2-1)^n(3^2-1)^n(5^2-1)^n\ldots}{(2^{2n}-1)(3^{2n}-1)(5^{2n}-1)\ldots} \quad \ldots\ldots\ldots\ldots (2);$$

and since, as shown above, $5B_n(5^{2n}-1)$ is always integral, the same is true of the expression on the right-hand side of (2), when the factor $5^{2n}-1$ is removed from the denominator, and the factor 5 introduced into the numerator. But $2^{2n}-1$ also is divisible by $\alpha+1$, $\beta+1$, &c. (these denoting, as before, primes, where α, β, ... are factors of $2n$), except for $\alpha+1=2$; similarly for $3^{2n}-1$, except for the prime 3, if it occur; and so on for all the other factors. Suppose now that q is the greatest, viz. the p^{th} prime, (p and q infinite); then the factors $\alpha+1$, $\beta+1$, &c. are involved at least $p-1$ times in the denominator of (2); and, since $\Pi(2n)$ contains them all once, except $2n+1$ if it occur, we see that the numerator $(2^2-1)^n(3^2-1)^n\ldots$ must contain $\alpha+1$, $\beta+1$, &c. at least $p-3$ times. We have therefore the following theorem of a curious kind connecting different orders of infinities, viz.:—That, p and q being as before, if any integer n be taken, a factor of which is α, then the product

$$1.3.2.4.4.6\ldots(q-1)(q+1)$$

(the factors being every number next above or below a prime) contains the factor $\alpha+1$, if prime, at least $\frac{p-3}{n}$ times; and $2n+1$, if prime, at least $\frac{p-2}{n}$ times.

Geometry on an Ellipsoid. *By* Prof. Clifford.

(*Read December 12th*, 1872.)

The metric properties of an ellipsoid are entirely determined by the four points in which it is met by the imaginary circle at infinity. I shall start, therefore, by assuming the existence of these four co-planar points o_1, o_2, o_3, o_4, which, taken all together, I call the absolute.

1. To represent the ellipsoid on a plane we require also two fixed points i, j; the plane sections of the ellipsoid are then represented by conics through these points, and the generating lines by lines on the plane through them. In fact, if we take a fixed point a on the ellipsoid E_2, and draw a line through a and a variable point x on the ellipsoid, this line will meet a plane L in one point y, which is the representative of x; the points i, j will then represent the generators through a. If we take the points i, j to be the absolute of the plane L, then all the plane sections will be represented by circles, the lines through i will represent one system of generating lines, and the lines through j the other. We shall have then, in addition, to consider the four points o; and the geometry of the ellipsoid will be merely the geometry of the plane considered in relation to these four points, which are concyclic.

2. We know, then, that the antipoints of the o lie upon three new circles, orthotomic of each other and of the first. These correspond to the principal sections of the ellipsoid. The antipoints themselves represent umbilici, four of which are real; I call these u_1, u_2, u_3, u_4, and we may now take the points u as our absolute instead of the points o.

3. What now are the directions of the lines of curvature at any point x of the ellipsoid? First, the indicatrix at x is represented by the point-circle at its corresponding point y, so that conjugate directions at x correspond to rectangular directions at y. Next, the tangent plane at x meets the plane at infinity in a line, say β. Through the points o can be drawn two conics to touch β, say at p, q. The lines xp, xq are tangents to the lines of curvature at x, since the points p, q are conjugates both of the section of the ellipsoid at infinity and of the imaginary circle. But now let o_1o_2 and o_3o_4 meet β in r, s respectively. Then the involution made by rs and the points where β meets the imaginary circle have p, q for double points. The interpretation of this on the plane is, that the directions through y corresponding to xp, xq make equal angles with the circles yo_1o_2, yo_3o_4. Hence—

The lines of curvature of the ellipsoid are represented by confocal anallagmatics having the u for foci.

Sections made by two conjugate planes of the ellipsoid are represented by orthotomic circles.

4. A straight line γ in space may be denoted by the two points c_1, c_2, where it meets the ellipsoid. The sections drawn through this line will be represented by the series of coaxal circles through c_1, c_2. The sections through δ, the polar of γ, will therefore be represented by the series of coaxal circles through d_1, d_2, the antipoints of c_1 c_2. Thus, a straight line being represented by a pair of points, its polar is represented by their antipoints, as is otherwise obvious.

I denote further the principal sections and the plane at infinity by XYZU, which notation will serve also for the circles which represent them. Now, in general, a section of E_2 passing through a fixed point of space is represented by a circle orthotomic of a fixed circle. In particular, the points $o_1\,o_2$, $o_3\,o_4$; $o_1\,o_3$, $o_4\,o_2$; $o_1\,o_4$, $o_2\,o_3$, correspond in this way to the circles XYZ. I want now to find the interpretation on the plane of the rectangularity of the lines γ and δ. The planes joining them to the point o_1o_2, o_3o_4 are harmonic of the lines o_1o_2, o_3o_4. Hence the circles c_1c_2X, d_1d_2X are harmonic of the circles coaxal with them and passing through o_1o_2, o_3o_4 respectively. This is to be true when X and Y are interchanged: the conditions may finally be written

$$\frac{c_1c_2\mathrm{YZ}}{d_1d_2\mathrm{YZ}} = \frac{c_1c_2\mathrm{ZX}}{d_1d_2\mathrm{ZX}} = \frac{c_1c_2\mathrm{XY}}{d_1d_2\mathrm{XY}}.$$

If now for d_1, d_2 we may substitute c_1, c_1' where c_1' is indefinitely near to c_1 *in any direction*, c_1c_2 represents the normal at c_1.

5. A circle P, orthotomic of U, represents a diametral section. Let the pole of this section be called p; p is a point at infinity. We know that it is always possible to find another point q at infinity, which is conjugate to p with respect both to the ellipsoid and to the imaginary circle. We may then endeavour to find the circle Q, of which q is the pole. Further, lines λ, μ can be drawn through p, q respectively, which are at right angles, and also conjugate polars of the ellipsoid. To represent these we must find a pair of points on P which have their antipoints on Q. These circles cut orthogonally; on each of them, then, there is a singly infinite number of point-pairs representing axes of the quadric, viz., the point-pairs determined by diameters of the other circle. That is to say, any circle P, orthotomic of U, being given, there can always be found a point q^0, such that the lines through q^0 determine on P point-pairs representing axes of the quadric.

The determination of q^0 depends on the position of the projecting-point a. The generators through a meet the diametral section Q in two points; the remaining generators through these intersect on the representative of q^0.

6. I now proceed to construct Q when P is given. In the first place, Q has to be orthotomic of P and U. Next, if we draw through P and Q two new circles, one of which has o_1 o_2 for harmonics, and the other

$o_3\ o_4$, these must be harmonics of P and Q. But a circle orthotomic of U and having $o_1\ o_2$ for harmonics, must have them for inverse points, and therefore have its centre on $o_1\ o_2$. Hence the line joining the centres of P and Q is cut harmonically by the lines o_1o_2, o_3o_4. Similarly, it is cut harmonically by o_1o_3, o_4o_2, and by oo_4, o_2o_3. Hence the centres of P and Q are polar opposites in regard to the quadrangle $o_1\ o_2\ o_3\ o_4$. They are therefore conjugate points in regard to the circle U.

7. Intersections of the ellipsoid by spheres are represented by anallagmatics *passing through* the four points o_1, o_2, o_3, o_4. There are two systems of real circles passing through pairs of them; these represent the circular sections. Sphero-conics are represented by such of these anallagmatics as have XYZU for focal circles. To find the axes of any circle P of the U system we must then draw two such anallagmatics having double contact with P; the point of contact in pairs will represent the axes of the corresponding section.

January 9th, 1873.

Dr. HIRST, F.R.S., President, in the Chair.

The following gentlemen were elected Members of the Society:—Mr. G. B. Finch, M.A., Mr. T. O. Harding, and Mr. John Macleod; the last named gentleman was subsequently admitted into the Society. Prof. R. Stawell Ball, of Dublin, and Dr. J. Hopkinson, B.A., Fellow of Trinity College, Cambridge, were proposed for election.

Papers were read by Mr. S. Roberts, V.P., "On Parallel Surfaces;" Prof. H. J. S. Smith, F.R.S., "On the greatest common divisors of the minor determinants of a rectangular matrix, of which the constituents are integral numbers," and "On an arithmetical demonstration of a theorem in the Integral Calculus;" Prof. Wolstenholme, "On the Summation of certain series."

The following presents were received:

"Reale Istituto Lombardo—Rendiconti," serie ii., vol. ii., fasc. xi.—xvii.; and "Memorie," vol. xi., ii della serie iii., fasc. ii. Milano, 1869.

"Monatsbericht," Aug., 1872.

A Catalogue of a collection of models of Ruled Surfaces, constructed by M. Fabre de Lagrange, with an appendix containing an account of the application of analysis to their investigation and classification, by C. W. Merrifield, 1872: from the author.

On Parallel Surfaces. *By* S. Roberts, M.A.

[*Read January 9th*, 1873.]

In the present paper the subject is treated with reference to order and class characteristics.

1. A parallel surface is enveloped by planes parallel to the tangent planes of a given surface at a constant distance k. The given surface will be called the primitive, and k is the modulus. It is hardly necessary to recapitulate the various equivalent ways in which a parallel may be defined.

If the primitive is a general surface of the order m, and $U=0$ is its equation in rectangular coordinates, the parallel, modulus k, has for its equation the net resultant, equated to zero, of the system

$$U = 0,$$
$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-k^2 = 0,$$
$$\begin{Bmatrix} x-\alpha, & y-\beta, & z-\gamma \\ U_1, & U_2, & U_3 \end{Bmatrix} = 0,$$

where U_1, U_2, U_3 are written for $\frac{dU}{dx}$, $\frac{dU}{dy}$, $\frac{dU}{dz}$, and the determinant matrix gives three equations equivalent to two conditions. The coordinates α, β, γ refer therefore to a point on the parallel corresponding to the point (x, y, z) on the primitive.

Instead of the above system, we may write

$$(U_1^2+U_2^2+U_3^2)(x-\alpha)^2-k^2U_1^2 = 0,$$
$$U_2(x-\alpha)-U_1(y-\beta) = 0,$$
$$U_3(y-\beta)-U_2(z-\gamma) = 0,$$
$$U = 0;$$

allowing for extraneous systems.

The primitive is supposed to have no special relation to the plane at infinity, and it will then be sufficient to determine the highest power of γ which enters the equation of the parallel. The axes of coordinates being arbitrary, except that they are supposed to be rectangular, it is clear that the infinitely distant points upon them will not in general lie upon the surface, and the highest power of γ in the equation will also give its degree.

The gross order of the resultant is $2m^3$, since γ only appears in the third equation. The system contains, however, twice over the extraneous system

$$x-\alpha = 0,$$
$$U_1 = 0,$$
$$U_3(y-\beta)-U_2(z-\gamma) = 0,$$
$$U = 0;$$

and there is no other such system containing γ. The order of the condition for the coexistence of the last system is $m(m-1)$.

The order of the parallel is therefore

$$2(m^3-m^2+m);$$

that is to say, it is equal to twice the number of normals which can be drawn from an arbitrary point to the primitive.

When we make α, β, γ constant in the equation of the parallel, it gives as an equation in k^2 the values of the squares of the normal distances from the point (α, β, γ) to the primitive. The number of parallels which can be drawn through a given point is obviously equal to the number of normals from that point for finite space.

2. Let r be the length of a principal radius of curvature of the primitive at the point (x, y, z). Then we have the condition

$$P\lambda^2-Q\lambda+R=0,$$

where $P = U_1^2 + U_2^2 + U_3^2$,

$$Q = \left(\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}\right)U_1^2 + \&c. - 2\frac{d^2U}{dy\,dz}U_2U_3 - \&c.,$$

$$R = \left\{\frac{d^2U}{dy^2}\frac{d^2U}{dz^2} - \left(\frac{d^2U}{dy\,dz}\right)^2\right\}U_1^2 + \&c.$$

$$+2\left\{\frac{d^2U}{dx\,dy}\frac{d^2U}{dx\,dz} - \frac{d^2U}{dx^2}\frac{d^2U}{dy\,dz}\right\}U_2U_3 + \&c.,$$

$$\lambda = \frac{P^{\frac{1}{2}}}{r}.$$

Also, if α, β, γ are the coordinates of the centre of curvature corresponding to r,

$$x-\alpha = \frac{rU_1}{P^{\frac{1}{2}}}, \quad y-\beta = \frac{rU_2}{P^{\frac{1}{2}}}, \quad z-\gamma = \frac{rU_3}{P^{\frac{1}{2}}}.$$

Substituting $\frac{U_1}{x-\alpha}$ for λ, we get the system

$$\left.\begin{aligned} R(x-\alpha)^2 - QU_1(x-\alpha) + PU_1^2 &= 0 \\ U_2(x-\alpha) - U_1(y-\beta) &= 0 \\ U_3(y-\beta) - U_2(z-\gamma) &= 0 \\ U &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (a).$$

If from these equations we eliminate x, y, z, and equate the resultant to zero, we shall have the equation of the surfaces of centres together with an extraneous factor.

The order of the gross resultant in γ is

$$4m^2(m-1),$$

but the extraneous system

$$x-\alpha = 0, \quad U_1 = 0,$$
$$U_3(y-\beta) - U_2(z-\gamma) = 0,$$
$$U = 0,$$

gives rise to the order $m(m-1)$, which has to be deducted twice. Hence the order of the surface of centres is

$$4m^2(m-1) - 2m(m-1) = 2m(m-1)(2m-1).$$

This order has been given by M. Darboux, "Comptes Rendus," 1870, p. 1828; and by Herr Löthar Marcks, "Math. Annal." b. v., h. 1.

Now we have to consider points of the parallel corresponding to points on the primitive at which a principal radius of curvature is equal to the modulus. Such points of the parallel are on the surface of centres when they lie on the same side of the primitive as the corresponding centre of curvature, and form a locus which is in fact the cuspidal curve of the parallel, since it is double and the tangent planes at its points coincide.

To obtain the order of the curve we must associate with the system (a) the equation $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-k^2 = 0$.

In this case also, since the axes of coordinates are not specially related to the curve, it is sufficient to find the order in γ of the conditions for the coexistence of the five equations now in question. By the general formula for the case of five equations homogeneous in four variables, the gross order in γ is

$$8m^2(m-1);$$

but the system

$$x-\alpha = 0, \quad U_1 = 0, \quad U_3(y-\beta) - U_2(z-\gamma) = 0, \quad U=0,$$
$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-k^2 = 0,$$

has to be twice excluded as before. The corresponding order for this system is $2m(m-1)$ and the required net order is

$$8m^2(m-1) - 2 \cdot 2m(m-1) = 4m(m-1)(2m-1);$$

that is to say, *the order of the cuspidal curve of the parallel is twice the order of the surface of centres.**

As, however, in the last deduction, the notion of the order of two conditions in one variable is introduced and may seem to call for considerable explanation, it is perhaps more satisfactory to obtain the result as follows.

If we consider the order of the conditions in β, γ (which is equivalent

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* In my paper on the Parallels of Conicoids (*Proceedings*, Jan. 1872), I was led by insufficient reasons to suppose that a further reduction should be made of $8m(m-1)(m-2)$.

to finding the number of points in which the curve in question is met by a plane) so that the system is of the form

Order in x, y, z	Order in β, γ
$4m-4$	0
m	1
m	1
m	0
2	2

we have to deduct for the system

$$y-\beta=0, \quad U_2=0, \quad R(x-a)^2-QU_1(x-a)+PU_1^2=0,$$
$$U=0, \quad \Sigma(x-a)^2-k^2=0,$$

as well as twice the order for the system

$$x-a=0, \quad U_1=0, \ \&c.$$

That is to say, we get, as a major limit,

$$\{16m^2(m-1)+8m(m-1)\}-8m(m-1)^2-2\,.\,2m(m-1),$$

or

$$8m^3+4m^2-12m.$$

The net order required cannot be more than this, since we have obtained the order as if the second equation of (a) contained γ as well as β in the order 1.

But if we treat the system as if the second equation did not contain either β or γ, we get the result

$$8m^3-12m^2+4m,$$

which is a minor limit. We can now infer that $8m^3$ is the true leading term, and since the value required vanishes for $m=0$, $m=1$, and for $m=2$ is 24, we have

$$m(m-1)(8m+A)$$
$$32+2A=24,$$

giving

$$4m(m-1)(2m-1).$$

There is reason to believe that the order of the cuspidal curve in question is twice the order of the surface of centres, much more generally than I can show analytically. For the question is to determine the order of the curve which is traced on the surface of centres by points where the moving normal length becomes a principal radius of curvature of the primitive at the corresponding point. This order is to be taken for a modulus k. Now, supposing the surface is not specially related to the plane at infinity, it can be shown that, in the limit when $k=\infty$, the curve in question becomes the section at infinity of the surface of centres taken twice over. In fact this section is the locus of centres for principal radii of curvature infinitely great, and must be taken twice, because we have $k=\pm\infty$. At all events, the conclusion seems to hold in a great many widely different cases, and even where singularities exist at infinity. Thus, in the case of the paraboloid, the

order is 18; and in that of Dupin's Cyclide, the surface of centres consists of two conics, and is therefore of the order 0, while the parallel, breaking up into two cyclides, has no cuspidal curve. In general, a surface of revolution comes within the result, if an axial plane section is not specially related to the plane at infinity. When there is no singularity at infinity, one might perhaps appeal to the law of continuity.

3. To any set of parallel tangent planes, drawn to the primitive, correspond twice the number of parallel tangent planes, similarly directed, of the parallel surface. *Hence the class of that surface is twice the class of the primitive.*

Also, since to a cylinder which envelopes the primitive corresponds a cylinder enveloping the parallel surface and being the parallel, modulus k, of the former cylinder, it follows that *the characteristics of the general tangent cone can be immediately derived from those of the parallel of a plane curve in plane space.*

We have now obtained a sufficient number of the characteristics of the parallel to determine the values in the following table:—

Parallel.	Reciprocal of Parallel.	
Order	Class	$2(m^3-m^2+m)$
Class	Order	$2m(m-1)^2$
Order of tangent cone	Class of section	$2m^2(m-1)$
Number of its double edges	Number of its double tangents	$2m(m-1)(m^4-m^3-7m+8)$
Number of its cuspidal edges	Number of its inflexional tangents	$2m(m-1)(4m-5)$
Order of nodal curve	Class of node couple torse	$2(m^3-m^2+m)^2-6m(m-1)(2m-1)-2m^3+2m^2-m$ $= m(2m^5-4m^4+6m^3-18m^2+22m-7)$
Order of cuspidal curve	Class of spinode torse	$4m(m-1)(2m-1)$
Class of node couple torse	Order of nodal curve	$m(m-1)(m-2)(m^3-m^2+m-12)+m(m-1)^2[m(m-1)^2-1]$ $= m(m-1)(2m^4-6m^3+6m^2-16m+25)$
Class of spinode torse	Order of cuspidal curve	$8m(m-1)(m-2)$

In other words, we possess the characteristics of the plane sections of the parallel and its reciprocal.

To every triple tangent plane of the primitive will correspond two for the parallel, but there may be more.*

* It seems probable, geometrically, that the spinode torse of the parallel is the parallel of the spinode torse of the primitive; and in like manner, that the node couple torse of the parallel directly corresponding to that of the primitive, is the parallel of the latter torse.

4. We may also write for the fundamental equations connecting a point on the parallel with the corresponding point on the primitive

$$U = 0, \quad \alpha = x \pm \frac{kU_1}{P^{\frac{1}{2}}}, \quad \beta = y \pm \frac{kU_2}{P^{\frac{1}{2}}}, \quad \gamma = z \pm \frac{kU_3}{P^{\frac{1}{2}}} \quad \ldots\ldots (b).$$

If therefore $\pm k = 0$, $P \neq 0$, we have $\alpha = x$, $\beta = y$, $\gamma = z$, or the primitive enters twice in the parallel, modulus 0. When $P = 0$, the system gives a distinct surface as part of the parallel. This surface is the developable which circumscribes the primitive and the imaginary circle at infinity, that is to say, the *focal torse* of the primitive.

To find the equation of this torse, we may consider that the tangent plane at (x, y, z) of U, *i. e.*, if α, β, γ are the current coordinates,

$$(\alpha - x)\,U_1 + (\beta - y)\,U_2 + (\gamma - z)\,U_3 = 0 \quad \ldots\ldots\ldots\ldots (c),$$

is parallel to the plane $\alpha U_1 + \beta U_2 + \gamma U_3 = 0$,

which is to be a tangent plane of the point cone

$$\alpha^2 + \beta^2 + \gamma^2 = 0.$$

Hence we have

$$U_1^2 + U_2^2 + U_3^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (d),$$

and with (c) and (d) must be combined

$$\left.\begin{aligned} (\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2 &= 0 \\ U &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (e).$$

The equation (e) expresses that the algebraic distance of any point on the torse from the point of the curve of contact $P = 0$, $U = 0$ on the same generator, is nil.* The system now arrived at is obtained directly from the system (b), when $k = 0$, $P = 0$. If we actually proceeded with the elimination, we should get the square of the equation of the developable, the order of the resultant being $4m^2(m-1)$. But we may avoid the doubling of the order by taking the derived system

$$\begin{aligned} U &= 0, \\ U_1^2 + U_2^2 + U_3^2 &= 0, \\ (x - \alpha)\,U_2 - (y - \beta)\,U_1 &= 0, \\ (x - \alpha)\,U_3 - (z - \gamma)\,U_1 &= 0. \end{aligned}$$

The order of the resulting equation in γ is $2m^2(m-1)$.

It is easy to see that this focal torse will form part of the parallel, modulus 0. In that case the generating sphere is in fact a cone enveloping the circle at infinity. The primitive is the locus of the nodes. The developable is more strictly the parallel, modulus 0, of the curve of contact.

5. Because a finite length is evanescent at infinity, all the parallel surfaces, to finite moduli, of a given primitive have the same section at infinity, and its nature may now be inferred.

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* According to French nomenclature, right lines meeting the imaginary circle at infinity have been usefully termed "lines of no length."

In the first place, the corresponding section of the primitive is a double curve on the surface.

The developable of which I have been treating is one which circumscribes a surface and a plane curve. Consider, then, the section by the plane of the curve. This section consists of (1) the plane curve which is multiple in the degree n, if n is the class of the surface; for through a tangent of the curve can be drawn n tangent planes to the surface; (2) the common tangents of the curve and the section of the surface by the plane of the curve. These are generators of the torse. Thus, for the general surface, the section is

$$2\,(\text{section of primitive}) + m(m-1)^2\,(\text{circle at infinity}) + 2m(m-1)\,\text{lines}.$$

We may also consider the section at infinity as a line of curvature on any surface (in so far that the normals along the section form an envelope which lies on the surface of centres); and a finite modulus at infinity being evanescent, the section will consist of the parallel, modulus 0, of the corresponding section of the primitive together with the locus of points corresponding to the curve of contact of the focal torse, *i. e.*, the multiple circle.

5. Probably the most symmetrical as it is the most obvious form of the equation of a parallel surface, is the tangential one

$$\phi\,(\alpha+k,\ \beta+k,\ \gamma+k,\ \delta+k) = 0,$$

where the coordinates are the perpendiculars from four fixed points of reference to the tangent planes, and

$$\phi\,(\alpha\,\beta\,\gamma\,\delta) = 0$$

is the corresponding tangential equation of the primitive. If Π is the expression in these coordinates for the imaginary circle at infinity, we must make the above equation homogeneous by writing $k\sqrt{\Pi}$ for k, and the result, when rationalised, is of the form

$$\Phi^2 - k^2\Pi\Psi^2 = 0.$$

It is easy to see that the point polar of the plane at infinity is the same for the parallel and the primitive.

For more detailed statement, however, I prefer correlatives of Cartesian coordinates, which I have hitherto employed: that is to say, if

$$\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z-1=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{I})$$

represents a plane in x, y, z; I take X, Y, Z, -1 as the coordinates of the plane. Let then $\phi\,(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, -1) = 0$ be the reciprocal of the equation of the primitive, so that the plane (I) touches the primitive.

A plane which is parallel at a distance k to the tangent plane of the parallel touches the primitive. Such a plane is represented by

$$\mathrm{X}'x+\mathrm{Y}'y+\mathrm{Z}'z-1\pm k\mathrm{R}'=0,$$

where R′ is written for $\sqrt{\mathrm{X}'^2+\mathrm{Y}'^2+\mathrm{Z}'^2}$.

The reciprocal of the parallel (or the parallel itself tangentially) is represented by $\phi(X, Y, Z, -1 \pm kR) = 0,$

which may be written $\Phi^2 - k^2R^2\Psi^2 = 0$ .....(*g*),

accents being omitted.

In other words, the reciprocal of the parallel is obtained by writing $\frac{1}{R} \mp k$ for $\frac{1}{R}$ in the reciprocal of the primitive expressed in polar coordinates.

6. We are now able to see that the spinode torse of the parallel is of twice the class of the spinode torse of the primitive, and that the node-couple torse consists of a surface of twice the class of the node-couple torse of the primitive together with another surface of the class

$$m(m-1)^2\{m(m-1)^2-1\}.$$

In fact, let us suppose that (X_1, Y_1, Z_1) is a stationary plane or a double tangent plane of the primitive. Then for the corresponding planes, which are stationary or doubly tangential with respect to the parallel, we have $\frac{X}{1 \mp kR} = X_1, \quad \frac{Y}{1 \mp kR} = Y_1, \quad \frac{Z}{1 \mp kR} = Z_1,$

and we are to take all the k's positive or all negative.

Hence to such planes of the primitive passing through the point

$$AX + BY + CZ = 0,$$

correspond twice the number of such planes of the parallel passing through the same point, or the class of the torse of the parallel is twice the class of the corresponding torse of the primitive.

But we have also, from the form of the equation, a node-couple torse of the order $m(m-1)^2\{m(m-1)^2-1\}.$

It appears further that if the primitive has a spinode torse of the class c', and a node-couple torse of the class b', the parallel will have in general a spinode torse of the class $2c'$, and a node-couple torse of the class $2b'$, together with one of the class $n(n-1)$, where n is the class of the primitive.*

The equation of the pth parallel of the primitive, the parallels being taken to successive moduli $k_1, k_2, \ldots k_p$, will be of the form

$$\phi(X, Y, Z, -1 \pm k_1R \pm k_2R \pm \ldots \pm k_pR) = 0.$$

It breaks up accordingly when rationalized.

It will be observed that if we make $Z = 0$ in the reciprocal of the parallel, the result is in the form of the (reciprocal of the parallel) of the (reciprocal of a plane section) of the (reciprocal of the primitive). The characteristics are therefore known. But this amounts to the same thing as taking the parallel of the enveloping cylinder as *ante*, and is

* See note, p. 222.

better expressed by saying that the equations (g) and $Z=0$ tangentially represent a corresponding enveloping cylinder.

7. When the primitive is general in its class n, we can proceed further in determining the characteristics of the parallel. For in this case there is no spinode torse, and the node-couple torse envelopes a surface of the class n and another of the class $n-1$. The primitive being still without special relation to the plane at infinity, the characteristics of the tangent cone can be obtained directly from those of the parallel of a curve in plane space.

We can determine a sufficient number of characteristics to enable us to tabulate the general characteristics of the family of surfaces to which the parallels belong. The results will, however, indicate rather the equivalents of the singularities than the singularities themselves. It must be borne in mind that the case of quadrics, to which it is natural to refer, is specialized in three particulars; viz., a quadric is (1) a ruled surface, (2) symmetrical about three planes, and (3) general in class and order.

8. Referring to the list of characteristics given in my paper "On the Parallels of Conicoids and Conics" (Proc., Jan. 1872), I obtain the following results:

n_1	$2(n^3-n^2+n)$,*
a	$2n^2$,
δ	$2n(n-1)(n^2+n-4)$,
κ	$6n(n-1)$,
b	$n(n-1)(2n^4-2n^3+4n^2-13n+7) = \frac{1}{2}[n_1(n_1-1)-3c-a]$,
$2k$	$n(n-1)(4n^{10}-12n^9+28n^8-96n^7+180n^6-260n^5+487n^4-577n^3+395n^2-514n+440)$ $= b^2-(2n_1-3)\,b+\frac{9}{2}\beta+3\gamma+\rho$,
$3t$	$2n(n-1)(2n^7-4n^6+8n^5-45n^4+60n^3-60n^2+274n-294) = b(n-2)-\rho-2\beta-3\gamma$,
q	$n(n-1)(4n^3-18n+9)$,
ρ	$2n(n-1)(2n^3+2n-11)$,
j	0,
c	$4n(n-1)(2n-1)$,
h	$2n(n-1)(16n^4-32n^3+20n^2-42n+49)$,
r	$6n(n-1)(2n-1)$,
σ	$8n(n-1)$,
β	$2n(n-1)(22n-31)$,
γ	$8n(n-1)(n-2)(2n^3+n^2-5n-30)$,

* For a general centre of reciprocation, the orders of the parallels of a surface and its reciprocal are therefore the same. The centre may, however, be so taken as to alter this, by introducing special relations to the plane at infinity.

n_1'	$2n$,
a'	$2n^2$,
δ'	$2n(n-1)(n^2-n-1)$,
κ'	$2n(n^2-1)$,
b'	$n(n-1)$,
k'	$\frac{n(n-1)^2(n-2)}{2}$,
t'	0,
q'	$n(n-1)(2n-3)$,
ρ'	$2n(n-1)^2$,
j'	$2n(n-1)$,
c'	0.

The form of equation (g) shows that there are $2n(n-1)$ pinch-points on the nodal curve of the reciprocal. They lie on the point-sphere $X^2+Y^2+Z^2=0$, and are independent of the modulus. We know also the values of k' and q' for a curve forming the complete intersection of two surfaces. The value of ρ' is also easily obtained from the equation.

In fact, treating (g) as a point-equation, we have for the first polar of a point $\quad 2\Phi D\Phi - 2k^2R^2\Psi D\Psi - k^2\Psi^2 DR^2 = 0,$

where D stands for $X_1\frac{d}{dX}+Y_1\frac{d}{dY}+Z_1\frac{d}{dZ}+W_1\frac{d}{dW}$, and the equation (g) is supposed to be made homogeneous by the introduction of W for negative unity.

By substitution, and dividing by Φ, we get

$$2D\Phi - 2kRD\Psi - \Phi DR^2 = 0;$$

and the intersections of the curve of contact of the tangent cone from $(X_1, Y_1, Z_1, -1)$ with the nodal curve are given by

$$\Phi = 0, \quad \Psi = 0, \quad (D\Phi)^2 - k^2R^2(D\Psi)^2 = 0.$$

9. If two surfaces touch, their reciprocals with regard to the same quadric surface also touch. Since a parallel is the locus of the centres of spheres of constant radius touching the primitive, its equation can be obtained by forming the condition that

$$\left.\begin{array}{c}\varphi(X, Y, Z, -1) = 0 \\ k^2(X^2+Y^2+Z^2) - (Xx+Yy+Zz-1)^2 = 0\end{array}\right\} \ldots\ldots\ldots (g')$$

may touch. The first equation is the reciprocal of the primitive, the second is that of the system of spheres.

This mode of proceeding is not, in fact, usually the most convenient; but it suggests some important inferences.

We may interpret the system (g') thus:

If a surface is touched by a series of quadrics of revolution with a

common focus, the eccentricity of each being proportional to the distance from the focus of the pole of the directrix with respect to a sphere about the focus as centre, the locus of the poles is a parallel of the reciprocal of the surface with reference to the same sphere.

To bring the subject under a more general point of view, we may consider the envelope

$$\phi(X, Y, Z, W) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (h),$$

$$k^2(aX^2+bY^2+cZ^2+dW^2)-(Xx+Yy+Zz+Ww)^2=0 \ldots\ldots (k).$$

This seems to be the general class of envelopes to which parallels naturally belong.

Taking the reciprocal of the second equation, we get

$$\left(\frac{X^2}{a}+\frac{Y^2}{b}+\frac{Z^2}{c}+\frac{W^2}{d}-k^2\right)\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}\right)-\left(\frac{Xx}{a}+\frac{Yy}{b}+\frac{Zz}{c}+\frac{Ww}{d}\right)^2=0 \quad \ldots\ldots\ldots (l).$$

When we combine with this

$$\phi(x, y, z, w) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (m),$$

the envelope bears the same relation to ϕ that the envelope $[(h), (k)]$ bears to the reciprocal of ϕ, and we see that in general the order of the one envelope is the same as that of the other; in fact the one can be derived from the other by obvious substitutions which do not affect the order.

For a parallel surface, it will be observed we can make $d=0$, $W=1$ at once in (k), but in (l) d enters as a factor, and must first be divided out, and then equated to zero in the residuum.

According to a remark by Mr. W. Roberts, the equation of a parallel surface will give the first negative pedal of the primitive by writing $x^2+y^2+z^2$ for k^2, the values x, y, z being each one half of the corresponding coordinate of the pedal. The form (g') becomes, in this case, remarkably symmetrical, and we have

$$\phi(X, Y, Z, -1) = 0,$$

$$(x^2+y^2+z^2)(X^2+Y^2+Z^2)-(Xx+Yy+Zz-1)^2=0;$$

the second equation represents a system of paraboloids with a common focus. The surface in (x, y, z) is such that one of the normal distances, from a point on it to the primitive, is equal to its distance from a fixed point. In other words, the surface is the locus of the centre of spheres passing through a fixed point, and touching the primitive. And it is therefore the reciprocal of the inverse of the primitive in accordance with known results. The order of the surface is, however, reduced by the special substitution for k^2.

10. It becomes evident, on looking at the system (g'), that if the tangential equation of the primitive is of the form

$$\Phi^2-(X^2+Y^2+Z^2)\Psi^2=0 \quad \ldots\ldots\ldots\ldots\ldots (n),$$

the parallels will break up. Eliminating $X^2+Y^2+Z^2$ by means of the equation $k^2(X^2+Y^2+Z^2)-(Xx+Yy+Zz-1)^2=0$, we get an equation

$$k^2\Phi^2-(Xx+Yy+Zz-1)^2\Psi^2=0,$$

which represents two surfaces having the same general characteristics. The two surfaces forming the compound parallel will therefore also possess in general the same characteristics.

The equation (n) may be interpreted in general thus. *If a surface of the class $2n$ has the imaginary circle for a multiple curve in the degree n and a node-couple torse of the class $n(n-1)$, and if the node-couple torse and the developable generated by the tangent planes at points on the imaginary circle, envelope the same surface of the class n, the node-couple torse also enveloping a surface of the class $n-1$, then the parallel of such a primitive will break up.*

But the result may be materially modified by special singularities arising from particular relations of Φ, Ψ, and $X^2+Y^2+Z^2$. The parallel will, however, still break up, but the nature of the primitive will be altered so that the general conditions will cease to apply.* It is plain that when the parallel surface breaks up, so must also the enveloping cylinder, and *vice versâ*.

11. Further, considering that the system (g') gives the corresponding system for plane space by simply removing Z and z, we see at once that the torse tangentially represented by

$$\phi(X, Y, Z)=0,$$
$$X^2+Y^2+Z^2=0,$$

will have for its sections, by any series of parallel planes, the parallels to imaginary moduli of curves in plane space.†

For, in taking the reciprocal by means of the condition that

$$Xx+Yy+Zz-1=0$$

may touch the curve which is represented when X, Y, Z are point coordinates, we may first eliminate Z, obtaining

$$\phi(X, Y, \sqrt{-X^2-Y^2})=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (p),$$
$$z^2(X^2+Y^2)+(Xx+Yy-1)^2=0;$$

and the further process is precisely that of obtaining the parallel of the reciprocal curve to (p), the modulus being $\sqrt{-z^2}$. So that planes

* As an example, take the tangential equation $R=a+\frac{bX}{R}$, or the surface of revolution, of which an axial section is the first negative focal pedal of a conic; *i.e.*, the reciprocal of a Limaçon of Pascal.

† Hence, if we take two parallels of a curve in plane space, and place them in parallel planes, joining at the same time corresponding points by threads, we get a homographic transformation of a torse of the kind mentioned in the text.

parallel to the plane $z=0$ cut the torse in sections which are parallels of the sections by $z=0$.

In general, however, it is necessary, on account of the radical, to associate with (p) the equation

$$\phi(X, Y, -\sqrt{-X^2-Y^2}) = 0,$$

so that the rationalised form is

$$\Phi^2-(X^2+Y^2)\Psi^2 = 0,$$

a form which, according to what has been stated with reference to space of three dimensions, gives rise to a broken parallel.

If a surface is symmetrical about a plane, its reciprocal with respect to any point in the plane will be also symmetrical about the plane. Or, as it may be stated, the tangential equation may be taken so as to be symmetrical with respect to a point.

We have then an equation of the form

$$\phi(X, Y, Z^2) = 0,$$

and the parallels in plane space forming the corresponding sections of the torse, when Z is eliminated, will not break up.

This will be made more clear by taking a simple case.

If we are given $$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0,$$

then the system, representing the focal torse

$$aX^2+bY^2+cZ^2-1=0,$$
$$X^2+Y^2+Z^2=0,$$

gives a series of parallel curves which do not break up, and the corresponding torse does not break up. But if, instead of taking the centre of the given surface as the centre of reciprocation, we take another point, the corresponding system is, say,

$$(AX+BY+CZ-1)^2-(aX^2+bY^2+cZ^2)=0,$$
$$X^2+Y^2+Z^2=0,$$

and we get, by the process indicated, two developables. These surfaces belong to distinct conicoids, but have of course the same characteristics. The present case, however, applies to a species of surface which is necessarily symmetrical about a plane. When this is not so, we cannot infer that a plane section of either of the developables is a complete parallel in itself.*

12. With regard now to the sections of the focal torse, let

$$\phi(X, Y, Z) = 0$$

* I have lately seen that M. Darboux has shown that the sections are parallels. His proof is elegant and simple, depending on the fact that the generators of the torse in question are normal to any curve on the surface. (*Mém. de la Soc. des Sciences de Bordeaux*, t. viii., p. 303. 1872.) The present point of view is altogether different. See also "Note on the Parallel Curves of Conics," (*Quarterly Journal of Mathematics*, March, 1872, pp. 64, 65.

be of the class n with a node-couple torse of the class b, and a cuspidal torse of the class c, then the result of eliminating Z from the above equation by means of $X^2+Y^2+Z^2=0$ will take in X, Y the form of the tangential equation of a plane curve, having $2b$ double tangents and $2c$ inflexional tangents, arising from the singular developables, and $n(n-1)$ double tangents due to the effect of rationalising the equation. The class of the curve is $2n$, and we can get its other characteristics. For instance, if the surface $\phi(X, Y, Z)=0$ possesses point generality, and is of the order m, the curve in question, which is in fact the envelope of projections on its plane of the tangent planes of the developable, is tangentially represented by

$$\Phi^2-(X^2+Y^2)\Psi^2=0,$$

and has the circular points at infinity for multiple points in the degree $m(m-1)^2$, the class is $2m(m-1)^2$, and the order $2m^2(m-1)$ is obtained by making use of the known values of b, c for a general surface. The order of the parallel of such a curve is $4m^2(m-1)$, the class is $4m(m-1)^2$; and if δ and κ are the double points of the primitive curve, $2\delta+4m^4(m-1)^2$ and 2κ are the corresponding numbers for the parallel. Of the double points $4m^4(m-1)^2$ are due to the intersections of the two curves into which the parallel breaks up. The characteristics of each distinct curve of the parallel are those of the primitive.

It will be noticed moreover that the characteristics of the primitive are those of a plane section of the tangent cone to a parallel of the general surface with which we set out.

A great deal of what has been said will apply to hyper-space. Thus, if we take the hyper-torse

$$\phi(X, Y, Z, W)=0,$$
$$X^2+Y^2+Z^2+W^2=0,$$

and eliminate W, while taking the condition that these equations and

$$Xx+Yy+Zz+Ww-1=0$$

shall have a double solution, it appears that parallel sections of the hyper-torse will be parallels of surfaces in three dimensions to imaginary moduli, and so forth.

13. When there are singularities in the primitive, the order of the parallel is still twice the number of normals which can be drawn to the surface from an arbitrary point.* Suppose, for instance, that the surface is still not specially related to the plane at infinity, and is of the order m and class n. Then, considering the section of the parallel at

* The number of normals from an arbitrary point to a general surface has long been known, and was first determined, I believe, by M. Terquem. I do not know how far the result may have been generalised. In determining the order of the parallel for surfaces with singularities, we also obtain the corresponding number of normals.

infinity, we have seen that it consists of the corresponding section of the primitive taken twice, of the common tangents of that section and the imaginary circle at infinity, and of the circle at infinity which is multiple in the degree n. *Hence the order of the parallel is*

$$2(m+a+n),$$

where a is the order of the tangent cone, or class of the general section of the reciprocal of the primitive.

The characteristics of the tangent cone are obtained in the manner already mentioned, by considering a cylinder enveloping the surface.

The class of the parallel is twice the class of the primitive in every case. A cnicnode diminishes the order by 4, and a binode by 6.

14. If the primitive has p contacts with the plane at infinity, the order of the focal developable, and therefore the order of the parallel, is reduced by $2p$. Obviously the order of the parallel cannot be less than $2m$, m being the order of the primitive.

The order of the tangent cone is not, however, reduced by contact at infinity, and its characteristics cannot now be directly obtained from the consideration of an enveloping cylinder, since the plane at infinity is specially related to the primitive. The contacts at infinity give rise to a singular point at the origin on the reciprocal surface of the parallel, but do not affect the orders of the nodal and cuspidal curves on that surface. Hence we may obtain the characteristics of the tangent cone, which are in fact the same as when there are no contacts at infinity. Considering, again, the section of the parallel at infinity, we can see that in fact the circle at infinity is multiple in the degree $n-p$ when there are p contacts with the plane at infinity, and the generators at infinity of the focal torse which pass through the p points of contact count twice over.

When, however, the primitive is specially related to the plane at infinity and the imaginary circle on that plane, it is best to consider how many normals can be drawn from an arbitrary point.

For the form of the equation (g') shows that the coordinates and k enter the equation in the same degree, but k^2 necessarily enters in a power equal to the number of normals from an arbitrary point. Thus, if the primitive is a surface of revolution, the number of normals is the number that can be drawn to an axial plane section from a point in its plane, and the order of the parallel is twice this number. The number of normals which can be drawn to Dupin's cyclide from an arbitrary point is 4, and the order of the parallel (which breaks up) is 8.*

15. With reference to skew surfaces, it is of importance to deter-

* I shall perhaps mention this subject again with reference to the number of normals which can be drawn to a surface with singularities.

mine the effect of a straight line contained in the primitive and meeting the imaginary circle at infinity. For such a line all the normals lie in one plane, namely, the plane which contains the line and touches the circle. This being so, it appears at once that, in general, to the line in question correspond two lines on the parallel also in the same plane and meeting the circle at infinity in the same point. But from the peculiarity of this imaginary plane, it is necessary to consider the multiplicity of these lines on the parallel. To determine this, we may refer to the case of quadrics, on the parallels of which such lines are double. *Hence, in the case of a general skew surface, for every generator which meets the circle at infinity there are on the parallels two nodal lines meeting the circle in the same point.**

The circle being already multiple, we have on it points of higher multiplicity by two. If, however, there are several such straight lines on the primitive intersecting in the same point of the circle, the points are of still higher multiplicity. This is the case in quadrics in which pairs of generators so intersect.

M. Darboux has found that the imaginary planes in question triply touch the surface of centres along certain curves, and correspond in this respect to the eight planes of Prof. Clebsch, containing cuspidal conics of the surface of centres of a quadric.

As a further analogy, it may be mentioned that, if the primitive is symmetrical about a plane, the section of the parallel by the plane will contain the parallel in plane space of the corresponding section of the primitive, together with a double curve, the order of which can be derived from that of the parallel.

16. A parallel of a primitive surface of revolution is plainly also a surface of revolution, and a section by a plane through the axis is a parallel of the corresponding section of the primitive to the same modulus. Thus the determination of the nodal and cuspidal curves depends on the general theory of surfaces of revolution.

If a section of a surface of revolution by a plane through the axis has δ double points and κ cusps, of which δ' double points and κ' cusps are on the axis, there will be

$$\frac{\delta-\delta'}{2} \text{ nodal circles}, \qquad \frac{\kappa-\kappa'}{2} \text{ cuspidal circles},$$

$$\delta' \text{ cnicnodes}, \qquad \kappa' \text{ binodes}.$$

The degree of the tangent cone will be

$$n+2\delta'+3\kappa',$$

where n is the class of the section in question.

* When the tangent planes at points on a line of no length contained in a surface touch at p points, the corresponding lines are generally $p+1$ fold.

If, then, we reciprocate with respect to a point on the axis, we shall have a surface whose axial section is of the degree n, with $3(n-m)+\kappa$ cusps, and $\frac{(n-m)(n+m-9)}{2}+\delta$ double points; but of these $\frac{n-(m-2\delta'-3\kappa')}{2}$ will be nodes on the axis corresponding to circles of plane contact on the original surface. These will be generated by the $\frac{n-(m-2\delta'-3\kappa')}{2}$ double tangents necessarily perpendicular to the axis. These results will have to be modified if inflexional tangents are perpendicular to the axis.

Two points of the imaginary circle at infinity are singular points upon any surface of revolution of a higher order than the second. The focal torse of a surface of revolution consists of the point-spheres at the axial foci of a section by a plane through the axis. The number of these foci is equal to the class of the section. We must remember that the parallel of any such section will have multiple points at infinity. It is useful to regard the singularities of a surface of revolution in connection with established formulæ for general surfaces; but the subject is only incidentally connected with that of this paper, and cannot now be dwelt on.

17. A developable of the order r may be regarded as an extreme case of an ordinary surface. The order of the tangent cone will be n, the number of tangent planes which can be drawn through an arbitrary point. The class of the surface is 0, and the order and class of the parallel surfaces are $2(r+n)$ and 0; or it is also a developable, as is otherwise evident. It is easy to show that the differential condition is satisfied.

The theory of developables is usually associated with that of curves of double curvature.

18. A curve of double curvature may also be regarded as an extreme case of a surface. The surface order in this view is 0, the class is r, the degree of the reciprocal torse, or what is the same thing, the degree of the torse generated by the tangents of the curve (rank of the system); the order of the tangent cone is m, the order of a cone enveloping the curve, that is to say, the order is the degree of the curve. The order of the parallel tubular surface is therefore $2(r+m)$, and the class is $2r$.

Although the general parallels of developables and curves are in point of fact surfaces, the specialities of the primitives are so great that it is desirable perhaps to treat them apart, and I do not therefore develope more fully this branch of the subject in the present paper.

19. If X, Y, Z are the coordinates of a point on a parallel corresponding to the point whose coordinates are x, y, z on the primitive, and p, q, r, s, t signify as usual the first and second differentials of z with respect to x and y; P, Q, R, S, T being the corresponding expressions for Z, X, Y; the following relations hold:—

$$p = \mathrm{P}, \quad q = \mathrm{Q},$$

$$\left(\frac{d\mathrm{X}}{dx}\cdot\frac{d\mathrm{Y}}{dy} - \frac{d\mathrm{X}}{dy}\cdot\frac{d\mathrm{Y}}{dx}\right)\left(\frac{dx}{d\mathrm{X}}\cdot\frac{dy}{d\mathrm{Y}} - \frac{dx}{d\mathrm{Y}}\cdot\frac{dy}{d\mathrm{X}}\right) = 1,$$

$$\Theta = \frac{d\mathrm{X}}{dx}\cdot\frac{d\mathrm{Y}}{dy} - \frac{d\mathrm{X}}{dy}\cdot\frac{d\mathrm{Y}}{dx}$$

$$= \frac{k^2(rt-s^2)+k(1+p^2+q^2)^{\frac{1}{2}}\{(1+q^2)r+(1+p^2)t-2pqs\}+(1+p^2+q^2)^2}{(1+p^2+q^2)^2},$$

$$(1+\mathrm{P}^2)\,\mathrm{S}-\mathrm{PQR} = \frac{1}{\Theta}\{(1+p^2)\,s-pqr\},$$

$$(1+\mathrm{Q}^2)\,\mathrm{S}-\mathrm{PQT} = \frac{1}{\Theta}\{(1+q^2)\,s-pqt\},$$

$$(1+\mathrm{Q}^2)\,\mathrm{R}-(1+\mathrm{P}^2)\,\mathrm{T} = \frac{1}{\Theta}\{(1+q^2)\,r-(1+p^2)\,t\},$$

$$\frac{d\mathrm{X}}{dx} - 1 = k\,\frac{(1+q^2)\,r-pqs}{(1+p^2+q^2)^{\frac{3}{2}}},$$

$$\frac{d\mathrm{Y}}{dy} - 1 = k\,\frac{(1+p^2)\,t-pqs}{(1+p^2+q^2)^{\frac{3}{2}}},$$

$$\frac{d\mathrm{X}}{dy} = k\,\frac{(1+q^2)\,s-pqt}{(1+p^2+q^2)^{\frac{3}{2}}},$$

$$\frac{d\mathrm{Y}}{dx} = k\,\frac{(1+p^2)\,s-pqr}{(1+p^2+q^2)^{\frac{3}{2}}}.$$

The immediate conclusions from these formulæ are, however, geometrically obvious. Thus an umbilicus on the primitive corresponds to two umbilici on the parallel. To a plane line of curvature corresponds a plane line of curvature; the differential equation determining the directions in which a given normal is intersected by successive normals is the same for the system, and so forth.

February 13*th*, 1873.

Dr. HIRST, F.R.S., President, in the Chair.

Prof. R. Stawell Ball and Dr. J. Hopkinson were elected Members; and Prof. A. G. Greenhill, of Cooper's Hill College, was proposed for election.

The following papers were read:—

Prof. H. J. S. Smith "On the Higher Singularities of Plane Curves," and "On Systems of Linear Congruences."

Mr. J. Macleod "On the Application of the Hodograph to the Solution of Problems on Projectiles."

The President, Mr. Roberts, and Mr. Cotterill, made remarks on the communications.

The following presents were received:—

"Annual of the Royal School of Naval Architecture," No. 3, Jan. 1873: from Mr. C. W. Merrifield, F.R.S.

"Bulletin des Sciences Mathématiques," table des matières, Vol. iii., and No. 1, Vol. iv., Jan. 1873.

"Crelle," 75 Band, 3es Heft.

"Monatsbericht," Sept. and Oct. 1872.

"Journal of the London Institution," No. 18, Jan. 27, 1873.

Arithmetical Notes.*

By Henry J. Stephen Smith, Savilian Professor of Geometry in the University of Oxford.

I.—*On the Arithmetical Invariants of a Rectangular Matrix, of which the Constituents are Integral Numbers.*

1. Let $\|a_{ij}\|$ represent a rectangular matrix of the type $n \times (n+m)$, and let ∇_s be the greatest common divisor of the $\frac{\Pi n}{\Pi s . \Pi (n-s)} \times \frac{\Pi (n+m)}{\Pi s . \Pi (n+m-s)}$ minor determinants of order s which appertain to

* Being three Papers read at the January and February (1873) Meetings of the Society.

it; so that, in particular, ∇_n is the greatest common divisor of the $\frac{\Pi(m+n)}{\Pi n \,.\, \Pi m}$ determinants of the matrix; the numbers ∇_n, ∇_{n-1}, ... ∇_1 are the *arithmetical invariants* of the matrix. We suppose that the matrix is asyzygetic; *i. e.*, that the $\frac{\Pi(m+n)}{\Pi n \,.\, \Pi m}$ determinants are not all equal to zero. We then have the theorem (Memoir, p. 318 *sqq.**):

Theorem (*a*). "The quotient $\frac{\nabla_s}{\nabla_{s-1}}$ is the greatest common divisor of the quotients obtained by dividing each minor determinant of order s by the greatest common divisor of its own first minors."

Let p be a prime dividing ∇_i, and let I_i be the exponent of the highest power of p which divides ∇_i; any minor of order i which is divisible by $p^{I_i - \sigma}$ may be said to be divisible *in excess* by p^σ. Using this abbreviated mode of expression, we may enunciate the two following corollaries:

Corollary (*b*). "Any minor determinant, which is not divisible in excess by p, contains at least one first minor which is not divisible in excess by p."

Corollary (*c*). "Any minor determinant, which is not divisible in excess by p^σ, contains at least one first minor which is not divisible in excess by p^σ."

Or, which is the same thing,

"If all the first minors of a given minor are divisible in excess by p^σ, the given minor is itself divisible in excess by p^σ."

Of these corollaries, the first is a particular case of the second; and the second is only a re-statement, in other words, of the theorem (*a*).

2. It is the object of this note to establish a theorem, which may be regarded as reciprocal to the theorem (*a*).

Theorem (A). "The fraction $\frac{\nabla_{s-1}}{\nabla_s}$ is the greatest common divisor of the fractions obtained by dividing each minor of order $s-1$ by the greatest common divisor of its first majors; or, which is the same thing, the integral number $\frac{\nabla_s}{\nabla_{s-1}}$ is the least common denominator of these fractions."

Any square matrix which contains a given square matrix is here called, for brevity, a major of that matrix; if the given square matrix be of order $s-1$, its *first* majors are the square matrices of order s which contain it.

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* The references in this and the two following notes are to a Memoir "On Systems of Linear Indeterminate Equations and Congruences" (Phil. Trans., vol. cli., pp. 293—326).

The theorem (A) admits of two corollaries, corresponding to the corollaries (*b*) and (*c*):

Corollary (B). "Any minor determinant, which is not divisible in excess by p, is contained in at least one first major which is not divisible in excess by p."

Corollary (C). "Any minor determinant, which is not divisible in excess by p^σ, is contained in at least one first major which is not divisible in excess by p^σ."

Or, which is the same thing,

"If all the first majors of a given minor are divisible in excess by p^σ, the given minor is itself divisible in excess by p^σ."

3. To prove the theorem (A), we consider, in the first place, a *square* matrix $\|a_{ij}\|$ of the type $n \times n$, so that $\nabla_n = \Sigma \pm a_{11} a_{22} \dots a_{nn}$; and we represent the minor determinant $\frac{d\,\nabla_n}{d\,a_{ij}}$ by A_{ij}, so that the reciprocal matrix is $\|A_{ij}\|$. The determinant of this matrix is ∇_n^{n-1}, and the greatest common divisor of its minor determinants of order i is $\nabla_n^{i-1} \times \nabla_{n-i}$.

Let $M = \frac{\Pi n}{\Pi(s-1) \,.\, \Pi(n-s+1)}$, $N = n-s+1$; let k_μ $[\mu = 1, 2, 3, \dots M^2]$ represent any one of the M^2 minors of order $s-1$ which appertain to the matrix $\|a_{ij}\|$; let $k_{\mu,\nu}$ $[\nu = 1, 2, 3, \dots N^2]$ represent any one of the first majors of k_μ; let d_μ be the greatest common divisor of these N^2 numbers; lastly, let K_μ and $K_{\mu,\nu}$ represent the minors which, in the reciprocal matrix $\|A_{ij}\|$, are reciprocal to the minors k_μ and $k_{\mu,\nu}$ in the matrix $\|a_{ij}\|$; so that, for example, if $k_\mu = \Sigma \pm a_{11} a_{22} \dots a_{s-1,s-1}$, $K_\mu = \Sigma \pm A_{s,s} A_{s+1,s+1} \dots A_{n,n}$; and if $k_{\mu,\nu} = \Sigma \pm a_{11} a_{22} \dots a_{s,s}$, $K_{\mu,\nu} = \Sigma \pm A_{s+1,s+1} \dots A_{n,n}$. Thus the N^2 first minors of K_μ are precisely the determinants $K_{\mu,\nu}$; we also have the equations $K_\mu = \nabla_n^{n-s} \times k_\mu$, $K_{\mu,\nu} = \nabla_n^{n-s-1} \times k_{\mu,\nu}$; so that, applying the theorem (*a*) to the matrix $\|A_{ij}\|$, we find that the greatest common divisor of the M^2 integral numbers, $[\nabla_n^{n-s} \times k_\mu] \div [\nabla_n^{n-s-1} \times d_\mu] = \nabla_n \times \frac{k_\mu}{d_\mu}$, is the quotient $[\nabla_n^{n-s} \times \nabla_{s-1}] \div [\nabla_n^{n-s+1} \times \nabla_s] = \nabla_n \times \frac{\nabla_{s-1}}{\nabla_s}$; that is to say, the greatest common divisor of the M^2 fractions $\frac{k_\mu}{d_\mu}$ is the fraction $\frac{\nabla_{s-1}}{\nabla_s}$, in accordance with the theorem (A). The corollaries (B) and (C) may be immediately verified by observing that, if either of them were supposed untrue for any given minor k_μ, the corresponding fraction $\frac{k_\mu}{d_\mu}$ would acquire a denominator which could not be a sub-multiple of $\frac{\nabla_s}{\nabla_{s-1}}$.

4. To extend the demonstration to the case of an oblong matrix of the type $n \times (m+n)$, we retain the preceding notations, so far as they are applicable; and we put $M_1 = \frac{\Pi(n+m)}{\Pi(s-1) \cdot \Pi(n+m-s+1)}$, $N_1 = n+m-s+1$; so that the values of μ and ν are now $[1, 2, \ldots M \times M_1]$ and $[1, 2, \ldots N \times N_1]$ respectively. Let Q be a common multiple of the $M \times M_1$ numbers d_μ, and let us complete the given rectangular matrix into a square one by adding rows of constituents each of which is divisible by Q. In the resulting square matrix, any given minor k_μ (appertaining to the given rectangular matrix) will have N_1^2 first majors; but it is evident that the greatest common divisor of these N_1^2 first majors will be the same as that of the $N \times N_1$ first majors which appertain to the given rectangular matrix. Hence the $M \times M_1$ fractions $\frac{k_\mu}{d_\mu}$, obtained by dividing each minor of order $s-1$ in the given matrix by the greatest common divisor of its first majors, occurs among the M_1^2 fractions similarly derived from the completed square matrix; also the numbers ∇_s and ∇_{s-1} are evidently the same for both matrices. Applying, therefore, the theorem (A) to the square matrix, we see that $\frac{\nabla_{s-1}}{\nabla_s}$ is a common divisor of the $M \times M_1$ fractions $\frac{k_\mu}{d_\mu}$; that it is the greatest common divisor of these fractions, may be proved by considering a minor determinant of order s in the given matrix, which is not divisible in excess by p; this minor determinant contains [by the corollary (b)] a first minor which is itself not divisible in excess by p; if k_μ be this first minor, the denominator of the corresponding fraction $\frac{k_\mu}{d_\mu}$ necessarily contains the prime p raised to the power $I_s - I_{s-1}$.

It will be observed that, whether we consider a rectangular or a square matrix, the corollary (C) is an immediate consequence from the theorem (A); but the theorem (A) does not follow conversely from the corollary (C). For the absence of factors prime to $\frac{\nabla_s}{\nabla_{s-1}}$ from the least common denominator of the fractions $\frac{k_\mu}{d_\mu}$ is asserted in the theorem, but is not asserted in the corollary.

5. Every common divisor of the first minors of a given minor is evidently a common divisor of the first majors of that minor; and, if the matrix be square, a consideration of the reciprocal matrix shows that, conversely, every common divisor, prime to the determinant of the matrix, of the first majors of any given minor, divides the first minors of that minor. We thus obtain the self-reciprocal theorem:

Theorem (*d*D). "In any square matrix, the greatest common divisor of the first minors of any given minor is identical with the greatest common divisor of its first majors, so far as factors prime to the determinant of the matrix are concerned."

This theorem is not universally true in the case of oblong matrices, at least if we suppose that, in its enunciation, the words "determinant of the matrix" are replaced by the words "greatest common divisor of the determinants of the matrix." If, for example, $\|a\|$ is a square matrix of order $n-1$, and $\|b\|$ is a matrix of the type $n\times(n+m)$, of which the determinants are relatively prime, the symbol $\left\|\begin{matrix}0 & b\\ a & \end{matrix}\right\|$ may serve to represent a matrix of the type $n\times(2n+m-1)$, in which the first $n-1$ constituents of the uppermost row are zeros. In this matrix the greatest common divisor of the first majors of $|a|$ is evidently $|a|$ itself; and this greatest common divisor is prime to the greatest common divisor of the determinants of the matrix, because these determinants are relatively prime; but unless $\|a\|$ is an unit matrix, its determinant cannot be equal to the greatest common divisor of its own first minors.

6. It may be added, that the properties to which this note refers admit of being stated in a generalized form. Thus, we may replace the enunciations of the theorems (*a*), (A), and (*d*D) by the following:

Theorem (*a'*). "The quotient $\frac{\nabla_s}{\nabla_{s-i}}$ is the greatest common divisor of the quotients obtained by dividing each minor determinant of order s by the greatest common divisor of its own first minors of order $s-i$."

Theorem (A'). "The fraction $\frac{\nabla_{s-i}}{\nabla_s}$ is the greatest common divisor of the fractions obtained by dividing each minor of order $s-i$ by the greatest common divisor of its own majors of order s."

Theorem (*d*D'). "In any square matrix, the greatest common divisor of the minors of order $s-i$ appertaining to a given minor of order s is identical with the greatest common divisor of the majors of order $s+i$ appertaining to the given minor, so far as factors prime to the determinant of the matrix are concerned."

A very slight modification of the proof of the theorem (*a*), (Memoir, pp. 318—320,) supplies a proof of the theorem (*a'*), and from it the theorems (A') and (*d*D') may be inferred by means of the methods employed in this note to establish the theorems (A) and (*d*D).

II.—*On Systems of Linear Congruences.*

1. Let

$$(1) \ldots A_{i,1}x_1 + A_{i,2}x_2 + \ldots + A_{i,n}x_n \equiv A_{i,n+1} \pmod{M},$$
$$i = 1, 2, 3, \ldots n,$$

represent a system of n linear congruences; let

$$D_n, D_{n-1}, \ldots\ldots D_1,$$
$$\nabla_n, \nabla_{n-1}, \ldots\ldots \nabla_1,$$

respectively denote the arithmetical invariants of the augmented and unaugmented matrices of the system; *i. e.*, of the matrices

$$(2) \ldots\ldots\ldots\ldots \|A_{ij}\| \quad \begin{matrix} i = 1, 2, \ldots n, \\ j = 1, 2, \ldots n+1, \end{matrix}$$

and

$$(3) \ldots\ldots\ldots\ldots \|A_{ij}\| \quad \begin{matrix} i = 1, 2, \ldots n, \\ j = 1, 2, \ldots n; \end{matrix}$$

let d_s and δ_s be the greatest common divisors of M with $\frac{D_s}{D_{s-1}}$, and of M with $\frac{\nabla_s}{\nabla_{s-1}}$ respectively; lastly, let $d = d_1 \times d_2 \times \ldots d_n$, $\delta = \delta_1 \times \delta_2 \times \ldots \delta_n$. We then have the two theorems (Memoir, p. 320):

"The necessary and sufficient condition for the resolubility of the system (1) is $d = \delta$."

"When this condition is satisfied, the number of its incongruous solutions is $d = \delta$."

There are similar theorems (Memoir, p. 323) relating to defective and redundant systems of congruences. Some observations which may serve in certain cases to facilitate the applications of the theory are contained in the present note.

2. Consider separately the powers of the different prime numbers dividing the modulus M. Let p be one of these primes; and let μ, α_s, a_s be the exponents of the highest powers of p which divide M, D_s, ∇_s respectively. Then, because $\frac{D_{s+1}}{D_s} \div \frac{D_s}{D_{s-1}}$, $\frac{\nabla_{s+1}}{\nabla_s} \div \frac{\nabla_s}{\nabla_{s-1}}$, $\nabla_s \div D_s$, and $\frac{\nabla_s}{\nabla_{s-1}} \div \frac{D_s}{D_{s-1}}$, are all integral (Memoir, pp. 317—320), we have the inequalities

$$(4) \ldots\ldots a_n - a_{n-1} \geqq a_{n-1} - a_{n-2} \geqq \ldots\ldots \geqq a_1 - a_0;$$

$$(5) \ldots\ldots a_n - a_{n-1} \geqq a_{n-1} - a_{n-2} \geqq \ldots\ldots \geqq a_1 - a_0;$$

$$(6) \ldots\ldots\ldots\ldots\ldots\ldots a_s \geqq a_s;$$

$$(7) \ldots\ldots\ldots\ldots a_s - a_{s-1} \geqq a_s - a_{s-1}.$$

Let $a_\sigma - a_{\sigma-1}$ be the first term in the series (4) which is less than μ; so that, if $a_n - a_{n-1} < \mu$, we have $\sigma = n$, and in every other case

$$(8) \quad \ldots\ldots\ldots\ldots\ a_{\sigma+1} - a_\sigma \geq \mu > a_\sigma - a_{\sigma-1}.$$

We may then replace the two theorems of the Memoir by the two following:

"The necessary and sufficient condition for the resolubility of the congruences (1), considered with regard to the modulus p^μ, is

$$(9) \quad \ldots\ldots\ldots\ldots\ a_\sigma = a_{\sigma'}."$$

"When this condition is satisfied, the number of incongruous solutions is $p^{a_\sigma + (n-\sigma)\mu}$."

For the condition of resolubility is (Memoir, p. 321) that the greatest common divisor of the numbers

$$(10) \quad \ldots\ldots\ p^a,\ p^{a_{n-1}+\mu},\ p^{a_{n-2}+2\mu},\ \ldots\ldots\ p^{n\mu},$$

should be the same as the greatest common divisor of the numbers

$$(11) \quad \ldots\ldots\ p^{a_n},\ p^{a_{n-1}+\mu},\ p^{a_{n-2}+2\mu},\ \ldots\ldots\ p^{n\mu}.$$

The greatest common divisor of the numbers (10) is $p^{a_\sigma + (n-\sigma)\mu}$; for it follows from the inequalities (8) and (4) that, in the series (10), the exponent $a_\sigma + (n-\sigma)\mu$ is less than any exponent which follows it, and not greater than any which precedes it. Again, if $a_\sigma = a_{\sigma'}$, $p^{a_\sigma + (n-\sigma)\mu}$ is also the greatest common divisor of the numbers (11); for it is equal to one of them, and it certainly divides all of them, because it divides all the numbers (10), each of which, by virtue of the inequalities (6), divides one of the numbers (11). But if $a_\sigma > a_{\sigma'}$, the greatest common divisor of the numbers (11) is a power of p having an exponent higher than $a_\sigma + (n-\sigma)\mu$. For the inequality $a_\sigma > a_{\sigma'}$, combined with the inequality (7), implies the successive inequalities $a_{\sigma+1} > a_{\sigma'+1}$, $a_{\sigma+2} > a_{\sigma'+2}$, $\ldots$ $a_n > a_{n'}$; so that every exponent in the series (11), which precedes $a_\sigma + (n-\sigma)\mu$, surpasses the corresponding exponent in (10), and therefore certainly surpasses $a_\sigma + (n-\sigma)\mu$. And again, the exponents following $a_\sigma + (n-\sigma)\mu$ in the series (11) also surpass $a_\sigma + (n-\sigma)\mu$, for these exponents are not less than the corresponding exponents in (10), each of which is greater than $a_\sigma + (n-\sigma)\mu$. Thus the condition for the equality of the two greatest common divisors, *i. e.*, for the resolubility of the congruences (1), is $a_\sigma = a_{\sigma'}$. And, by transforming the proposed system (Memoir, p. 322) into an equivalent system of the type

$$(12) \quad \ldots\ldots\ldots\ldots\ \frac{\nabla_{n-s+1}}{\nabla_{n-s}} b_s \equiv c_s \pmod{p^\mu},$$

it is immediately ascertained that, when this condition is satisfied, the number of incongruous solutions is equal to $p^{a_\sigma + (n-\sigma)\mu}$, *i. e.*, to the common value of the two greatest common divisors.

3. The criterion of resolubility just established differs in form only from that given in the Memoir; and the coincidence of the two may be shown by observing that the equation $\alpha_\sigma = \alpha_\sigma$, combined with the inequalities (7), implies the equations $\alpha_{\sigma-1} = \alpha_{\sigma-1}$, $\alpha_{\sigma-2} = \alpha_{\sigma-2}$, ... $\alpha_1 = \alpha_1$. As here stated, the criterion of resolubility, and the expression for the number of solutions, depend only on the numbers σ, ∇_σ, D_σ, and do not involve explicitly the consideration of the two complete series of invariants. When μ surpasses $\alpha_n - \alpha_{n-1}$, the condition of resolubility is $\alpha_n = \alpha_n$, and the number of solutions is p^{α_n}. In this case we only need to determine the invariants ∇_n and D_n, or rather the exponents of the highest powers of p dividing those numbers. But when $\mu \leqq \alpha_n - \alpha_{n-1}$, it is necessary to calculate successively the invariants D_n, D_{n-1}, D_{n-2}, ... in order to ascertain the index σ for which the inequalities (8) are satisfied. It will be observed, however, that since every minor of order s, which is not divisible in excess by p, contains a minor of order $s-1$ which is itself not divisible in excess by p, it is always possible to restrict the examination of the minors of order $s-1$ (which is required in order to obtain the value of D_{s-1}) to those which are contained in a single minor of order s.

We may also notice that if, in the series of differences

$$\alpha_n - \alpha_n, \quad \alpha_{n-1} - \alpha_{n-1}, \quad \ldots\ldots \quad \alpha_1 - \alpha_1,$$

$\alpha_s - \alpha_s$ is the first which is equal to zero, $p^{\alpha_{s+1} - \alpha_s}$ is the highest power of p for which the proposed congruences are resoluble.

4. In the case of a redundant or defective system, as well as in the case of a system such as (1), which is neither defective nor redundant, it will be found that the equation of condition $d = \delta$ may be replaced by the equation $\alpha_\sigma = \alpha_\sigma$. And in all these cases alike, whenever the system admits of any solution at all, the number of incongruous solutions is $p^{\alpha_\sigma + (n-\sigma)\mu}$, n denoting the number of the indeterminates.

5. In the case of a redundant system, the condition $d = \delta$, or $\alpha_\sigma = \alpha_\sigma$, although necessary, is not sufficient (Memoir, p. 324). The complete condition of resolubility in this case is that the greatest common divisor of the numbers

$$(13) \ldots\ldots\ldots p^{\alpha_{n+1}}, \quad p^{\alpha_n + \mu}, \quad p^{\alpha_{n-1} + 2\mu}, \quad \ldots\ldots \quad p^{(n+1)\mu},$$

should be the same as the greatest common divisor of the numbers

$$(14) \ldots\ldots\ldots p^{\alpha_n + \mu}, \quad p^{\alpha_{n-1} + 2\mu}, \quad \ldots\ldots \quad p^{(n+1)\mu}.$$

And the identity of these two greatest common divisors implies not only (as in art. 2) the equation $\alpha_\sigma = \alpha_\sigma$, but also the inequality

$$(15) \ldots\ldots\ldots\ldots \mu \leqq \alpha_{n+1} - \alpha_n;$$

since, if this condition be not satisfied, a_{n+1} will be less than any of the exponents following it in the series (13), and therefore less than any exponent in the series (14).

The two conditions expressed by the equation $\alpha_\sigma = \alpha_s$, and the inequality (15), are sufficient as well as necessary. But it is remarkable that the exponent μ not only satisfies the inequality (15), but also the inequality

$$(16) \ldots\ldots\ldots\ldots \quad \mu \leqq a_{n+1} - a_n,$$

which is, in general, closer than (15), because $\alpha_n \geqq a_n$. To prove this, consider, in the given redundant system, any set of $n+1$ congruences such that the determinant of their augmented matrix is not divisible in excess by p; and multiply these $n+1$ congruences, taken in order, by the determinants of their unaugmented matrix. The sum of the products is the determinant of the augmented matrix; we infer that this determinant is divisible by $p^{\alpha_n + \mu}$, *i.e.* that $a_{n+1} \geqq \alpha_n + \mu$.

Since (as has been said) the two conditions of resolubility, taken together, are sufficient, they must involve the inequality (16). To deduce it from them, we may employ the following lemma:

"Let $$\| Q_{ij} \| \quad \begin{matrix} i = 1, 2, \ldots n, \\ j = 1, 2, \ldots n+m, \end{matrix}$$

represent any square or oblong matrix of integral numbers, and let $\| q \|$ be a partial matrix consisting of any s of the n horizontal rows of $\| Q \|$; also let $Q_n, Q_{n-1}, \ldots$; $q_s, q_{s-1}, \ldots$ be the arithmetical invariants of $\| Q \|$ and $\| q \|$ respectively; then, if r is any number not surpassing s, $\dfrac{Q_n}{Q_{n-r}}$ is divisible by $\dfrac{q_s}{q_{s-r}}$."

To establish this lemma, let us suppose that $\| q \|$ consists of the last s rows of $\| Q \|$, and let us replace $\| Q \|$ by a compound matrix of the form $\| Q' \| \times \| Q'' \|$, in which $\| Q' \|$ is a prime matrix of the same type as $\| Q \|$, and $\| Q' \|$ is a square matrix of the form

$$\begin{Vmatrix} h_1, & k_{1,2}, & k_{1,3}, & \ldots\ldots & k_{1,n} \\ 0, & h_2, & k_{2,3}, & \ldots\ldots & k_{2,n} \\ 0, & 0, & h_3, & \ldots\ldots & k_{3,n} \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ 0, & 0, & 0, & \ldots\ldots & h_n \end{Vmatrix}$$

Then the greatest common divisor of any horizontal row of minors in $\| Q \|$ is the same as the greatest common divisor of the corresponding row of minors in $\| Q' \|$ (Memoir, p. 311); it is sufficient, therefore, to verify the lemma in the case of this last matrix. In it we have

$$h_1 \times h_2 \times \ldots \times h_n = Q_n,$$
$$h_{n-s+1} \times h_{n-s+2} \times \ldots \times h_n = q_s;$$

and if we multiply $h_1 \times h_2 \times \ldots h_{n-s} = \frac{Q_n}{q_s}$ by any minor of order $s-r$ contained in the last s rows of $\|Q'\|$, we obtain a product which is either a minor of $\|Q'\|$ of the order $n-r$, or else is equal to zero; this product is therefore in every case divisible by Q_{n-r}; *i. e.*, $\frac{Q_n}{q_s} \times q_{s-r}$ is divisible by Q_{n-r}, or $\frac{Q_n}{Q_{n-r}}$ is divisible by $\frac{q_s}{q_{s-r}}$.

To apply this lemma to the case which we are considering here, we change n into $n+1$, and we put $s=n$, $r=n-\sigma$. Attending only to the powers of p, we thus find

$$\alpha_{n+1}-\alpha_n \geqq \alpha_{\sigma+1}-\alpha_\sigma.$$

But $\alpha_\sigma = \alpha_\sigma$, and $\alpha_{\sigma+1}-\alpha_\sigma \geqq \mu$; therefore $\alpha_{n+1}-\alpha_n \geqq \mu$.

6. When the determinants of the augmented or unaugmented matrix of a proposed system of congruences vanish, we may add multiples of the modulus to the constituents of these matrices, so as to obtain determinants which do not vanish. We proceed, however, to show that this preliminary operation is unnecessary, and that the preceding theory is immediately applicable, notwithstanding the evanescence of the determinants.

If, in any matrix of which the arithmetical invariants are ∇_n, ∇_{n-1}, ... ∇_1, the minor determinants of order $s+1$ are all equal to zero, we must attribute the same value to the invariant ∇_{s-1}; and *vice versâ*, if $\nabla_{s+1}=0$, the minor determinants of order $s+1$ must themselves be all equal to zero. Let $\nabla_{s+1}=0$, $\nabla_s>0$, so that $\nabla_{s+r}=0$ $(r=1, 2, \ldots n-s)$, $\nabla_{s-r}>0$ $(r=0, 1, 2, \ldots s-1)$. The quotients $\frac{\nabla_s}{\nabla_{s-1}}, \frac{\nabla_{s-1}}{\nabla_{s-2}}, \ldots \frac{\nabla_1}{\nabla_0}$ are determinate positive integers, and the quotient $\frac{\nabla_{s+1}}{\nabla_s}$ is zero; but the quotients $\frac{\nabla_{s+2}}{\nabla_{s+1}}, \ldots \frac{\nabla_n}{\nabla_{n-1}}$, assume the indeterminate form $\frac{0}{0}$. To these quotients we attribute the value zero: this may be regarded as an arbitrary definition of the value of these indeterminate numbers; it is, however, suggested by the observation that if the constituents of the matrix, instead of being integral numbers, were rational and integral functions of an indeterminate quantity, every factor of ∇_i would also be a factor of ∇_{i+1}, and would be contained in ∇_{i+1} oftener than in ∇_i; *i.e.*, $\frac{\nabla_{i+1}}{\nabla_i}$ would vanish with ∇_i.

7. Let us first suppose that the proposed system of congruences,

$$(17) \ldots\ldots A_{i,1}x_1 + A_{i,2}x_2 + \ldots\ldots + A_{i,n+m}x_{n+m} \equiv A_{n+m+1} \pmod{p^\mu},$$
$$i = 1, 2, 3, \ldots n, \qquad m \geqq 0,$$

is not redundant; and let the invariants ∇_n, ∇_{n-1}, ... of its unaugmented matrix vanish as far as ∇_{s+1} inclusively, but let ∇_s not vanish. The integers,

$$\begin{aligned} &a_n,\ a_{n-1},\ \ldots\ldots\ a_{s+1},\\ &a_n - a_{n-1},\ \ldots\ldots\ a_{s+1} - a_s,\\ &a_n,\ a_{n-1},\ \ldots\ldots\ a_{s+2},\\ &a_n - a_{n-1},\ \ldots\ldots\ a_{s+2} - a_{s+1}, \end{aligned}$$

may then be regarded as greater than any assigned number; and if, in the inequalities (8), which determine the index σ, we adopt this interpretation of these symbols, the criterion of resolubility, and the expression for the number of solutions, will continue applicable to the system (17) without any modification whatever. For the criterion of resolubility this is evident, because the demonstration of art. 2 subsists unchanged. To prove the same thing for the formula which expresses the number of solutions, we employ the following lemma:

"The matrix $\|\mathrm{A}_{i,j}\|$ $\begin{matrix} i = 1, 2, \ldots n, \\ j = 1, 2, \ldots n+m, \end{matrix}$ (in which ∇_{s+1} vanishes, but ∇_s does not vanish,) satisfies an equation of the form

$$(\mathrm{A}) \ldots\ldots\ldots \|\beta\| \times \|\mathrm{A}_{i,j}\| \times \|\gamma\| = \left\| \begin{matrix} q_{i,j}, & 0 \\ 0, & 0 \end{matrix} \right\|,$$

in which $\|\beta\|$ is an unit matrix of the type $n \times n$, $\|\gamma\|$ an unit matrix of the type $(n+m) \times (n+m)$, $\left\| \begin{matrix} q_{ij}, & 0 \\ 0, & 0 \end{matrix} \right\|$ a matrix of the type $n \times (n+m)$, which has all its constituents equal to zero, except those composing the partial matrix $\|q_{ij}\|$; this partial matrix being of the type $s \times s$, and having for its arithmetical invariants ∇_s, ∇_{s-1}, ... ∇_0."

Consider the system,

$$(18) \ldots\ldots \mathrm{A}_{i,1} x_1 + \mathrm{A}_{i,2} x_2 + \ldots\ldots + \mathrm{A}_{i,n+m} x_{n+m} = 0, \qquad i = 1, 2, 3, \ldots n,$$

which is equivalent to s independent equations; and take for $\|\gamma\|$ any unit matrix of the type $(n+m) \times (n+m)$, in which the last $n+m-s$ columns are the $n+m-s$ rows of a fundamental set of solutions of the system (18). The compound matrix $\|\mathrm{A}\| \times \|\gamma\|$ will be of the form $\|c_{ij}, 0\|$, where $\|c_{ij}\|$ is a partial matrix of the type $n \times s$, and $\|0\|$ is a partial matrix, composed entirely of zeros, of the type $n \times (n+m-s)$; so that $\|c_{ij}, 0\|$ is of the same type as $\|\mathrm{A}\|$. Again, consider the system

$$(19) \ldots\ldots\ldots c_{1,j} x_1 + c_{2,j} x_2 + \ldots\ldots + c_{n,j} x_n = 0, \qquad j = 1, 2, 3, \ldots s,$$

of which the equations are independent; and take for $\|\beta\|$ any unit matrix of the type $n \times n$, in which the $n-s$ lowest rows are the $n-s$

rows of a fundamental set of solutions of the system (19). The compound matrix $\|\beta\| \times \|A\| \times \|\gamma\|$ is then of the form $\left\| \begin{matrix} q_{ij}, & 0 \\ 0, & 0 \end{matrix} \right\|$.

If therefore, in the system (17), we transform the indeterminates by the substitution

$$(x_1, x_2, \ldots x_{n+m}) = \|\gamma\| \times (y_1, y_2, \ldots y_{n+m}),$$

and at the same time replace the congruences themselves by others linearly derived from them by multiplication with the constituents of the rows of $\|\beta\|$, we obtain an equivalent system of the form

$$(20) \ldots\ldots q_{1,j}\, y_1 + q_{2,j}\, y_2 + \ldots\ldots q_{s,j}\, y_s \equiv h_j \pmod{p^\mu},$$
$$j = 1, 2, \ldots s,$$

in which the indeterminates $y_{s+1}, \ldots y_{n+m}$ do not appear explicitly. By the theorem of art. 2, the number of solutions of (20) is $p^{a_\sigma + (s-\sigma)\mu}$ and to each of these solutions there correspond $p^{(n+m-s)\mu}$ solutions of (17), because each of the indeterminates $y_{s+1}, \ldots y_{n+m}$ may have any one of p^μ values. Thus the number of solutions of (17) is $p^{a_\sigma + (n+m-\sigma)\mu}$.

8. Next, let the proposed system of congruences be redundant; and, as before, let ∇_{s+1} vanish, but not ∇_s. In this case also the demonstration relating to the criteria of resolubility subsists unchanged; to determine the number of solutions, we select from the proposed system a partial system of s equations such that the determinants of its unaugmented matrix are not divisible in excess by p. Every solution of this partial system satisfies the remaining congruences of the proposed system; *i. e.*, the number of solutions of the proposed system is $p^{a_\sigma + (n-\sigma)\mu}$, if n be the number of indeterminates.

9. It is worth while to observe that, in the equation (A), we may attribute to the partial matrix $\|q_{ij}\|$ the simple form $\left\| \frac{\nabla_s}{\nabla_{s-1}}, \frac{\nabla_{s-1}}{\nabla_{s-2}}, \ldots \frac{\nabla_1}{\nabla_0} \right\|$. For if $\|u\|$ and $\|v\|$ are reducing units of $\|q_{ij}\|$, so that

$$\|u\| \times \|q_{ij}\| \times \|v\| = \left\| \frac{\nabla_s}{\nabla_{s-1}}, \frac{\nabla_{s-1}}{\nabla_{s-2}}, \ldots \frac{\nabla_1}{\nabla_0} \right\|,$$

we may replace $\|u\|$ and $\|v\|$, which are both of the type $s \times s$, by unit matrices of the types $n \times n$ and $(n+m) \times (n+m)$ respectively; and may then compound these substituted units with the unit matrices $\|\beta\|$ and $\|\gamma\|$. The unit matrices thus substituted for $\|u\|$ and $\|v\|$ contain $\|u\|$ and $\|v\|$ as partial matrices; their forms are sufficiently indicated by the symbols $\left\| \begin{matrix} u, \\ 0, \end{matrix} \begin{matrix} \\ \mathrm{I} \end{matrix} \right\|$, $\left\| \begin{matrix} v, & 0 \\ \mathrm{J} & \end{matrix} \right\|$; where I and J are matrices of the types $n \times (n-s)$ and $(n+m-s) \times (n+m)$ respectively, and are subject to no restriction other than that implied by the equations

$$\left| \begin{matrix} u, \\ 0, \end{matrix} \begin{matrix} \\ \mathrm{I} \end{matrix} \right| = 1, \quad \left| \begin{matrix} v, & 0 \\ \mathrm{J} & \end{matrix} \right| = 1.$$

Another transformation of a matrix, in which the first $n-s$ invariants vanish, is sometimes useful; viz., we may write

$$\|A\| = \|\omega\| \times \left\| 0, 0, \ldots 0, \frac{\nabla_s}{\nabla_{s-1}}, \frac{\nabla_{s-1}}{\nabla_{s-2}}, \ldots \frac{\nabla_1}{\nabla_0} \right\| \times \|Q\|,$$

where $\|\omega\|$ is an unit matrix of the type $n \times n$, and $\|Q\|$ is a matrix of the same type as $\|A\|$, of which the s lowest rows form a prime matrix, while the others are arbitrary. The proof of this transformation presents no difficulty, and may be omitted here.

10. When the absolute terms in a linear system of congruences are all congruous to zero, the criteria of resolubility are always satisfied, as they ought to be, because the system is satisfied by attributing to the indeterminates values congruous to zero, and therefore certainly admits of one solution at least. In this case, however, it is usually of importance to determine, not the whole number of solutions, but the number of solutions prime to the modulus (*i.e.*, the number of solutions in which one at least of the indeterminates is prime to the modulus). Reducing the proposed congruences to an equivalent system of the form (12), which can always be done by the transformations indicated in the Memoir (p. 322), and in art. 7 of the present note, we find that if N is the whole number of solutions, $N \div p^{n-\sigma}$ is the number of solutions in which every indeterminate has a value divisible by p, so that $N\left(1-\frac{1}{p^{n-\sigma}}\right)$ is the number of solutions prime to the modulus. And, in general, if k_i is the number of terms in the series (5) which satisfy the inequality $\alpha_{s+1}-\alpha_s \geqq \mu-i$, so that, in particular, $k_0 = n-\sigma$, the number of solutions in which every indeterminate has a value divisible by p^i will be found to be $N \div p^{k_0+k_1+\ldots+k_{i-1}}$.

11. The transformation of a given square matrix indicated by the equation

$$(21)\ldots\ldots \|A_{ij}\| = \|\alpha\| \times \left\| \frac{\nabla_n}{\nabla_{n-1}}, \frac{\nabla_{n-1}}{\nabla_{n-2}}, \frac{\nabla_{n-2}}{\nabla_{n-3}}, \ldots \frac{\nabla_2}{\nabla_1}, \frac{\nabla_1}{\nabla_0} \right\| \times \|\beta\|$$

forms the basis of many of the demonstrations in this and the preceding note. It has been already pointed out (Memoir, p. 314) that this equation always admits of an infinite number of solutions: this observation may be verified as follows. If $\|u_{ij}\|$ is any unit matrix of which the constituents are defined by the equations

$$u_{i,j} = \lambda_{i,j}, \qquad j < i,$$

$$u_{i,j} = \lambda_{i,j} \frac{\nabla_{n-i+1}}{\nabla_{n-i}} \div \frac{\nabla_{n-j+1}}{\nabla_{n-j}}, \qquad j \geq i.$$

the matrix $\|v_{ij}\|$, of which the constituents are defined by the equations

$$v_{ij} = \lambda_{ij}, \quad j \geq 1,$$

$$v_{ij} = \lambda_{ij} \frac{\nabla_{n-j+1}}{\nabla_{n-j}} \div \frac{\nabla_{n-i+1}}{\nabla_{n-i}}, \quad j \leq i,$$

will also be an unit matrix; and, if $\|\alpha\|$, $\|\beta\|$ be two given unit matrices satisfying the equation (21), that equation will also be satisfied by the unit matrices comprehended in the formulæ

$$\|\alpha\| \times \|u\|, \quad \|v\|^{-1} \times \|\beta\|,$$

and by them only.

III.—*On an Arithmetical Demonstration of a Theorem in the Integral Calculus.*

1. Let

$$\left.\begin{matrix} a_{i1}y_1 + a_{i2}y_2 + \ldots\ldots + a_{in}y_n = b_i \\ i = 1, 2, 3, \ldots n \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots \text{(A)}$$

represent a system of linear equations, in which the numbers a_{ij} and b_i are integral, the determinant $\nabla = |a_{ij}|$ being different from zero and positive. If, considering the numbers b_i with regard to the modulus ∇, we attribute to them in succession the ∇^n systems of values of which they are susceptible with regard to that modulus, we have the arithmetical theorem (Memoir, p. 325):

(i.) "The system (A) is resoluble in integral numbers y for precisely ∇^{n-1} of the ∇^n systems b."

2. Let Σ represent a complex, or multiplicity, of order n, of which the indeterminates $y_1, y_2, \ldots y_n$ are independent. Any system of values assigned to the indeterminates is called a point of Σ; and if the values are integral, the point is an integer point. Let

$$[x_{i1}, x_{i2}, x_{i3}, \ldots\ldots x_{in}]$$
$$i = 1, 2, 3, \ldots n,$$

represent n given integer points of Σ; we shall suppose these n points to be asyzygetic, so that the determinant $\nabla = |x_{ij}|$ is different from zero, and may be assumed to be positive. We then obtain the theorem:

(ii.) "The number of integer points $[X_1, X_2, \ldots X_n]$ which satisfy the inequalities

$$0 < X_1 \frac{d\nabla}{dx_{i1}} + X_2 \frac{d\nabla}{dx_{i2}} + \ldots\ldots + X_n \frac{d\nabla}{dx_{in}} \leq \nabla,$$
$$i = 1, 2, 3, \ldots n,$$

is precisely ∇."

For, by the theorem (i.), the equations

$$\left.\begin{gathered} X_1 \frac{d\nabla}{dx_{i1}} + X_2 \frac{d\nabla}{dx_{i2}} + \ldots\ldots + X_n \frac{d\nabla}{dx_{in}} = B_i \\ i = 1, 2, 3, \ldots n \end{gathered}\right\} \ldots\ldots\ldots\ldots (B),$$

of which the determinant is ∇^{n-1}, and in which each of the numbers B_i is supposed to receive in succession every value from 1 to ∇^{n-1} inclusive, are resoluble in integral numbers X for $\nabla^{(n-1)^2}$ of these $\nabla^{n(n-1)}$ systems of values. But the solution of the equations (B) is given by the system

$$\begin{gathered} \nabla X_j = B_1 x_{1j} + B_2 x_{2j} + \ldots\ldots + B_n x_{nj}, \\ j = 1, 2, 3, \ldots n. \end{gathered}$$

If therefore the equations (B) are resoluble in integral numbers for any particular system of values of the numbers B_i, considered with regard to the modulus ∇, they are resoluble for the $\nabla^{n(n-2)}$ systems of values congruous to that particular system for the modulus ∇, but incongruous to one another for the modulus ∇^{n-1}. Hence, if k be the number of systems of values, taken with regard to the modulus ∇, for which the equations (B) are resoluble in integral numbers, we have $k \times \nabla^{n(n-2)} = \nabla^{(n-1)^2}$, or $k = \nabla$; a result which establishes the theorem (ii.).

3. Another proof of the same theorem may be obtained as follows: The number of systems of values for which the equations (B) are resoluble in integral numbers is equal to the number of solutions of the congruences

$$\begin{gathered} B_1 x_{1j} + B_2 x_{2j} + \ldots\ldots + B_n x_{nj} \equiv 0, \pmod{\nabla}, \\ j = 1, 2, 3, \ldots n, \end{gathered}$$

in which $B_1, B_2, \ldots B_n$ are the indeterminates. And the number of solutions of this system of congruences is ∇. (Memoir, p. 320, or arts. 1—3 of the preceding note.)

4. Let

$$\begin{gathered} [y_{i1}, y_{i2}, \ldots\ldots y_{in}] \\ i = 1, 2, 3, \ldots n, \end{gathered}$$

represent any n asyzygetic points of Σ; and let λ_j be a positive common measure of the n quantities $y_{1,j}, y_{2,j}, \ldots y_{n,j}$ so that we have n^2 equations of the form $y_{ij} = x_{ij} \times \lambda_j$, in which the numbers x_{ij} are integral, and $|x_{ij}| = \nabla$ is different from zero and positive.

Let $D = |y_{ij}| = \nabla \times \prod_{j=1}^{j=n} \lambda_j$; the number of points of Σ, which are of the type $[X_1\lambda_1, X_2\lambda_2, \ldots X_n\lambda_n]$, $X_1, X_2, \ldots$ denoting integral numbers, and which satisfy the inequalities

$$0 < (X_1\lambda_1) \frac{dD}{dy_{i1}} + (X_2\lambda_2) \frac{dD}{dy_{i2}} + \ldots\ldots (X_n\lambda_n) \frac{dD}{dy_{in}} \leqq D \ldots\ldots\ldots (C),$$

is, by the theorem (ii.), equal to $D \div \Pi\lambda = \nabla$; that is to say, the product $\Pi\lambda$, taken as many times as there are points $(X_1\lambda_1, X_2\lambda_2, \ldots X_n\lambda_n)$ satisfying the conditions (C), is equal to D. Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be infinitesimal, and let U represent the integral $\int \ldots \iint dY_1\, dY_2 \ldots dY_n$, extended to all values of $Y_1, Y_2, \ldots Y_n$ which satisfy the inequalities

$$\left.\begin{gathered} 0 < Y_1 \frac{dD}{dy_{i1}} + Y_2 \frac{dD}{dy_{i2}} + \ldots\ldots + Y_n \frac{dD}{dy_{in}} \leqq D \\ i = 1, 2, 3, \ldots n \end{gathered}\right\};$$

we have, by the definition of a definite integral, $U = D$.

5. This well-known result (which may be described as a generalization of the expression for the volume of a parallelepiped in space referred to three rectangular axes) may of course be otherwise obtained by employing the formula for the transformation of a multiple integral; viz., if we put

$$\begin{gathered} u_i = Y_1 \frac{dD}{dy_{i1}} + Y_2 \frac{dD}{dy_{i2}} + \ldots\ldots + Y_n \frac{dD}{dy_{in}}, \\ i = 1, 2, 3, \ldots n, \end{gathered}$$

the limits of u_i are 0 and D, and the functional determinant is D^{n-1}; hence $\int_0^D \cdots \int_0^D \int_0^D du_1\, du_2 \ldots du_n = D^{n-1} \times U$, or $D = U$.

But as we have obtained the value of the definite integral U by a direct process, we may employ the result to demonstrate the formula for the transformation of a multiple integral; and this it may be worth while to do, because some of the demonstrations which have been given of that formula are not free from obscurity.

6. For convenience of expression, we may regard the indeterminates $X_1, X_2, X_3, \ldots X_n$ as rectangular coordinates of a homogeneous and isotropic space of n dimensions. On this supposition we may define the volume unit of Σ to be that portion of Σ which is comprised between the limits

$$\begin{gathered} 0 \leqq X_i \leqq 1, \\ i = 1, 2, 3, \ldots n; \end{gathered}$$

so that $dX_1\, dX_2 \ldots dX_n$ represents an element of Σ having the form (in n dimensions) of a rectangular parallelepiped. In any multiple integral $\int V d\Sigma$, of order n, we must regard the integration as extending over a certain "integral space," or portion of Σ; this integral space is to be resolved into infinitesimal elements, which may be of any form whatever, but which must exhaust the whole of it; each element is to be multiplied by the function V of its coordinates; and the definite in-

tegral is the sum obtained by adding the results. When the indeterminates are transformed by the equations

$$\left.\begin{array}{c} X_i = \phi_i(x_1, x_2, x_3, \ldots x_n) \\ i = 1, 2, 3, \ldots n \end{array}\right\} \quad \ldots\ldots\ldots\

correspond for elements which do correspond is admissible, because the value of a definite integral is not altered by substituting one mode of decomposition into elements for another. It may be added that the symbol $\int \mathrm{U}\, d\Sigma$, which involves no hypothesis as to the form of the element $d\Sigma$, supplies the most general and abstract expression of the definite integral; while the symbol $\int \ldots \iint \mathrm{U}\, d\mathrm{X}_1\, d\mathrm{X}_2 \ldots d\mathrm{X}_n$, which indicates a particular mode of decomposition into elements, suggests at the same time a particular method of obtaining its value by successive integration.

8. It will be seen that, when a definite order is established among the indeterminates (X) and (x), the ratio of corresponding elements in the two spaces is determined in sign as well as in magnitude; but that if two indeterminates in either set be interchanged, the sign of this ratio is also changed. The Jacobian loci $\left|\frac{d\mathrm{X}_i}{dx_j}\right| = 0$ and $\left|\frac{d\mathrm{X}_i}{dx_j}\right| = \infty$ (which for our present purpose we take together) may be regarded, in general, as dividing the space Σ into two regions A and B, in the first of which $\left|\frac{d\mathrm{X}_i}{dx_j}\right|$ is positive, and in the second negative. Similarly, σ is divided by the corresponding loci $\left|\frac{dx_j}{d\mathrm{X}_i}\right| = \left|\frac{d\mathrm{X}_i}{dx_j}\right|^{-1} = \infty, = 0$, into two regions a and b respectively corresponding to A and B, in such a manner that corresponding elements have the same sign in the regions A, a, and opposite signs in the regions B, b. But in using the formula for the transformation of a multiple integral, it is in general convenient to give to the functional determinant its absolute value, thus considering the elements of both spaces as positive throughout. And when the space over which the integration extends is traversed or bounded by the Jacobian loci, it is always necessary to examine the circumstances which present themselves at those loci.

Investigation of the Disturbance produced by a Spherical Obstacle on the Waves of Sound. *By the* Hon. J. W. Strutt, M.A.

[*Read November 14th*, 1872.]

Introduction.

When a train of plane waves otherwise unimpeded impinge on a space occupied by matter, whose mechanical properties differ from those

of the surrounding fluid, secondary waves are thrown off, which may be regarded as a disturbance due to the change in the nature of the medium—a point of view more especially appropriate, when the *region of disturbance*, as well as the alteration of mechanical properties, is small. If the obstacle be fluid, the mechanical properties spoken of are two—the *compressibility* and the *density*: no account is taken in the present paper of any friction or viscosity.

The problem to which the most attention has been given is that of a rigid spherical obstacle, which is either fixed, or (more generally) so supported that, when disturbed from the position of equilibrium, it is urged back by a force proportional to the displacement. The mathematical solution is worked out without any limitation as to the size of the sphere; but in drawing conclusions from it, I have confined myself for the most part to the case when the diameter of the sphere is small compared to the length of the sound waves. In the course of the work it became necessary to expand a^n in a series of La Place's (or rather, I believe, Legendre's) functions,—giving a result which has probably other important applications.*

I next consider the problem of a fluid spherical obstacle, working it out in full for a very small sphere; and afterwards investigate anew the same problem by a very different analysis, not restricted to the case of a sphere or an abrupt variation of mechanical properties on the one hand, but on the other, less general, in requiring the variation, and the region over which it occurs, to be small.

Finally, I have indicated the solution of the problem when the source of sound is at a finite distance from the obstacle, the primary waves being accordingly spherical instead of plane.

Expansion of the Velocity-potential in a Harmonic Series.

If ψ denote the velocity-potential of the motion in a gas, and a the velocity of transmission of sound, the differential equation satisfied by ψ is

$$\frac{d^2\psi}{dt^2} = a^2\left(\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz^2}\right) \quad \ldots\ldots\ldots\ldots\ldots (1),$$

while the variable part of the pressure is given by

$$\delta p = -\sigma\frac{d\psi}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

σ being the undisturbed density.

If we suppose that the motion corresponds to a simple tone of wave-

* In the theory of the vibrations of a solid body, the dilatation, and the three component distortions, are subject to a differential equation identical in form with that satisfied by the velocity-potential with which we have to deal in treating of fluid motions. The results of the present paper are at once applicable to the more general problem, so far as they go; but, as the boundary conditions are not completely expressible in terms of the four above mentioned quantities, further complication is introduced in deriving the values of the boundary displacements and their differentials.

length λ, ψ may be taken to be proportional to ϵ^{ikat}, where $k = 2\pi \div \lambda$, and $i = \sqrt{-1}$. The exponential factor will run through all the expressions containing ψ, and will sometimes be omitted for brevity. Thus (1) and (2) become

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\psi + k^2\psi = 0 \quad \ldots\ldots\ldots\ldots \quad (3),$$

$$\delta p = -ika\sigma\psi \quad \ldots\ldots\ldots\ldots \quad (4).$$

With any arbitrary point O for pole, ψ may be expanded in a series of surface harmonics $\psi = \psi_0 + \psi_1 + \ldots + \psi_n + \ldots$ ..... (5).

Considered, however, as a function of r (the radius vector), ψ_n is different from a solid harmonic of order n:* it satisfies the equation

$$\frac{d^2\psi_n}{dr^2} + \frac{2}{r}\frac{d\psi_n}{dr} - \frac{n(n+1)}{r^2}\psi_n + k^2\psi_n = 0 \ldots\ldots\ldots\ldots (6),$$

differing from that belonging to solid harmonics by the presence of the term $k^2\psi_n$.

In order to solve this equation, we may observe that when r is very great it passes into $\frac{d^2}{dr^2}(r\psi_n) + k^2(r\psi_n) = 0$ ..... (7).

of which the solution is $r\psi_n = A\epsilon^{ikr} + B\epsilon^{-ikr}$ ..... (8).

We may assume the same form for the solution of the complete equation (6), if we look upon A and B no longer as constants, but as functions of r to be determined. Substituting in (6), we find for B

$$\frac{d^2B}{dr^2} - 2ik\frac{dB}{dr} - \frac{n(n+1)}{r^2}B = 0 \ldots\ldots\ldots\ldots (9).$$

Now assume $B = B_0 + B_1 r^{-1} + B_2 r^{-2} + \ldots + B_s r^{-s} + \ldots$ ..... (10),

and substitute. Equating to zero the coefficient of r^{-s-2}, we obtain

$$B_{s+1} = B_s \frac{n(n+1) - s(s+1)}{2ik(s+1)} = B_s \frac{(n-s)(n+s+1)}{2ik(s+1)} \ldots\ldots (11).$$

Thus $$B_1 = \frac{n(n+1)}{2ik}B_0,$$

$$B_2 = B_1\frac{(n-1)(n+2)}{2ik\,.\,2} = \frac{(n-1)\ldots(n+2)}{2\,.\,4\,.\,(ik)^2}B_0;$$

so that

$$B = B_0\left\{1 + \frac{n(n+1)}{2\,.\,ikr} + \frac{(n-1)n(n+1)(n+2)}{2\,.\,4\,.\,(ikr)^2} + \ldots \right.$$

$$\left. \ldots + \frac{1\,.\,2\,.\,3\ldots 2n}{2\,.\,4\,.\,6\ldots 2n\,.\,(ikr)^n}\right\} \ldots\ldots (12).\dagger$$

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* Stokes, Phil. Mag., Dec. 1868; or Phil. Trans., 1868.

† Equation (6) may be advantageously put into the form

$$\frac{d^2(r\psi)}{d(kr)^2} - \frac{n(n+1)}{(kr)^2}(r\psi) + (r\psi) = 0.$$

Denoting, with Prof. Stokes, the series within brackets by $f_n(ikr)$,

$$B = B_0 f_n(ikr) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

A can be derived from B by changing the sign of i; thus,

$$A = A_0 f_n(-ikr) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

A_0 and B_0, though independent of r, are functions of the angular co-ordinates: in the most general case, any two spherical harmonics of order n. Equation (8) may therefore be written

$$r\psi_n = S_n \epsilon^{-ikr} f_n(ikr) + S_n' \epsilon^{+ikr} f_n(-ikr) \quad \ldots\ldots\ldots\ldots\ldots (15).$$

On differentiation of (15), there results

$$\frac{d\psi_n}{dr} = -\frac{S_n}{r^2} \epsilon^{-ikr} F_n(ikr) - \frac{S_n'}{r^2} \epsilon^{+ikr} F_n(-ikr) \quad \ldots\ldots\ldots\ldots (16),$$

where

$$F_n(ikr) = (1+ikr) f_n(ikr) - ikr f_n'(ikr) \quad \ldots\ldots\ldots\ldots (17).$$

The functions f and F may be expressed by means of the zonal harmonic functions Q, defined by the equation

$$\{1-2e\mu+e^2\}^{-\frac{1}{2}} = 1+Q_1 e+Q_2 e^2+\ldots+Q_n e^n+\ldots \quad \ldots\ldots\ldots (18).$$

It is known* that

$$Q_n(\mu) = 1-\frac{n}{1}\cdot\frac{n+1}{1}\cdot\frac{1-\mu}{2}+\frac{n(n-1)}{1.2}\cdot\frac{(n+1)(n+2)}{1.2}\left(\frac{1-\mu}{2}\right)^2-\ldots$$
$$\ldots\ldots\ldots (19);$$

or, changing μ into $1-\mu$,

$$Q_n(1-\mu) = 1-\frac{n}{1}\cdot\frac{n+1}{1}\cdot\frac{\mu}{2}+\frac{n(n-1)}{1.2}\cdot\frac{(n+1)(n+2)}{1.2}\cdot\frac{\mu^2}{2^2}+\ldots$$
$$\ldots\ldots\ldots (20).$$

Consider now the symbolic operator $Q_n\left(1-\frac{d}{dy}\right)$, and let it operate on $\frac{1}{y}$. Since $\left(\frac{d}{dy}\right)^s \frac{1}{y} = (-1)(-2)(-3)\ldots(-s)\, y^{-s-1}$,

$$Q_n\left(1-\frac{d}{dy}\right)\frac{1}{y} = y^{-1}+\frac{n(n+1)}{1.2}y^{-2}+\frac{(n-1)\ldots(n+2)}{2.4}y^{-3}+\ldots.$$

A comparison with (12) now shows that

$$f_n(y) = y\, Q_n\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y} \quad \ldots\ldots\ldots\ldots\ldots (21).$$

Again, from (17) we see that

$$\frac{F_n(y)}{y^2} = \left(1-\frac{d}{dy}\right)\cdot\frac{f_n(y)}{y} \quad \ldots\ldots\ldots\ldots (22);$$

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* Thomson and Tait's Natural Philosophy, § 782 (quoted from Murphy).

Whence $$F_n(y) = y^2 Q_n\left(1 - \frac{d}{dy}\right)\left(1 - \frac{d}{dy}\right)\cdot\frac{1}{y} \quad \ldots\ldots\ldots (23).$$

The equations (21), (23) originally presented themselves by an indirect process, which will be indicated in the proper place; but I have preferred to give here an analytical proof, in order to introduce them at once, without anticipation of future results.

Motion at the pole finite.

If it be part of the conditions of the question that the motion be continuous through the pole itself, equation (15) assumes a special form. Reversing the series for f_n, we have

$$r\psi_n = \frac{1.3.5\ldots(2n-1)}{(ikr)^n}\{S_n\epsilon^{-ikr}(1+ikr+\ldots) + (-1)^n S_n'\epsilon^{+ikr}(1-ikr+\ldots)\};$$

showing that, as r diminishes without limit, $r\psi_n$ approximates to

$$\frac{1.3.5\ldots(2n-1)}{(ikr)^n}\{S_n+(-1)^n S_n'\}.$$

In order therefore that ψ_n may be finite at the pole,

$$S_n+(-1)^n S_n' = 0 \quad \ldots\ldots\ldots (24)$$

is a necessary condition; that it is sufficient will be seen a little further on. Accordingly (15) becomes

$$r\psi_n = S_n\{\epsilon^{-ikr} f_n(ikr) - (-1)^n \epsilon^{+ikr} f_n(-ikr)\} \quad \ldots\ldots\ldots (25).$$

If by a separation of real and imaginary parts we put

$$f_n = \alpha' + i\beta' \quad \ldots\ldots\ldots (26),$$

(25) becomes

$$r\psi_n = -2i^{n+1} S_n\left\{\alpha' \sin\left(kr + n\frac{\pi}{2}\right) - \beta' \cos\left(kr + n\frac{\pi}{2}\right)\right\} \quad \ldots\ldots (27).$$

Another form of (25) may be derived by means of (21). If $ikr = y$,

$$\psi_n = ikS_n\left\{\epsilon^{-y} Q_n\left(1 - \frac{d}{dy}\right)\cdot\frac{1}{y} - (-1)^n \epsilon^{+y} Q_n\left(1 + \frac{d}{dy}\right)\cdot\frac{1}{y}\right\}$$

$$= (-1)^n ikS_n\left\{Q_n\left(\frac{d}{dy}\right)\cdot\frac{\epsilon^{-y}}{y} - Q_n\left(\frac{d}{dy}\right)\cdot\frac{\epsilon^{+y}}{y}\right\}$$

$$= -2ik(-1)^n S_n Q_n\left(\frac{d}{d.ikr}\right)\cdot\frac{\sin kr}{kr} \quad \ldots\ldots\ldots (28).$$

A comparison of (28) and (27) shows that

$$Q_n\left(\frac{d}{d.ikr}\right)\cdot\frac{\sin kr}{kr} = \frac{i^{-n}}{kr}\left\{\alpha' \sin\left(kr + n\frac{\pi}{2}\right) - \beta' \cos\left(kr + n\frac{\pi}{2}\right)\right\} \ldots (29).$$

Again, introducing (24) into (16), the latter becomes

$$\frac{d\psi_n}{dr} = -\frac{S_n}{r^2}\{\epsilon^{-ikr} F_n(ikr) - (-1)^n \epsilon^{+ikr} F_n(-ikr)\} \quad \ldots\ldots (30).$$

If $F = \alpha + i\beta$, this may be written

$$\frac{d\psi_n}{dr} = \frac{2i^{n+1}S_n}{r^2}\left\{\alpha \sin\left(kr + n\frac{\pi}{2}\right) - \beta\cos\left(kr + n\frac{\pi}{2}\right)\right\} \quad \ldots\ldots (31).$$

By differentiating (28) we obtain a third form for $\frac{d\psi_n}{dr}$,

$$\frac{d\psi_n}{dr} = -2ik^2(-1)^n S_n Q_n\left(\frac{d}{d\,.\,ikr}\right)\frac{d}{d\,.\,kr}\cdot\frac{\sin kr}{kr} \quad \ldots\ldots (32).$$

From a comparison between (32) and (31), we infer

$$Q_n\left(\frac{d}{d\,.\,ikr}\right)\frac{d}{d\,.\,kr}\cdot\frac{\sin kr}{kr} = -\frac{i^{-n}}{k^2r^2}\left\{\alpha\sin\left(kr + n\frac{\pi}{2}\right) - \beta\cos\left(kr + n\frac{\pi}{2}\right)\right\} \quad \ldots\ldots (33).$$

These equations constitute the solution of the problem of determining the *proper tones* of any spherical cavity bounded by rigid walls. In a paper published in the Proceedings of the Society for 1872, I have tabulated the roots of the equation

$$Q_n\left(\frac{d}{d\,.\,ikr}\right)\frac{d}{d\,.\,kr}\cdot\frac{\sin kr}{kr} = 0 \quad \ldots\ldots (34),$$

which gives the values of k or λ for any assigned value of r.

When n is considerable, ψ_n and $\frac{d\psi_n}{dr}$ are of high order in r; we shall presently see how to find the leading terms.

Expansion in Spherical Harmonics of the Velocity-potential of Plane Waves.

This necessarily comes under the head already considered; so that we might assume as the expression for the velocity-potential,

$$\phi_n = \text{const.}\ Q_n(\mu)\,.\,Q_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr} \quad \ldots\ldots (35).$$

The plane waves are supposed to be normal to the axis from which μ is measured; the symmetry then requires that the general harmonic S_n reduce to a simple multiple of $Q_n(\mu)$. But it will be more instructive to work this case out independently.

If ϕ be the velocity-potential, we have

$$\phi = \epsilon^{ik(at+x)} = \epsilon^{ikat}\,.\,\epsilon^{ikr\mu} \quad \ldots\ldots (36).$$

It is required to expand $\epsilon^{ikr\mu}$ in spherical harmonics.

Taking then

$$\epsilon^{ikr\mu} = A_0 + A_1Q_1 + A_2Q_2 + \ldots + A_nQ_n + \ldots \quad \ldots\ldots (37),$$

multiplying by Q_n, and integrating with respect to μ between the limits -1 and $+1$, we find

$$\int_{-1}^{+1} Q_n(\mu)\,\epsilon^{ikr\mu}\,d\mu = A_n\int_{-1}^{+1}(Q_n)^2\,d\mu = \frac{2A_n}{2n+1} \quad \ldots (38).$$

by a known theorem. Accordingly

$$A_n = \frac{2n+1}{2}\int_{-1}^{+1} Q_n(\mu)\, \epsilon^{ikr\mu}\, d\mu \quad \ldots\ldots\ldots\ldots\ldots (39).$$

Putting n equal to zero, and remembering that $Q_0 = 1$, we see that

$$A_0 = \tfrac{1}{2}\int_{-1}^{+1} (\cos kr\mu + i \sin kr\mu)\, d\mu$$

$$= \int_0^1 \cos kr\mu\, d\mu = \frac{\sin kr}{kr} \quad \ldots\ldots\ldots\ldots\ldots (40).$$

Now since

$$\int_{-1}^{+1} \epsilon^{ikr\mu}\, d\mu = \frac{2 \sin kr}{kr},$$

and

$$\left(\frac{d}{d\,.\,ikr}\right)^n \int_{-1}^{+1} \epsilon^{ikr\mu}\, d\mu = \int_{-1}^{+1} \mu^n\, \epsilon^{ikr\mu}\, d\mu,$$

it follows that

$$\int_{-1}^{+1} \mu^n \epsilon^{ikr\mu}\, d\mu = 2\left(\frac{d}{d\,.\,ikr}\right)^n \cdot \frac{\sin kr}{kr},$$

and therefore, if $Q_n\left(\frac{d}{d\,.\,ikr}\right)$ denote the same function of $\frac{d}{d\,.\,ikr}$ that $Q_n(\mu)$ is of μ,

$$\int_{-1}^{+1} Q_n(\mu)\, \epsilon^{ikr\mu}\, d\mu = 2 Q_n\left(\frac{d}{d\,.\,ikr}\right) \cdot \frac{\sin kr}{kr} \quad \ldots\ldots\ldots\ldots (41),$$

so that from (39)

$$A_n = (2n+1)\, Q_n\left(\frac{d}{d\,.\,ikr}\right) \cdot \frac{\sin kr}{kr} \quad \ldots\ldots\ldots\ldots\ldots (42).$$

$Q_n(\mu)$ being a rational and integral function of μ, A_n is satisfactorily given by (42), and we obtain for ϕ, from (36),

$$\phi = \epsilon^{ikat} \Sigma\, (2n+1)\, Q_n(\mu)\, Q_n\left(\frac{d}{d\,.\,ikr}\right) \cdot \frac{\sin kr}{kr} \quad \ldots\ldots\ldots (43),$$

$$\frac{d\phi}{dr} = k\epsilon^{ikat} \Sigma (2n+1)\, Q_n(\mu)\, Q_n\left(\frac{d}{d\,.\,ikr}\right) \cdot \frac{d}{d\,.\,kr}\frac{\sin kr}{kr} \quad \ldots\ldots (44).$$

It was from these equations so obtained that I first perceived the relations expressed in (22) and (23). From the nature of the case it was evident that by a proper determination of S_n, the expression for ψ_n in (27) must be capable of being identified with that for φ_n in (43). By trigonometrical transformation we may reduce $Q_n\left(\frac{d}{d\,.\,ikr}\right) \cdot \frac{\sin kr}{kr}$ to

$$\frac{1}{2i^n}\left\{ \begin{array}{l} i \sin\left(kr + n\frac{\pi}{2}\right)\left\{ Q_n\left(1+\frac{d}{dy}\right)\cdot\frac{1}{y} + Q_n\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y}\right\} \\ +\cos\left(kr+n\frac{\pi}{2}\right)\left\{ Q_n\left(1+\frac{d}{dy}\right)\cdot\frac{1}{y} - Q_n\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y}\right\} \end{array}\right\},$$

where $ikr = y$. A comparison with (27) now proves that

$$\left.\begin{aligned} \alpha' &= \mathrm{H}ir\left\{\mathrm{Q}_n\left(1+\frac{d}{dy}\right)+\mathrm{Q}_n\left(1-\frac{d}{dy}\right)\right\}\cdot\frac{1}{y}\\ \beta' &= -\mathrm{H}r\left\{\mathrm{Q}_n\left(1+\frac{d}{dy}\right)-\mathrm{Q}_n\left(1-\frac{d}{dy}\right)\right\}\cdot\frac{1}{y}\end{aligned}\right\},$$

where H is some constant independent of r, but possibly dependent on n.

We deduce $f_n = \alpha' + i\beta' = 2i\mathrm{H}r\mathrm{Q}_n\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y}$;

H is now easily determined by considering in this equation the term of highest order in r and y. On the left-hand side we have, from (12),

$$f_n = 1 + \ldots\ldots;$$

on the right the term of highest order will be obtained by leaving out the symbol of differentiation. Remembering that $\mathrm{Q}_n(1) = 1$, we have

$$2i\mathrm{H}r\cdot\frac{1}{y} = 2\frac{\mathrm{H}}{k};$$

H therefore equals $\frac{1}{2}k$; whence

$$f_n(y) = y\mathrm{Q}_n\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

Expansion in powers of r.

Expanding the exponential in (39), we get

$$\frac{2\mathrm{A}n}{2n+1} = \int_{-1}^{+1} d\mu\,\mathrm{Q}_n(\mu)\left\{1 + ikr\,.\,\mu + \frac{(ikr)^2}{1\,.\,2}\mu^2 + \ldots + \frac{(ikr)^n}{1\,.\,2\,\ldots\,n}\mu^n + \frac{(ikr)^{n+1}}{1\,.\,2\,\ldots\,(n+1)}\mu^{n+1} + \frac{(ikr)^{n+2}}{1\,.\,2\,\ldots\,(n+2)}\mu^{n+2} + \ldots\right\}.$$

Now any positive integral power of μ, such as μ^p, can be expanded in a terminating series of the functions Q, the function of highest order being Q_p. It follows that if $p < n$,

$$\int_{-1}^{+1}\mu^p\mathrm{Q}_n(\mu)\,d\mu = 0,$$

by known properties of these functions. And further, whether p be less than n or not, if they be such that one is odd while the other is even, we still have the same integral equal to zero. We therefore write

$$\frac{2\mathrm{A}_n}{2n+1} = \int_{-1}^{+1} d\mu\,\mathrm{Q}_n(\mu)\left\{\frac{(ikr)^n}{1\,.\,2\,\ldots\,n}\mu^n + \frac{(ikr)^{n+2}}{1\,.\,2\,\ldots\,(n+2)}\mu^{n+2} + \ldots\right\} \quad \ldots\ldots\ldots (45).$$

The expression for $\mathrm{Q}_n(\mu)$ in powers of μ is

$$\mathrm{Q}_n(\mu) = \frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{1\,.\,2\,.\,3\,\ldots\,n}\left\{\mu^n - \frac{n\,(n-1)}{2\,(2n-1)}\mu^{n-2} + \frac{n\,(n-1)\,(n-2)\,(n-3)}{2\,.\,4\,.\,(2n-1)\,(2n-3)}\mu^{n-4} - \ldots\right\} \quad \ldots\ldots (46).$$

so that $\mu^n = \dfrac{1.2.3 \ldots n}{1.3.5 \ldots (2n-1)} Q_n +$ terms in μ of lower order than μ^n; and therefore

$$\int_{-1}^{+1} \mu^n Q_n(\mu)\, d\mu = \frac{1.2.3 \ldots n}{1.3.5 \ldots (2n-1)} \cdot \frac{2}{2n+1} \quad \ldots\ldots\ldots (47).$$

Again, from (46), changing n into $n+2$,

$$Q_{n+2} = \frac{1.3.5 \ldots (2n+3)}{1.2.3 \ldots (n+2)} \left\{ \mu^{n+2} - \frac{(n+2)(n+1)}{2(2n+3)} \mu^n + \ldots \right\}.$$

Multiplying by Q_n, and integrating between -1 and $+1$, there results

$$0 = \int_{-1}^{+1} \mu^{n+2} Q_n\, d\mu - \frac{(n+2)(n+1)}{2(2n+3)} \int_{-1}^{+1} \mu^n Q_n\, d\mu \quad \ldots\ldots\ldots (48).$$

Thus

$$A_n = \frac{2n+1}{2} \frac{(ikr)^n}{1.2 \ldots n} \int_{-1}^{+1} \mu^n Q_n\, d\mu \left\{ 1 + \frac{(ikr)^2}{2(2n+3)} + \ldots \right\}$$

$$= \frac{(ikr)^n}{1.3.5 \ldots (2n-1)} \left\{ 1 + \frac{(ikr)^2}{2(2n+3)} + \ldots \right\} \quad \ldots\ldots\ldots (49).$$

Comparing with (42),

$$Q_n \left(\frac{d}{d \cdot ikr} \right) \frac{\sin kr}{kr} = \frac{(ikr)^n}{1.3.5 \ldots (2n+1)} \left\{ 1 - \frac{k^2 r^2}{2(2n+3)} + \ldots \right\} \ldots (50).$$

The expansion of ψ_n, when the motion at the pole is finite, is therefore, from (28),

$$\psi_n = -2ik\,(-1)^n S_n \frac{(ikr)^n}{1.3 \ldots (2n+1)} \left\{ 1 - \frac{k^2 r^2}{2(2n+3)} + \ldots \right\} \ldots (51),$$

showing that, when r vanishes, ψ_n is not only not infinite, but an infinitesimal quantity of order r^n.

Another method, which is especially to be preferred if more terms in the series be desired, leads to an expression for ψ_n in terms of a Bessel's function of fractional order. Consider the differential equation satisfied by Bessel's functions

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{m^2}{z^2}\right) y = 0 \quad \ldots\ldots\ldots (a),$$

and write it in the form

$$\frac{d^2(yz^{\frac{1}{2}})}{dz^2} + \left(1 - \frac{4m^2-1}{4z^2}\right)(yz^{\frac{1}{2}}) = 0 \quad \ldots\ldots\ldots (b).$$

It is known that the solution of (a), subject to the condition of finiteness, when $z=0$, is $\qquad y = A J_m(z)$,

where

$$J_m(z) = \frac{z^m}{2^m \Gamma(m+1)} \left\{ 1 - \frac{z^2}{2(2m+2)} + \frac{z^4}{2.4(2m+2)(2m+4)} \right\} \ldots (c),$$

is the Bessel's function of order m. When m is integral,

$$\Gamma(m+1) = \lfloor m ;$$

but here we shall have to do with m fractional and of the form $n+\frac{1}{2}$, n being integral. In this case,

$$\Gamma(m+1) = \frac{1.3.5 \ldots (2n+1)}{2^{n+1}} \sqrt{\pi}.$$

Referring now to (6), we see that the solution of

$$\frac{d^2\theta}{dz^2} + \left(1 - \frac{4m^2-1}{4z^2}\right)\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (d),$$

under the above-mentioned condition, is

$$\theta = Az^{\frac{1}{2}} J_m(z).$$

Now the function ψ_n, with which we are here concerned, satisfies

$$\frac{d^2(r\psi_n)}{d(kr)^2} + \left(1 - \frac{n(n+1)}{(kr)^2}\right) r\psi_n = 0 \quad \ldots\ldots\ldots\ldots\ldots (e),$$

which is of the same form as (d), if $m = n+\frac{1}{2}$; so that

$$r\psi_n \propto (kr)^{\frac{1}{2}} J_{n+\frac{1}{2}}(kr),$$

or

$$\psi_n = A(kr)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(kr)$$

$$= A\frac{(kr)^n \sqrt{2}}{1.3 \ldots (2n+1)\sqrt{\pi}} \left\{1 - \frac{z^2}{2(2n+3)} + \frac{z^4}{2.4(2n+3)(2n+5)} - \ldots\right\}.$$

Determining the constant by a comparison with (51), we find

$$\psi_n = -2(-1)^n i^{n+1} k S_n \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(kr)$$

$$= -2ik(-1)^n S_n \frac{(ikr)^n}{1.3.5 \ldots (2n+1)}$$

$$\times \left\{1 - \frac{k^2r^2}{2(2n+3)} + \frac{k^4r^4}{2.4(2n+3)(2n+5)} - \ldots\right\} \quad (f).$$

It may be remarked that the function $J_m(z)$ can be expanded in an always convergent ascending series, and also in descending series multiplied by exponential (or circular) factors, which belong to the semi-convergent class. The ascending series are always infinite; but when, as here, $m-\frac{1}{2}$ is integral, the other series terminate, and may then, of course, be inverted, so as to proceed by ascending powers. In the general case,

$$J_m(z) = \sqrt{\left(\frac{2}{\pi z}\right)} \left\{1 - \frac{(1^2-4m^2)(3^2-4m^2)}{1.2(8z)^2} + \ldots\right\} \cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$$

$$+ \sqrt{\left(\frac{2}{\pi z}\right)} \left\{\frac{1^2-4m^2}{1.8z} - \frac{(1^2-4m^2)(3^2-4m^2)(5^2-4m^2)}{1.2.3(8z)^3} + \ldots\right\}$$

$$\times \sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right).$$

Similar modes of expression exist for $\frac{d\psi_n}{dr}$. We have

$$\frac{d\psi_n}{dr} = Ak\left\{(kr)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(kr) - \tfrac{1}{2}(kr)^{-\frac{3}{2}}J_{n+\frac{1}{2}}(kr)\right\}.$$

Now
$$J_{n+\frac{1}{2}} = \frac{n+\frac{1}{2}}{kr}J_{n+\frac{1}{2}} - J_{n+\frac{3}{2}},$$

by a property of these functions, so that

$$\frac{d\psi_n}{dr} = Ak(kr)^{-\frac{1}{2}}\left\{\frac{n}{kr}J_{n+\frac{1}{2}}(kr) - J_{n+\frac{3}{2}}(kr)\right\}.$$

But, if the series is required, the simplest plan is to differentiate (f). It may be noticed that if $n=0$, the expression for $\frac{d\psi_n}{dr}$ is of the same form as ψ_{n+1}; but that this relation does not obtain in any other case, except for very large values of kr. A comparison of the different expressions for ψ_n shows that

$$Q_n\left(\frac{d}{d\,.\,ikr}\right)\cdot\frac{\sin kr}{kr} = i^n\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr).$$

Differentiation of (51) gives

$$\frac{d\psi_n}{dr} = \frac{2n\,(-1)^n k^2 S_n(ikr)^{n-1}}{1\,.\,3\,.\,5\,\ldots\,(2n+1)}\left\{1 - \frac{n+2}{2n\,(2n+3)}k^2r^2 + \ldots\right\} \quad \ldots(52).$$

In calculating the roots of the equation

$$\tan\left(kr + n\frac{\pi}{2}\right) = \beta \div \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots(53)$$

for my former paper, I noticed, without then seeing the reason, that the fractions $\beta \div \alpha$ were, for small values of r, nearer and nearer approximations to $\tan\left(kr + n\frac{\pi}{2}\right)$ as n increased. After what precedes, the explanation is obvious. From (33) and (50),

$$\alpha\, S_n\left(kr + n\frac{\pi}{2}\right) - \beta\cos\left(kr + n\frac{\pi}{2}\right) = -(-1)^n\frac{n\,(kr)^{n+1}}{1\,.\,3\,.\,5\,\ldots\,(2n+1)} + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots

$\cos n\frac{\pi}{2} = i^n$ (n even); and, as we shall see a little later (61), the leading term in a is $1.3.5\ldots(2n-1)(n+1)(ikr)^{-n}$, so that

$$\tan kr - \frac{\beta}{a} = -\frac{n}{(n+1)(2n+1)}\frac{(kr)^{2n+1}}{\{1.3.5\ldots(2n-1)\}^2}\ldots(56).$$

For instance, if $n=2$,

$$\tan kr - \left(\frac{\beta}{a}\right)_2 = -\frac{2k^5r^5}{3^3.5} + \ldots.$$

The values of $\frac{\beta}{a}$ will be given presently.

When n is odd, we have from (55)

$$-\cot kr - \frac{\beta}{a} = -\frac{n(kr)^{n+1}}{1.3\ldots(2n+1)\,a\sin n\frac{\pi}{2}.kr} + \ldots;$$

where $a = 1.3.5\ldots(2n-1)(n+1)(ikr)^{-n+1} + \ldots.$

Therefore

$$\cot kr + \frac{\beta}{a} = \frac{n(kr)^{2n-1}}{(n+1)(2n+1)\{1.3.5\ldots(2n-1)\}^2} + \ldots;$$

from which it follows that

$$\tan kr + \frac{a}{\beta} = -\frac{n(kr)^{2n+1}}{(n+1)(2n+1)\{1.3.5\ldots(2n-1)\}^2} + \ldots..(57),$$

for the leading term in $a \div \beta$ is $-kr$. For example, if $n=3$,

$$\tan kr + \left(\frac{a}{\beta}\right)_3 = -\frac{(kr)^7}{2100} + \ldots.$$

Vibrations propagated outwards.

This case has been treated by Prof. Stokes. When r is sufficiently great, $f_n(ikr)$ reduces to unity, and then (15) becomes, on introduction of the time factor,

$$r\psi_n = S_n e^{ik(at-r)} + S'_n e^{ik(at+r)},$$

the second term of which represents a disturbance travelling inwards from infinity towards the pole. When such a component is excluded, $S'_n = 0$, and we have simply

$$r\psi_n = S_n e^{-ikr} f_n(ikr) \quad \ldots\ldots\ldots\ldots(58),$$

$$\frac{d\psi_n}{dr} = -\frac{S_n}{r^2} e^{-ikr} F_n(ikr) \quad \ldots\ldots\ldots\ldots(59).$$

With regard to the values of $F_n(ikr)$, the following table (in which I have added the two last equations) is given by Stokes:

$$F_0(ikr) = ikr + 1,$$

$$F_1(ikr) = ikr + 2 + \frac{2}{ikr},$$

$$F_2(ikr) = ikr + 4 + \frac{9}{ikr} + \frac{9}{(ikr)^2},$$

$$F_3(ikr) = ikr + 7 + \frac{27}{ikr} + \frac{60}{(ikr)^2} + \frac{60}{(ikr)^3},$$

$$F_4(ikr) = ikr + 11 + \frac{65}{ikr} + \frac{240}{(ikr)^2} + \frac{525}{(ikr)^3} + \frac{525}{(ikr)^4},$$

$$F_5(ikr) = ikr + 16 + \frac{135}{ikr} + \frac{735}{(ikr)^2} + \frac{2625}{(ikr)^3} + \frac{5670}{(ikr)^4} + \frac{5670}{(ikr)^5},$$

$$F_6(ikr) = ikr + 22 + \frac{252}{ikr} + \frac{1890}{(ikr)^2} + \frac{9765}{(ikr)^3} + \frac{34020}{(ikr)^4} + \frac{72765}{(ikr)^5} + \frac{72765}{(ikr)^6}.$$

In order to find the leading terms in $F_n(ikr)$ when ikr is small, we may proceed thus. Reversing the series in (12),

$$f_n(ikr) = 1 \cdot 3 \cdot 5 \ldots (2n-1)(ikr)^{-n} \left\{ 1 + ikr + \frac{n-1}{2n-1}(ikr)^2 + \ldots \right\} \quad \ldots\ldots\ldots (60),$$

whence, from (17), we find

$$F_n(ikr) = 1 \cdot 3 \cdot 5 \ldots (2n-1)(n+1)(ikr)^{-n} \times \left\{ 1 + ikr + \frac{n^2(ikr)^2}{(n+1)(2n-1)} + \ldots \right\} \quad \ldots\ldots (61).$$

The way is now clear for the consideration of the special problem of this paper: the disturbance produced when a train of plane waves falls on an obstructing sphere. The whole motion external to the sphere may be divided into two parts; that belonging to the plane waves supposed to be undisturbed, and secondly a motion due to the presence of the sphere and radiating outwards from it. The first has a velocity-potential ϕ, which we have already investigated (43), (44). The second, with velocity-potential ψ, is included under the forms given in (58), (59). In order to adapt them to our present purpose, we have only to replace S_n by $a_n Q_n(\mu)$; for a spherical surface harmonic of order n, which is symmetrical round the axis of μ, can only be of the form $a_n Q_n(\mu)$. Thus

$$r\psi_n = a_n Q_n(\mu) \cdot \epsilon^{-ikr} f_n(ikr) \quad \ldots\ldots\ldots\ldots\ldots (60),$$

$$\frac{d\psi_n}{dr} = -\frac{a_n}{r^2} Q_n(\mu) \cdot \epsilon^{-ikr} F_n(ikr) \quad \ldots\ldots\ldots\ldots\ldots (61).$$

The velocity-potential of the whole motion $(\varphi+\psi)$ is expressed by (43) and (60); the constants a_n depend on the character of the obstruction presented by the sphere. The simplest case which we can consider is that of

A rigid and fixed Sphere.

The condition to be satisfied is simply that at the surface of the sphere $(r=c)$ for each harmonic element separately

$$\frac{d\phi}{dr}+\frac{d\psi}{dr}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (62),$$

which for the element of order n gives

$$a_n=(2n+1)\frac{kc^2\epsilon^{ikc}}{F_n(ikc)}Q_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\cdot\frac{\sin kc}{kc} \quad \ldots\ldots\ldots (63).$$

The disturbance due to the presence of the sphere is therefore

$$\psi=\frac{kc^2}{r}\epsilon^{ik(at-r+c)}\times\Sigma\frac{2n+1}{F_n(ikc)}Q_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\frac{\sin kc}{kc}\cdot Q_n(\mu)\,.\,f_n(ikr) \quad \ldots\ldots\ldots (64),$$

corresponding to the plane waves

$$\phi=\epsilon^{ik(at+x)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36),$$

which, it will be observed, is true without any restriction as to the magnitude of the sphere in relation to λ. At a sufficient distance from the source of disturbance, we may take

$$f_n(ikr)=1.$$

In order to pass to the solution of a real problem, we have to separate the real and imaginary parts, and throw away the one or the other. If we elect to retain the real parts, we have the plane waves represented by

$$[\phi]=\cos k(at+x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (65).$$

Confining ourselves, for simplicity's sake, to parts of space at a great distance from the sphere, where $f_n(ikr)=$ unity, we proceed to extract the real part of (64). Since the functions Q are either wholly even or wholly odd,

$$Q_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\frac{\sin kc}{kc}$$

is wholly real, or wholly imaginary; so that this factor presents no difficulty. $\{F_n(ikc)\}^{-1}$, however, is complex; we have

$$\{F_n(ikc)\}^{-1}=\frac{\alpha-i\beta}{\alpha^2+\beta^2}=\frac{1}{\sqrt{\alpha^2+\beta^2}}\cdot\frac{\alpha-i\beta}{\sqrt{\alpha^2+\beta^2}}=\frac{1}{\sqrt{\alpha^2+\beta^2}}\epsilon^{i\gamma},$$

if $\tan\gamma=-\dfrac{\beta}{\alpha}$. Thus

$$\psi=\Sigma(2n+1)\frac{kc^2}{r}\epsilon^{i[k(at-r+c)+\gamma]}\times\{\alpha^2+\beta^2\}^{-\frac{1}{2}}Q_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\frac{\sin kc}{kc}\cdot Q_n(\mu) \quad \ldots\ldots (66).$$

When therefore n is even,

$$[\psi] = (2n+1)\frac{kc^2}{r}\cos\{k(at-r+c)+\gamma\}$$
$$\times\{\alpha^2+\beta^2\}^{-\frac{1}{2}}\,Q_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\,\frac{\sin kc}{kc}\cdot Q_n(\mu)\ldots\ldots\ldots(67),$$

while if n be odd,

$$[\psi] = (2n+1)\frac{kc^2}{r}\,i\sin\{k(at-r+c)+\gamma\}$$
$$\times\{\alpha^2+\beta^2\}^{-\frac{1}{2}}\,Q_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\,\frac{\sin kc}{kc}\cdot Q_n(\mu)\ldots\ldots\ldots(68).$$

As examples I will write down the terms in $[\psi]$ involving harmonics of orders 0, 1, 2. The following table of the functions Q is required for the purpose :—

From (46),
$$Q_0 = 1,$$
$$Q_1 = \mu,$$
$$Q_2 = \frac{3}{2}\left(\mu^2-\frac{1}{3}\right),$$
$$Q_3 = \frac{5}{2}\left(\mu^3-\frac{3}{5}\mu\right),$$
$$Q_4 = \frac{35}{8}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right),$$
$$Q_5 = \frac{63}{8}\left(\mu^5-\frac{10}{9}\mu^2+\frac{5}{21}\mu\right).$$

We have—

$n=0$,
$$\alpha^2+\beta^2 = 1+k^2c^2,$$
$$\tan\gamma = -kc,$$
$$[\psi_0] = \frac{kc^2}{r}\{1+k^2c^2\}^{-\frac{1}{2}}\frac{d}{d\,.\,kc}\,\frac{\sin kc}{kc}\,.\cos\{k(at-r+c)+\gamma\}\ldots(69);$$

$n=1$,
$$\alpha^2+\beta^2 = k^2c^2+\frac{4}{k^2c^2},$$
$$\tan\gamma = -\frac{kc-\dfrac{2}{kc}}{2},$$
$$[\psi_1] = \frac{3kc^2}{r}\left\{k^2c^2+\frac{4}{k^2c^2}\right\}^{-\frac{1}{2}}\frac{d^2}{d\,(kc)^2}\,\frac{\sin kc}{kc}\,\mu\,.\sin\{k(at-r+c)+\gamma\}\ldots(70);$$

$n=2$,
$$\alpha^2+\beta^2 = k^2c^2-2+\frac{9}{k^2c^2}+\frac{81}{k^4c^4},$$
$$\tan\gamma = -\frac{kc-\dfrac{9}{kc}}{4-\dfrac{9}{k^2c^2}},$$

$$[\psi_2] = -\frac{45kc^2}{r}\left\{k^2c^2 - 2 + \frac{9}{k^2c^2} + \frac{81}{k^4c^4}\right\}^{-\frac{1}{2}}$$

$$\times \left\{\frac{d^3}{d\,(kc)^3} + \frac{1}{3}\,\frac{d}{d\,(kc)}\right\}\frac{\sin kc}{kc}\cdot\left(\mu^2 - \frac{1}{3}\right)\cos\{k(at-r+c)+\gamma\} \quad \ldots(71).$$

In applications where kc is a small quantity, we shall require to expand these expressions in rising powers of kc. It will be found without difficulty that

$$[\psi_0] = -\frac{k^2c^3}{3r}\left(1 - \frac{3}{5}\,k^2c^2 + \frac{3}{7}\,k^4c^4 - \frac{19}{54}\,k^6c^6 + \ldots\right)$$

$$\times \cos\{k\,(at-r+c)+\gamma_0\} \quad \ldots(72),$$

$$[\psi_1] = -\frac{k^2c^3}{2r}\left(1 - \frac{3}{10}\,k^2c^2 - \frac{3}{28}\,k^4c^4 + \frac{1}{27}\,k^6c^6 + \ldots\right)$$

$$\times \mu\,.\sin\{k\,(at-r+c)+\gamma_1\} \quad \ldots(73),$$

$$[\psi_2] = -\frac{k^4c^5}{9r}\left(1 - \frac{25}{126}\,k^2c^2 + \frac{13}{567}\,k^4c^4 + \ldots\right)$$

$$\times \left(\mu^2 - \frac{1}{3}\right)\cos\{k\,(at-r+c)+\gamma_2\} \quad \ldots(74).$$

It appears that while $[\psi_0]$ and $[\psi_1]$ are of the same order in the small quantity kc, $[\psi_2]$ is two orders higher. We shall find presently that the higher harmonic components in $[\psi]$ contain still more elevated powers of kc. For a first approximation, then, we may confine ourselves to the elements of orders 0 and 1.

Although $[\psi_0]$ contains a cosine and $[\psi_1]$ a sine, they nevertheless differ in phase by a small quantity only. For, when n is even, we have from (56)

$$\tan\gamma = -\tan kc - \frac{n}{(n+1)\,(2n+1)}\,\frac{(kc)^{2n+1}}{\{1\,.\,3\,\ldots\,(2n-1)\}^2} + \ldots \quad \ldots(75),$$

so that γ is approximately (when n is great, very approximately) equal to $-kc$. On the other hand, when n is odd,

$$\tan\gamma = \cot kc - \frac{n\,(kc)^{2n-1}}{(n+1)\,(2n+1)\,\{1\,.\,3\,\ldots\,(2n-1)\}^2} \quad \ldots\ldots\ldots (76),$$

giving $$\gamma = \frac{\pi}{2} - kc \quad \text{approx.}$$

The velocity-potential of the disturbance due to a small rigid and fixed sphere is therefore approximately

$$[\psi_0]+[\psi_1] = -\frac{k^2c^3}{3r}\left(1 + \frac{3}{2}\,\mu\right)\cos k\,(at-r)$$

$$= -\frac{\pi T}{r\lambda^2}\left(1 + \frac{3}{2}\,\mu\right)\cos k\,(at-r) \quad \ldots\ldots\ldots\ldots(77),$$

if T be the volume of the obstacle.

It appears from (48) that, when the sphere is small compared with the wave-length, the disturbance is in general nearly proportional to the volume, and to the inverse square of the wave-length, as might have been anticipated;* but that in a certain direction, namely, when $\mu = -\frac{2}{3}$, this law fails, the leading term vanishing through the opposition of its two parts. In this case the disturbance will ultimately vary as k^4, or the inverse fourth power of the wave-length.

I now proceed to investigate the leading terms in the value of ψ_n in (66), when kc is small.

From (49),

$$(2n+1)\,Q_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\frac{\sin kc}{kc} = \frac{dA_n}{d\,.\,kc}$$

$$= \frac{n i^n (kc)^{n-1}}{1\,.\,3\,\ldots\,(2n-1)}\left\{1 - \frac{(n+2)\,k^2c^2}{2\,.\,n\,(2n+3)} + \ldots\right\} \quad \ldots\ldots\ldots (78).$$

Again, from (61),

$$\alpha^2+\beta^2 = F_n(ikc)\times F_n(-ikc)$$

$$= \{1\,.\,3\,.\,5\,\ldots\,(2n-1)\,(n+1)\,(kc)^{-n}\}^2 \times \left\{1+ikc+\frac{n^2(ikc)^2}{(n+1)\,(2n-1)}\right\}\left\{1-ikc+\frac{n^2(ikc)^2}{(n+1)\,(2n-1)}\right\}$$

$$= \{1\,.\,3\,.\,5\,\ldots\,(2n-1)\,(n+1)\,(kc)^{-n}\}^2\left\{1+\frac{(n-1)\,k^2c^2}{(n+1)\,(2n-1)}+\ldots\right\} \quad \ldots\ldots\ldots (79);$$

so that

$$\{\alpha^2+\beta^2\}^{-\frac{1}{2}} = \frac{(kc)^n}{1\,.\,3\,\ldots\,(2n-1)\,(n+1)}\left\{1-\frac{(n-1)\,k^2c^2}{2\,(n+1)\,(2n-1)}+\ldots\right\} \quad \ldots\ldots\ldots (80).$$

(66) now gives

$$\psi_n = \frac{c\,(kc)^{2n}\,n\,i^n\,Q_n(\mu)}{r\,\{1\,.\,3\,\ldots\,(2n-1)\}^2\,(n+1)}\,e^{i[k(at-r+c)+\gamma_n]} \times\left\{1-k^2c^2\left(\frac{n-1}{(2n+2)\,(2n-1)}+\frac{n+2}{2n\,(2n+3)}\right)\right\} \quad \ldots\ldots\ldots (81).$$

When n is even, $\gamma_n = -kc$ approx., and then

$$[\psi_n] = \frac{c\,(kc)^{2n}\,n\,i^n\,Q_n(\mu)}{r\,\{1\,.\,3\,\ldots\,(2n-1)\}^2\,(n+1)}\cos k\,(at-r) \times\left\{1-k^2c^2\left(\frac{n-1}{(2n+2)\,(2n-1)}+\frac{n+2}{2n\,(2n+3)}\right)\right\} \quad \ldots\ldots\ldots (82),$$

while if n be odd, we have merely to replace i^n by i^{n+1}, the result being in this case also real.

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* See a paper "On the Light from the Sky," Phil. Mag., Feb. 1871.

By means of (82) we may verify the two first terms in the expressions for $[\psi_1]$, $[\psi_2]$, in (73), (74). To the case of $n=0$, (82) does not apply.

Calculating from (81),

$$[\psi_3] = \frac{k^6 c^7}{120 r}\left\{1 - \frac{77}{540}k^2c^2\right\}\left\{\mu^3 - \frac{3}{5}\mu\right\}\sin\{k(at-r+c)+\gamma_3\} \quad \ldots\ldots\ldots (83),$$

$$[\psi_4] = \frac{k^8 c^9}{3150 r}\left\{\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}\right\}\cos\{k(at-r+c)+\gamma_4\} \quad \ldots\ldots (84).$$

Combining (73), (74), (83), (84), we have the value of $[\psi]$ complete so far as the terms which are of the order k^6c^6 compared with the two leading terms given in (77). In compounding the partial expressions, it is as necessary to be exact with the phases of the components as with their amplitudes; but for purposes requiring only one harmonic element at a time, the phase is often of subordinate importance. In such cases the values of γ given by (75), (76) may be used,

$$\left.\begin{array}{ll} (n \text{ even}) & \gamma = -kc \\ (n \text{ odd}) & \gamma = \frac{\pi}{2} - kc \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (85).$$

From (81) or (82) it appears that the leading term in ψ_n rises two orders in (kc) with each step in the order of the harmonic; and that ψ_n is itself expressed by a series containing only even powers of (kc). But besides being of higher order in kc, the leading term becomes rapidly smaller, as n increases on account of the other factors which it contains. This is evident, because for all values of n and μ, $Q_n(\mu)$ is less than unity; the same is true of $n \div (n+1)$; while i^n affects only the phase.

In particular cases any one of the harmonic elements of $[\psi]$ may vanish. From (67), (68), since $\alpha^2+\beta^2$ cannot vanish, we have in such a case

$$Q_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\,\frac{\sin kc}{kc} = 0,$$

the same equation as gives the periods of the vibrations of order n in a closed sphere of radius c. A little consideration shows that this result might have been anticipated. The table given in my former paper applies to this question, and shows, among other things, that when kc is small, no harmonic element in $[\psi]$ can vanish.

If preferred, the value of ψ can be expressed in terms of the functions α and β:

$$\psi_n = -\frac{1}{kr}\,\frac{2n+1}{\alpha+i\beta}\left\{\alpha\sin\left(kc+n\frac{\pi}{2}\right) - \beta\cos\left(kc+n\frac{\pi}{2}\right)\right\}$$
$$\times\, \epsilon^{i[k(at-r+c)-n.\frac{1}{2}\pi]}.\, Q_n(\mu) \quad \ldots\ldots\ldots\ldots\ldots (86).$$

From this equation it would probably not be impracticable to calculate the complete value of ψ, when kc has some moderate value (such as 3 or 4). After a few terms, the components ψ_n would rapidly diminish; the faster, the smaller is the value of kc. For the more important directions, represented by $\mu=0$ or $\mu=\pm1$, the calculation would be simpler still.

For any specified value of n, it is easy to find the limiting value of ψ_n, when kc is very *large*. We may put

$$\alpha=0, \quad \beta=kc, \quad \mathrm{F}=ikc;$$

so that $$\psi_n=\frac{2n+1}{ikr}\,\epsilon^{ik(at-r+c)-n.\frac{1}{2}\pi}\cos\left(kc+n\frac{\pi}{2}\right).\mathrm{Q}_n(\mu)\ldots\ldots\ldots(87),$$

of which the real part is

$$[\psi_n]=\frac{2n+1}{kr}\sin\left\{k\left(at-r+c\right)-n\frac{\pi}{2}\right\}\cos\left(kc+n\frac{\pi}{2}\right).\mathrm{Q}_n(\mu)\ldots(88).$$

The values used for α and β are legitimate for any value of n, when kc is sufficiently large. But it is at the same time true that, whatever may be the value of kc, however large, a value of n can be found beyond which the substitution fails to be permissible. For instance, in the expression for $\mathrm{F}_n(ikc)$, the last term, $1.3.5\ldots(2n-1)(n+1)(ikc)^{-n}$, may always be made greater than the first ikc, however large kc may be, by merely taking n sufficiently large. We cannot therefore, by means of (87), (88), write down a complete expression for ψ, even for the limit $kc=\infty$. We may indeed, when n is very great, form an approximate expression for ψ_n, by taking the last terms, instead of the first, in the expressions for F_n, α, β; but there would necessarily be a gap, where neither method of approximation would be applicable.

It does not seem very easy to deduce the consequences of (64) when kc is large, even when they are known beforehand. We know, for example, that immediately in the rear of a sufficiently large sphere there will be a sound *shadow*, where almost all motion ceases. Putting $r=c$, the expressions for ϕ and ψ become

$$\phi=\epsilon^{ikat}\Sigma(2n+1)\,\mathrm{Q}_n(\mu).\mathrm{Q}_n\left(\frac{d}{d.ikc}\right)\frac{\sin kc}{kc},$$

$$\psi=\epsilon^{ikat}\Sigma(2n+1)\,\mathrm{Q}_n(\mu).\mathrm{Q}_n\left(\frac{d}{d.ikc}\right)\frac{d}{d.kc}.\frac{\sin kc}{kc}.\frac{f_n(ikc)}{\mathrm{F}_n(ikc)}.$$

Now we know that

$$\Sigma(2n+1)\,\mathrm{Q}_n(\mu).\mathrm{Q}_n\left(\frac{d}{d.ikc}\right)\frac{\sin kc}{kc}=\epsilon^{ikc.\mu},$$

and therefore

$$\Sigma(2n+1)\,kc\,\frac{d}{d.kc}.\mathrm{Q}_n\left(\frac{d}{d.ikc}\right)\frac{\sin kc}{kc}.\mathrm{Q}_n(\mu)=ikc.\mu.\epsilon^{ikc.\mu}.$$

If we were to take the ratio $f_n(ikc) : F_n(ikc)$ as equal to $1 : ikc$ for *all* values of n, provided only kc be large enough, we should have

$$\phi+\psi = \epsilon^{ikat}.\epsilon^{ikc\mu}(1+\mu);$$

so that, when $\mu=-1$, $\phi+\psi=0$. It is probable that $\phi+\psi$ would vanish approximately for a considerable range of μ in the neighbourhood of -1, if the exact value of $f_n(ikc) : F_n(ikc)$ could be used. When $\mu=+1$, that is, on the side facing the waves,

$$\phi+\psi = 2\phi \text{ approximately,}$$

the sound being reflected nearly as from a plane wall.

Sphere rigid, but not fixed.

Instead of supposing the sphere fixed, let us consider what modification will be introduced if the manner of suspension be such as to permit of oscillations being produced under the influence of the train of waves. We may conceive the sphere to preserve its rigidity, so that no deformation of its surface is possible; while the centre is no longer absolutely fixed, but merely urged by a spring towards a position of equilibrium. Under these circumstances, the effect of the train of waves and the accompanying variable pressure on the sphere's surface, will be to set it in vibration parallel to the axis of x. Now such a motion is equivalent to a normal velocity at any point of the surface expressed simply by the harmonic term of order one. For every other harmonic element, the condition (62) remains as true as if the sphere were fixed. The components of ψ are therefore exactly the same as before, with the simple exception of ψ_1.

Let X denote the displacement of the centre of the sphere, reckoned negative in the direction along which the waves advance. Then, if we consider only motions produced by the waves, we may put

$$X = X_0 e^{ikat} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(89),$$

where X_0 is a constant (complex).

The first condition to be satisfied relating to the normal motion is

$$\frac{d\phi_1}{dr}+\frac{d\psi_1}{dr} = \dot{X}\mu \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(90).$$

If σ denote the undisturbed density, the variable part of the pressure is

$$\delta p = -\sigma\left(\frac{d\phi_1}{dt}+\frac{d\psi_1}{dt}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(91);$$

hence the force impelling the sphere along x is

$$-\int_{-1}^{+1} 2\pi c^2 \delta p\, \mu\, d\mu = 2\pi c^2\sigma \int_{-1}^{+1}\left(\frac{d\phi_1}{dt}+\frac{d\psi_1}{dt}\right)\mu\, d\mu,$$

and the equation of motion

$$M\ddot{X}+HX = 2\pi c^2\sigma\int_{-1}^{+1}\left(\frac{d\phi_1}{dt}+\frac{d\psi_1}{dt}\right)\mu\, d\mu \quad \ldots\ldots\ldots\ldots(92).$$

Now from (43) and (44),

$$\phi_1 = \frac{3}{i}\,\epsilon^{ikat}\,\frac{d}{d\,.\,kr}\left(\frac{\sin kr}{kr}\right).\mu,$$

$$\frac{d\phi_1}{dt} = 3ka\,\epsilon^{ikat}\,\frac{d}{d\,.\,kr}\left(\frac{\sin kr}{kr}\right).\mu,$$

$$\frac{d\phi_1}{dr} = \frac{3k}{i}\,\epsilon^{ikat}\,\frac{d^2}{d(kr)^2}\left(\frac{\sin kr}{kr}\right).\mu;$$

and from (58) and (59),

$$\psi_1 = \frac{a_1}{r}\,\epsilon^{ik(at-r)}\left(1+\frac{1}{ikr}\right).\mu,$$

$$\frac{d\psi_1}{dt} = ika\,\frac{a_1}{r}\,\epsilon^{ik(at-r)}\left(1+\frac{1}{ikr}\right).\mu,$$

$$\frac{d\psi_1}{dr} = -\frac{a_1}{r^2}\,\epsilon^{ik(at-r)}\left(ikr+2+\frac{2}{ikr}\right).\mu;$$

so that (90) and (92) become

$$\frac{3k}{i}\,\frac{d^2}{d(kc)^2}\,\frac{\sin kc}{kc} - \frac{a_1}{c^2}\,\epsilon^{-ikc}\left(ikc+2+\frac{2}{ikc}\right) = ika\mathrm{X}_0 \quad \ldots\ldots(93),$$

$$\mathrm{X}_0\left\{-\mathrm{M}k^2a^2+\mathrm{H}\right\} = \frac{4\pi c^2\sigma ka}{3}\left\{3\,\frac{d}{d\,.\,kc}\,.\,\frac{\sin kc}{kc} + i\,\frac{a_1}{c^2}\,\epsilon^{-ikc}\left(1+\frac{1}{ikc}\right)\right\} \quad \ldots\ldots\ldots(94),$$

two simple equations giving X_0 and a_1.

If, in order to see whether it is possible that the sphere may remain at rest, we put $\mathrm{X}_0 = 0$, and eliminate a_1, we find

$$kc\,\frac{d^2}{d(kc)^2}\,.\,\frac{\sin kc}{kc} : \frac{d}{d\,.\,kc}\,.\,\frac{\sin kc}{kc} = \frac{2+k^2c^2+ik^3c^3}{1+k^2c^2},$$

which cannot be satisfied, since the second side is essentially imaginary.

It may readily be shown, too, that $\frac{d}{dx}\left(\frac{\sin x}{x}\right)$ and $\frac{d^2}{dx^2}\left(\frac{\sin x}{x}\right)$ cannot vanish together. The first

$$= \frac{\cos x}{x} - \frac{\sin x}{x^2},$$

and does not (after $x=0$) vanish at the same time with either $\sin x$ or $\cos x$. The second $= \sin x\left(-\frac{1}{x}+\frac{2}{x^3}\right) - \frac{2\cos x}{x^2}$.

Hence, if there be a common root,

$$\tan x = x = \frac{2x}{2-x^2},$$

which cannot be true.

We conclude therefore that, whatever may be its size, the sphere will be set into vibration.

If there be a complete synchronism, so that the period of the waves is the same as that of the free vibration of the sphere (supposed *in vacuo*),

$$-\mathrm{M}k^2a^2+\mathrm{H}=0;$$

and

$$-\frac{kca\mathrm{X}_0}{3}=kc\frac{d^2}{d(kc)^2}\left(\frac{\sin kc}{kc}\right)+\frac{2+k^2c^2+ik^3c^3}{1+k^2c^2}\cdot\frac{d}{d\,.\,kc}\cdot\frac{\sin kc}{kc} \quad \ldots(95),$$

showing that X_0 remains finite. It might perhaps have been expected that in this case, as in some others bearing a resemblance to it, the motion would have increased without limit. But it must be remembered that, although we have introduced no special frictional force to retard the sphere's motion, yet the reaction of the surrounding air produces an effect of the same nature; for it is obvious that, independently of any viscosity (of which no account is here taken), a body set to vibrate in an infinite fluid medium otherwise undisturbed, and then left to itself, must finally come to rest, having used up its energy in the production of waves propagated outwards to an infinite distance.

If kc be very small, we obtain from (95) $a\mathrm{X}_0=3$, so that at the time t the velocity of the sphere is $3ik\,\epsilon^{ikat}$. At the same moment the velocity of the air at the point occupied by the centre, if the sphere were away, would be $ik\,\epsilon^{ikat}$. The former is therefore three times the latter.

In the general case, if kc be sufficiently small, we find

$$\left.\begin{aligned}\frac{2a_1}{c}-k^2c^2a\mathrm{X}_0&=-k^2c^2\\ \frac{a_1}{c}-\frac{3}{4}\frac{-\mathrm{M}k^2a^2+\mathrm{H}}{\pi c\sigma a}\mathrm{X}_0&=k^2c^2\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (96),$$

from which

$$a\mathrm{X}_0\left\{k^2c^2-\frac{3}{2}\frac{-\mathrm{M}k^2a^2+\mathrm{H}}{\pi\sigma ca^2}\right\}=3k^2c^2 \quad \ldots\ldots\ldots\ldots(97).$$

If there be no force urging the sphere to a position of equilibrium, $\mathrm{H}=0$, and

$$a\mathrm{X}_0\left(1+2\frac{\mathrm{M}}{\mathrm{M}'}\right)=3 \quad \ldots\ldots\ldots\ldots (98),$$

M′ denoting the mass of the air displaced by the sphere.

Fluid Spherical Obstacle.

Having now considered at some length the case of a rigid sphere, whether fixed or moveable, I pass on to that of a gaseous disturbing sphere, whose density and compressibility may be different from those of the surrounding medium through which the plane waves advance. The problem proposed has a less direct physical bearing than the pre-

ceding, because, in the first place, all gases have the same compressibility, and, in the second, two gases cannot be prevented from mixing by diffusion, unless separated by a membrane of some kind, whose presence would alter the mechanical conditions of the question. Nevertheless the solution is in several ways instructive.

Exterior to the sphere, we have ϕ the same exactly, and ψ the same in form, as before; so that equations (43), (44), (58), and (59) still apply. For the motion within the sphere, we have, from equations (25), (31),

$$\psi_n = \frac{a'_n Q_n}{r}\left\{\epsilon^{-ik'r} f_n(ik'r) - (-1)^n \epsilon^{+ik'r} f_n(-ik'r)\right\} \quad \ldots\ldots\ldots\ldots (99),$$

$$\frac{d\psi_n}{dr} = \frac{2a'_n Q_n}{r^2} \cdot i^{n+1}\left\{\alpha \sin\left(k'r + \frac{n\pi}{2}\right) - \beta \cos\left(k'r + \frac{n\pi}{2}\right)\right\} \quad \ldots (100),$$

S_n being replaced by $a'_n Q_n$; k' stands for $2\pi \div \lambda'$, λ' being the new wave-length.

If σ, σ' be the normal densities, m, m' the compressibilities,

$$\frac{k'^2}{k^2} = \frac{\sigma'}{\sigma} \cdot \frac{m}{m'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (101),$$

and the conditions, to be satisfied separately for each harmonic element, are

$$\frac{d\phi}{dr} + \frac{d\psi}{dr}\ (\text{outside}) = \frac{d\psi}{dr}\ (\text{inside}) \quad \ldots\ldots\ldots\ldots (102),$$

$$\sigma\{\phi + \psi\ (\text{outside})\} = \sigma'\{\psi\ (\text{inside})\} \quad \ldots\ldots\ldots\ldots (103),$$

expressing respectively the equality of the normal motion and pressure on the two sides of the bounding surface. From these equations the complete solution may be worked out; but I shall confine myself to finding the value of the leading terms when kc and $k'c$ are very small. In this case we have, when $r = c$,

$$\left.\begin{aligned} \psi_0\ (\text{inside}) &= -2ik'a'_0 \\ \frac{d\psi_0}{dr}\ (\text{inside}) &= \frac{2}{3} ik'^3 c a'_0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (104),$$

$$\left.\begin{aligned} \phi_0 &= 1 \\ \frac{d\phi_0}{dr} &= -\frac{k^2 c}{3} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (105),$$

$$\left.\begin{aligned} \psi_0\ (\text{outside}) &= \frac{a_0}{c} \\ \frac{d\psi_0}{dr}\ (\text{outside}) &= -\frac{a_0}{c^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (106).$$

Using these in (102), (103), and eliminating a'_0, retaining only the principal term,

$$a_0 = -\frac{k^2 c^3}{3} \cdot \frac{m' - m}{m'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (107).$$

In like manner, for the term of the first order,

$$\left.\begin{aligned}\psi_1 \text{ (inside)} &= -\frac{2}{3}\alpha_1' k'^2 c\mu \\ \frac{d\psi_1}{dr} \text{ (inside)} &= -\frac{2}{3}\alpha_1' k'^2 \mu\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (108),$$

$$\left.\begin{aligned}\phi_1 &= -\frac{kc}{i}\mu \\ \frac{d\phi_1}{dr} &= -\frac{k}{i}\mu\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (109),$$

$$\left.\begin{aligned}\psi_1 \text{ (outside)} &= \frac{a_1}{ikc^2}\mu \\ \frac{d\psi_1}{dr} \text{ (outside)} &= -\frac{2a_1}{ikc^3}\mu\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (110),$$

which give finally $$a_1 = \frac{k^2c^3(\sigma-\sigma')}{\sigma+2\sigma'} \quad \ldots\ldots\ldots\ldots (111).$$

At a distance from the sphere the disturbance is given by

$$\begin{aligned}\psi &= \frac{1}{r}\epsilon^{ik(at-r)}\{a_0+a_1\mu\} \\ &= -\frac{k^2c^3}{3r}\epsilon^{ik(at-r)}\left\{\frac{m'-m}{m'}+3\frac{\sigma'-\sigma}{\sigma+2\sigma'}\cdot\mu\right\} \quad \ldots\ldots\ldots (112).\end{aligned}$$

If we introduce the relations

$$\mathrm{T} = \frac{4\pi c^3}{3} \quad \text{and} \quad k = \frac{2\pi}{\lambda},$$

and throw away the imaginary part, we obtain

$$\psi = -\frac{\pi \mathrm{T}}{\lambda^2 r}\left\{\frac{m'-m}{m'}+3\frac{\sigma'-\sigma}{\sigma+2\sigma'}\cdot\mu\right\}\cdot\cos k(at-r) \ldots (113)$$

as the expression for the most important part of the disturbance, corresponding to (77) for a fixed rigid sphere. It appears, as might have been expected, that the term of zero order is due to the variation of compressibility, and that of order one to the variation of density. With regard to the latter, we have virtually the same problem as that already solved, when the sphere, supposed to preserve the shape of its surface, was allowed to move under the influence of the variable pressures due to the waves. Thus the value of a_1 in (111) is identical with that given by (98) and the first of the two equations numbered (96).

From (113) we may fall back on the case of a rigid and fixed sphere by putting σ' and m' both infinite. It is not sufficient to make σ' by itself infinite, apparently because, if m' at the same time remain finite, $k'c$ would not be small, as has been assumed.

In actual gases $m'=m$, and the term of zero order disappears. If

the gas occupying the spherical space be incomparably lighter than the other, $\sigma' = 0$, and

$$\psi = 3\,\frac{\pi T}{\lambda^2 r}\,\mu\,\cos k\,(at - r) \quad \ldots\ldots\ldots\

if Δm, $\Delta\sigma$ stand respectively for $m-m_0$, $\sigma-\sigma_0$. The two other equations are

$$\left.\begin{aligned} \frac{d}{dy}(m\delta)+\sigma k^2\alpha^2\beta+\frac{d}{dy}(\Delta m\,.\,\delta_0)=0 \\ \frac{d}{dz}(m\delta)+\sigma k^2\alpha^2\gamma+\frac{d}{dz}(\Delta m\,.\,\delta_0)=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (118).$$

It is to be observed that Δm, $\Delta\sigma$ vanish, except through a small space, which is considered as the region of disturbance; α, β, γ, δ, being the result of the disturbance, are to be considered as small quantities; so that, in our approximate treatment of the question, the variation of m and σ in the first two terms of equations (117), (118) is to be neglected, being there multiplied by the small quantities. We may therefore deduce

$$\nabla^2(m\delta)+\sigma k^2\alpha^2\,.\,\delta+\nabla^2(\Delta m\,.\,\delta_0)+k^2\alpha^2\frac{d}{dx}(\Delta\sigma\,.\,\alpha_0)=0,$$

or
$$\nabla^2\delta+k^2\delta=-\frac{1}{m}\nabla^2(\Delta m\,.\,\delta_0)-\frac{k^2\alpha^2}{m}\frac{d}{dx}(\Delta\sigma\,.\,\alpha_0)=0 \ldots\ldots (119).$$

Accordingly,

$$4\pi m\delta=\iiint\frac{\epsilon^{-ikr}}{r}\left\{\nabla^2(\Delta m\,.\,\delta_0)+k^2\alpha^2\frac{d}{dx}(\Delta\sigma\,.\,\alpha_0)\right\}dv=\mathrm{P}+\mathrm{Q}\ (\text{say}) \quad \ldots\ldots\ldots (120),$$

the integration going over a volume S which completely includes the region of disturbance.

Now, by Green's theorem,

$$\begin{aligned}\mathrm{P}&=\iiint\frac{\epsilon^{-ikr}}{r}\nabla^2(\Delta m\,.\,\delta_0)\,dv \\ &=\iint\left\{\frac{\epsilon^{-ikr}}{r}\frac{d}{dn}(\Delta m\,.\,\delta_0)-\Delta m\,.\,\delta_0\frac{d}{dn}\left(\frac{\epsilon^{-ikr}}{r}\right)\right\}dS \\ &\qquad+\iiint\Delta m\,.\,\delta_0\nabla^2\left(\frac{\epsilon^{-ikr}}{r}\right)dv.\end{aligned}$$

But on S, Δm and $\frac{d}{dn}(\Delta m\,.\,\delta_0)$ vanish; so that both the surface integrals disappear. Moreover,

$$\nabla^2\left(\frac{\epsilon^{-ikr}}{r}\right)=\frac{1}{r}\frac{d^2}{dr^2}\epsilon^{-ikr}=-k^2\frac{\epsilon^{-ikr}}{r};$$

and thus

$$\mathrm{P}=-k^2\iiint\Delta m\,.\,\delta_0\frac{\epsilon^{-ikr}}{r}\,dv=-k^2\Delta m\,.\,\delta_0\mathrm{T}\frac{\epsilon^{-ikr}}{r} \quad \ldots\ldots (121),$$

if T be the volume of a space throughout which Δm has a constant value.

Also, integrating by parts,

$$Q = k^2 a^2 \iiint \frac{\epsilon^{-ikr}}{r} \frac{d}{dx} (\Delta\sigma \,.\, \alpha_0)\, dv$$

$$= -k^2 a^2 \iiint \Delta\sigma \,.\, \alpha_0 \frac{d}{dx} \left(\frac{\epsilon^{-ikr}}{r}\right) dv = \frac{-ik^3 a^2 \,.\, \Delta\sigma \,.\, \alpha_0 \mathrm{T}}{r} \epsilon^{-ikr} \mu \ldots (122),$$

the distance r being so great in relation to the linear dimension of T that $\frac{1}{r^2}$ can be neglected.

Putting these values of P and Q in (120), we find

$$\delta = -\frac{\mathrm{T}k^2 \epsilon^{-ikr}}{4\pi mr} \{\Delta m \,.\, \delta_0 + ika^2 \Delta\sigma \,.\, \alpha_0 \mu\} \quad \ldots\ldots\ldots\ldots (123).$$

In order to compare (123) with our previous results, δ_0 and α_0 must be expressed in terms of ϕ, and δ in terms of ψ. We have

$$\phi = \epsilon^{ik(at+x)} \quad \ldots\ldots\ldots\ldots (124),$$

$$\frac{d\alpha_0}{dt} = \frac{d\phi}{dx} = ik\, \epsilon^{ik(at+x)},$$

$$\left.\begin{aligned} \alpha_0 &= \frac{1}{a}\, \epsilon^{ik(at+x)} \\ \delta_0 &= \frac{d\alpha_0}{dx} = \frac{ik}{a}\, \epsilon^{ik(at+x)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (125),$$

which, substituted in (123), give

$$\delta = \frac{-ik^3 \mathrm{T}\, \epsilon^{ik(at-r)}}{4\pi mar} \{\Delta m + a^2 \Delta\sigma \,.\, \mu\} \quad \ldots\ldots\ldots\ldots (126).$$

From δ we may pass to ψ by means of $\frac{d\delta}{dt} = \frac{d^2\psi}{dr^2}$, which gives

$$\psi = -\frac{k^2 \mathrm{T}\, \epsilon^{ik(at-r)}}{4\pi mr} \left\{ \Delta m + a^2 \Delta\sigma \,.\, \mu \right\}$$

$$= -\frac{\pi\, \mathrm{T}\, \epsilon^{ik(at-r)}}{\lambda^2 r} \left\{ \frac{\Delta m}{m} + \frac{\Delta\sigma}{\sigma} \,.\, \mu \right\} \quad \ldots\ldots\ldots\ldots (127).$$

Thus, for a train of plane waves $\phi = \cos k\,(at+x)$, we have a disturbance

$$\psi = -\frac{\pi \mathrm{T}}{\lambda^2 r} \left\{ \frac{\Delta m}{m} + \frac{\Delta\sigma}{\sigma} \,.\, \mu \right\} \cos k\,(at-r) \quad \ldots\ldots\ldots\ldots (128),$$

a result identical with (113) so far as the first powers of Δm, $\Delta\sigma$.

In obtaining (128), we have neglected the effect of the variable nature of the medium *on the disturbance*. We might, when the disturbance on this supposition is thoroughly known, approximate again in the same manner. The additional terms so found are necessarily of

the second order in Δm, $\Delta\sigma$, so that our expression must be correct as far as the first powers, as we see to be the case.

It must be remarked that in our last investigation we are not limited to a *sphere;* T is a small space of any shape, all that is required of it being that in all its dimensions it be small compared to λ. It is not even necessary that the changes of mechanical properties of the medium should be abrupt, or the properties themselves constant through T; only in such a case we must substitute for $T\Delta m$, $T\Delta\sigma$, $\iiint \Delta m\, dv$, $\iiint \Delta\sigma\, dv$ respectively, or (which means the same thing) understand by Δm, $\Delta\sigma$ their *average* values over the space T, where they differ from zero. Although, therefore, the first method has the advantage of giving the leading term correct to all powers of Δm, $\Delta\sigma$, and of allowing the approximation depending on the smallness of the sphere to be more easily pushed further, yet in two other respects the second method of investigation is the more general.*

Comparison of Intensities.

It is sometimes important to institute a comparison between the intensities of the secondary sound emanating from an obstacle, and that of the primary sound itself. For this purpose we may suppose the primary sound to originate in a point, provided that it be sufficiently distant (R) from T. Thus, for a rigid and fixed sphere,

$$\phi = \frac{\cos k\,(at-\mathrm{R})}{\mathrm{R}},$$

$$\psi = -\frac{\pi \mathrm{T}}{r\mathrm{R}\lambda^2}\left(1+\frac{3}{2}\mu\right)\cos k\,(at-r),$$

so that at equal distances from their sources the secondary and primary waves are in the ratio

$$-\frac{\pi \mathrm{T}}{\mathrm{R}\lambda^2}\left(1+\frac{3}{2}\mu\right). \qquad \text{(129)}.$$

The intensities of the sounds are therefore in the ratio

$$\frac{\pi^2 \mathrm{T}^2}{\mathrm{R}^2\lambda^4}\left(1+\frac{3}{2}\mu\right)^2,$$

* An analytical expression for the case of a sphere not very small may be best obtained by the first method; but for the not less important purpose of forming a clear mental image of what goes on, the advantage seems to lie with the second. For if the squares of the variations of mechanical properties can be neglected, the effect of a large obstructing body can be calculated by simple integration from those of its parts. By means of this idea the transition from the case of a small particle, whose *surface* does not come into consideration, to that of a thin plate a few or a great many square wave-lengths in area, may be followed in imagination, and is very instructive.

which in the case of $\mu = +1$ becomes approximately

$$\frac{61{\cdot}72\,T^2}{R^2\lambda^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (130).$$

In order that this may apply, it must be well understood that λ must be great compared with the linear dimension of T, and that R must be great compared with λ.

Appendix.

Source of Primary Waves at a Finite Distance.

It may be well to add here that the analytical solution of the problem of the disturbance caused by an obstructing sphere can be obtained, even when the source of the primary waves be not infinitely distant. If b be the distance between the centre of the sphere and the source, the velocity-potential may be expressed by

$$\phi = \frac{\epsilon^{ik(at-R)}}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (131),$$

where

$$R^2 = b^2 - 2rb\mu + r^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (132).$$

Let ϕ be expanded in a series of harmonics

$$\phi = \epsilon^{ikat}\{A'_0 + A'_1 Q_1 + \ldots\ldots + A'_n Q_n + \ldots\} \quad \ldots\ldots\ldots \quad (133),$$

the coefficients being functions of b and r. But, as we have seen, the motion through any space not including the source is expressible by

$$\phi = \epsilon^{ikat}\Sigma A''_n Q_n \left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr}\,.\,Q_n \quad \ldots\ldots\ldots\ldots \quad (134),$$

where the coefficients A″ *are independent of* r. They are functions of b which have to be found.

From (133) and (134),

$$A'_n = \frac{2n+1}{2}\int_{-1}^{+1}\frac{\epsilon^{-ikR}}{R}\,Q_n d\mu = A''_n Q_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr} \quad \ldots \quad (135).$$

We may determine A''_n by a comparison of the leading terms on the two sides of equation (135), when r is made infinitely small. On the second side we have already seen (50) that the leading term is

$$A''_n \frac{(ikr)^n}{1\,.\,3\,.\,5\,\ldots\ldots\,(2n+1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (136).$$

On the first side we have to evaluate

$$\frac{2n+1}{2}\int_{-1}^{+1}\frac{\epsilon^{-ikb\left(1+\frac{r^2}{b^2}-2\frac{r}{b}\mu\right)^{\frac{1}{2}}}}{b\left(1+\frac{r^2}{b^2}-2\frac{r}{b}\mu\right)^{\frac{1}{2}}}\times Q_n d\mu,$$

so far as the leading term (containing r^n). If b were indefinitely great, the integral would become

$$\frac{2n+1}{2}\int_{-1}^{+1}\frac{\epsilon^{-ikb}}{b}\epsilon^{ikr\mu}\,Q_n d\mu = \frac{(ikr)^n\,\epsilon^{-ikb}}{1.3.5\ldots\ldots(2n-1)\,b}, \text{ from (50).}$$

Comparing this with (136), we find

$$A_n'' = \frac{(2n+1)\,\epsilon^{-ikb}}{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (137),$$

whence ϕ is given by (134).

But, if b be not infinitely great, a different process is required. We have

$$\frac{\epsilon^{-ikR}}{R} = \left\{1 - ikR + \ldots\ldots + \frac{(-ikR)^p}{\lfloor p} + \ldots\ldots\right\} \div R = \Sigma\frac{(-ik)^p R^{p-1}}{1.2\ldots\ldots p},$$

and

$$R^{p-1} = b^{p-1}\left\{1 + \frac{r^2}{b^2} - 2\frac{r}{b}\mu\right\}^{\frac{p-1}{2}}.$$

Now if $\frac{\epsilon^{-ikR}}{R}$ be expanded in integral powers of μ, every term below μ^n will vanish when multiplied by Q_n and integrated; while the terms above μ^n, though not vanishing, will contribute nothing to the leading term, which alone we require. Let us put for the present

$$\frac{\epsilon^{-ikb\left(1-2\frac{r}{b}\mu\right)^{\frac{1}{2}}}}{b\left(1-\frac{2r}{b}\mu\right)^{\frac{1}{2}}} = \ldots\ldots + H\mu^n + \ldots\ldots;$$

then we may take

$$\frac{2n+1}{2}\int_{-1}^{+1}\frac{\epsilon^{-ikR}}{R}Q_n d\mu = \frac{2n+1}{2}\int_{1}^{+1} H\mu^n Q_n d\mu$$

$$= \frac{1.2.3\ldots\ldots n}{1.3.5\ldots\ldots(2n-1)}H,$$

which is to be equated to (136); so that

$$A_n'' = \frac{1.2.3\ldots\ldots n\,(2n+1)\,H}{(ikr)^n} \quad \ldots\ldots\ldots\ldots \quad (138).$$

It remains to determine H, or the coefficient of μ^n in

$$\Sigma\frac{(-ik)^p R^{p-1}}{1.2\ldots p}.$$

Now, in the expansion of

$$\frac{(-ik)^p R^p}{1.2\ldots p}, \text{ where } R = b\left(1-\frac{2r}{b}\mu\right)^{\frac{1}{2}},$$

the term sought is

$$\frac{(-ik)^p}{1.2\ldots\ldots p}\,b^{p-1}\,\frac{\frac{p-1}{2}\,\frac{p-3}{2}\ldots\ldots\frac{p-2n+1}{2}}{1.2\ldots\ldots n}\left(-\frac{2r}{b}\mu\right)^n$$

$$=\frac{(-ikb)^p}{1.2\ldots\ldots p}\,\frac{(p-1)(p-3)\ldots\ldots(p-2n+1)}{1.2\ldots\ldots n}\,(-1)^n\,\frac{r^n\mu^n}{b^{n+1}},$$

so that

$$\mathrm{H}=(-1)^n\,\frac{r^n}{1.2\ldots n.b^{n+1}}\sum_{p=-1}^{p=\infty}\frac{(p-1)(p-3)\ldots(p-2n+1)}{1.2.3\ldots\ldots p}\,(-ikb)\,;$$

and from (104)

$$\mathrm{A}_n''=(-1)^n\,\frac{2n+1}{b\,(ikb)^n}\sum_{p=-1}^{p=\infty}\frac{(p-1)(p-3)\ldots\ldots(p-2n+1)}{1.2.3\ldots p}\,(-ikb)^p \qquad \ldots\ldots\ldots (139).$$

The value of ϕ due to a source at a distance b from the pole is thus determined by (134), (139). The remainder of the investigation proceeds as before.*

On the Summation of Certain Series. *By* Prof. Wolstenholme.

(*Read January 9th*, 1873.)

Suppose a person to draw successively from a number n of different vessels, each containing white and black balls, and let p_r be the chance that a drawing from the rth vessel gives a white ball, then his whole expectation of drawing a white ball is

$$1-(1-p_1)(1-p_2)(1-p_3)\ldots\ldots(1-p_n)\,;$$

for he must either draw a white ball sometime or other, or fail in every attempt. But the sum of his several expectations at the first, second, ... nth drawings, is

$$p_1+(1-p_1)\,p_2+(1-p_1)(1-p_2)\,p_3+\ldots+(1-p_1)(1-p_2)\ldots(1-p_{n-1})\,p_n\,;$$

whence we have the algebraical identity

$$p_1+(1-p_1)\,p_2+(1-p_1)(1-p_2)\,p_3+\ldots+(1-p_1)(1-p_2)\ldots(1-p_{n-1})\,p_n$$
$$=1-(1-p_1)(1-p_2)\ldots(1-p_n).$$

This method of proof assumes that $p_1, p_2, \ldots$ are all less than 1, but it is obvious that such a restriction is altogether unnecessary. In fact, this identity, when obtained, is obviously true, but it is one which includes several series of interest, some examples of which are appended.

$$(1.)\ \frac{n}{m+n}+\frac{m}{m+n}\,\frac{n}{m+n-1}+\frac{m\,(m-1)\,n}{(m+n)(m+n-1)(m+n-2)}+\ldots$$
$$\ldots\text{ to }(m+1)\text{ terms}\equiv 1,$$

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* This Paper appears as it was originally read before the Society, without alterations other than purely verbal.

or

$$\frac{1}{m+n}+\frac{m}{(m+n)(m+n-1)}+\frac{m(m-1)}{(m+n)(m+n-1)(m+n-2)}+\ldots=\frac{1}{n}.$$

(2.) $$\frac{n}{m+n}+\frac{m}{m+n}\frac{n}{m+n+1}+\frac{m(m+1)}{(m+n)(m+n+1)}\frac{n}{m+n+2}+\ldots$$

$$\ldots \text{ to } r \text{ terms}=1-\frac{m(m+1)\ldots(m+r-1)}{(m+n)(m+n+1)\ldots(m+n+r-1)};$$

or $$1+\frac{m}{m+n+1}+\frac{m(m+1)}{(m+n+1)(m+n+2)}+\ldots\ldots \text{ to } r \text{ terms}$$

$$=\frac{m+n}{n}\left\{1-\frac{m(m+1)\ldots(m+r-1)}{(m+n)(m+n+1)\ldots(m+n+r-1)}\right\};$$

or $$1+\frac{m}{m+n}+\frac{m(m+1)}{(m+n)(m+n+1)}+\ldots\ldots \text{ to } r \text{ terms}$$

$$=\frac{m+n-1}{n-1}\left\{1-\frac{m(m+1)\ldots(m+r-1)}{(m+n-1)(m+n)\ldots(m+n+r-2)}\right\}.$$

(3.) $$\frac{p-q}{p}+\frac{q}{p}\frac{p-q}{p+r}+\frac{q(q+r)}{p(p+r)}\frac{p-q}{p+2r}+\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}\frac{p-q}{p+3r}$$

$$+\ldots\ldots \text{ to } n \text{ terms}=1-\frac{q(q+r)(q+2r)\ldots(q+\overline{n-1}r)}{p(p+r)(p+2r)\ldots(p+\overline{n-1}r)};$$

or $$1+\frac{q}{p+r}+\frac{q(q+r)}{(p+r)(p+2r)}+\frac{q(q+r)(q+2r)}{(p+r)(p+2r)(p+3r)}+\ldots\ldots$$

$$=\frac{p}{p-q}\left\{1-\frac{q(q+r)(q+2r)\ldots(q+\overline{n-1}r)}{p(p+r)\ldots(p+\overline{n-1}r)}\right\};$$

or $$1+\frac{q}{p}+\frac{q(q+r)}{p(p+r)}+\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\ldots\ldots \text{ to } n \text{ terms}$$

$$=\frac{p-r}{p-q-r}\left\{1-\frac{q(q+r)\ldots(q+\overline{n-1}r)}{(p-r)p(p+r)\ldots(p+\overline{n-2}r)}\right\};$$

whence we see that this well known series is convergent when $p>q+r$, and divergent when $p<q+r$; also when $p=q+r$ it becomes the common harmonical series, and is divergent.

In the case when this series is convergent, we get

$$1+\frac{q}{p}+\frac{q(q+r)}{p(p+r)}+\ldots\ldots \text{ to } \infty=\frac{p-r}{p-q-r},$$

$$\frac{q}{p}\left(1+\frac{q+r}{p+r}+\frac{(q+r)(q+2r)}{(p+r)(p+2r)}+\ldots\ldots\right)=\frac{q}{p}\frac{p}{p-q-r},$$

$$\frac{q(q+r)}{p(p+r)}\left(1+\frac{q+2r}{p+2r}+\ldots\ldots\right)=\frac{q(q+r)}{p(p+r)}\frac{p+r}{p-q-r},$$

$$\&c. \qquad = \qquad \&c.;$$

therefore $1+2\frac{q}{p}+3\frac{q(q+r)}{p(p+r)}+4\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\dots$ to ∞

$$=\frac{1}{p-q-r}\left\{p-r+q\left(1+\frac{q+r}{p}+\frac{(q+r)(q+2r)}{p(p+r)}+\dots\right)\right\}$$

$$=\frac{1}{p-q-r}\left\{p-r+q\frac{p-r}{p-q-2r}\right\}=\frac{(p-r)(p-2r)}{(p-q-r)(p-q-2r)},$$

provided that $p>q+2r$. These results suggest the general law.

Assuming that, when m is an integer, and $p>q+mr$,

$$1+m\frac{q}{p}+\frac{m(m+1)}{2}\frac{q(q+r)}{p(p+r)}+\frac{m(m+1)(m+2)}{\lfloor 3}\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\dots$$

..... to ∞

$$=\frac{(p-r)(p-2r)\dots(p-mr)}{(p-q-r)(p-q-2r)\dots(p-q-mr)},$$

we shall then have

$$\frac{q}{p}\left(1+m\frac{q+r}{p+r}+\frac{m(m+1)}{\lfloor 2}\frac{(q+r)(q+2r)}{(p+r)(p+2r)}+\dots \text{ to } \infty\right)$$

$$=\frac{q}{p}\cdot\frac{p(p-r)\dots p-(m-1)r}{(p-q-r)\dots(p-q-mr)},$$

$$\frac{q(q+r)}{p(p+r)}\left(1+m\frac{q+2r}{p+2r}+\dots \text{ to } \infty\right)$$

$$=\frac{q(q+r)}{p(p+r)}\cdot\frac{(p+r)p\dots p-(m-2)r}{(p-q-r)\dots(p-q-mr)},$$

&c. = &c.;

or adding,

$$1+(m+1)\frac{q}{p}+\frac{(m+1)(m+2)}{\lfloor 2}\frac{q(q+r)}{p(p+r)}$$

$$+\frac{(m+1)(m+2)(m+3)}{\lfloor 3}\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\dots$$

$$=\frac{(p-r)(p-2r)\dots(p-mr)}{(p-q-r)(p-q-2r)\dots(p-q-mr)}$$

$$\times\left\{1+\frac{q}{p-mr}+\frac{q(q+r)}{(p-mr)\{p-(m-1)r\}}+\dots \text{ to } \infty\right\};$$

or if $p-mr>q+r$, and therefore $p>q+(m+1)r$,

$$=\frac{(p-r)(p-2r)\dots(p-mr)}{(p-q-r)(p-q-2r)\dots(p-q-mr)}\cdot\frac{p-mr-r}{p-mr-q-r}$$

$$=\frac{(p-r)(p-2r)\dots\{p-(m+1)r\}}{(p-q-r)(p-q-2r)\dots\{p-q-(m+1)r\}}.$$

Hence, when $p > q+mr$, and m is a whole number, we have the limit of the sum of the series

$$1+m\frac{q}{p}+\frac{m(m+1)}{2}\frac{q(q+r)}{p(p+r)}+\frac{m(m+1)(m+2)}{\lfloor 3}\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\ldots \text{ to } \infty$$

equal to $$\frac{(p-r)(p-2r)\ldots(p-mr)}{(p-q-r)(p-q-2r)\ldots(p-q-mr)}.$$

Hence, putting $r=1$, we get the result

$$\frac{\Gamma(q)}{\Gamma(p)}+m\frac{\Gamma(q+1)}{\Gamma(p+1)}+\frac{m(m+1)}{\lfloor 2}\frac{\Gamma(q+2)}{\Gamma(p+2)}+\ldots \text{ to } \infty$$
$$=\frac{\Gamma(q)}{\Gamma(p)}\frac{\Gamma p}{\Gamma(p-m)}\cdot\frac{\Gamma(p-q-m)}{\Gamma(p-q)};$$

and putting $-m$ for m in this result, and striking out corresponding factors, we have, when m is a whole number,

$$1-m\frac{q}{p}+\frac{m(m-1)}{\lfloor 2}\frac{q(q+1)}{p(p+1)}+\frac{m(m-1)(m-2)}{\lfloor 3}\frac{q(q+1)(q+2)}{p(p+1)(p+2)}-\ldots$$
$$\equiv\frac{(p-q)(p-q+1)\ldots(p-q+m-1)}{p(p+1)\ldots(p+m-1)}$$
$$\equiv\left(1-\frac{q}{p}\right)\left(1-\frac{q}{p+1}\right)\left(1-\frac{q}{p+2}\right)\ldots\ldots\left(1-\frac{q}{p+m-1}\right).$$

Strictly this is not a proof of the result; but, as used to be continually said of the use of $\sqrt{-1}$, only serves to suggest a result which should be proved independently.

This equality may in this case be very readily proved, either by the method of so-called "induction," or by proving that $q-p$, $q-p-1$, ... are factors of the sinister, since the coefficients of q^m are manifestly the same.

It has, however, been pointed out to me that the simplest proof is by equating coefficients of x^m in the identity

$$(1+x)^{p+q}=(1+x)^q\left(1-\frac{x}{1+x}\right)^{-p}.$$

It may of course be written equally in the form

$$1-m\frac{q}{p}+\frac{m(m-1)}{\lfloor 2}\frac{q(q-1)}{p(p-1)}-\ldots \quad \text{(where } m \text{ is a whole number)}$$
$$\equiv\left(1-\frac{q}{p}\right)\left(1-\frac{q}{p-1}\right)\left(1-\frac{q}{p-2}\right)\ldots \text{ to } m \text{ factors,}$$

in which it appeared as a Senate House Problem in 1869. It is really only another form of Vandermonde's Theorem, which is that, if a_n denote $a(a-1)\ldots(a-n+1)$, then

$$(a+b)_n\equiv a_n+n\,a_{n-1}\,b_1+\frac{n(n-1)}{\lfloor 2}a_{n-2}\,b_2+\ldots;$$

whence, dividing by a_n,

$$\left(1+\frac{b}{a}\right)\left(1+\frac{b}{a-1}\right)\ldots\ldots\left(1+\frac{b}{a-n+1}\right)$$
$$\equiv 1+n\frac{b}{a-n+1}+\frac{n(n-1)}{\underline{|2}}\frac{b(b-1)}{(a-n+1)(a-n+2)}+\ldots\ldots;$$

or putting $a-n+1=-c$, and reversing the order of the factors,

$$\left(1-\frac{b}{c}\right)\left(1-\frac{b}{c-1}\right)\left(1-\frac{b}{c-2}\right)\ldots\ldots\left(1-\frac{b}{c-n+1}\right)$$
$$=1-n\frac{b}{c}+\frac{n(n-1)}{\underline{|2}}\frac{b(b-1)}{c(c-1)}-\ldots\ldots.$$

I believe I was the first to discover that my question was really only a very well-known theorem, slightly disguised; and I did penance accordingly by appending the proper title to the question when reprinted in the Calendar.

The series which can be brought under the general formula may be written as follows:

$$1+\frac{x}{y+a}+\frac{x(x+a)}{(y+a)(y+b)}+\frac{x(x+a)(x+b)}{(y+a)(y+b)(y+c)}+\ldots \text{ to } r \text{ terms},$$

the sum of which is

$$\frac{y}{y-x}\left\{1-\frac{x(x+a)(x+b)\ldots \text{ to } r \text{ factors}}{y(y+a)(y+b)\ldots \text{ to } r \text{ factors}}\right\}.$$

They are all included in the formulæ given by Gauss in his Memoir "Disquisitiones generales circa seriem infinitam

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\ldots\ldots,"$$

Göttingen Comm. 1812, and Werke, Band iii. See Todhunter's "Algebra," fifth edition, p. 511. The above reference was kindly supplied by Mr. J. W. L. Glaisher.

March 13th, 1873.

Dr. HIRST, F.R.S., President, in the Chair.

Prof. A. G. Greenhill, M.A., of Cooper's Hill College, was elected a Member.

The following papers were read:—

Mr. R. B. Hayward, M.A., "On an extension of the term *Area* to any closed circuit in space."

Mr. J. W. L. Glaisher "On the Evaluation of a class of Definite Integrals involving circular functions in the Numerator and powers of the variable only in the Denominator."

Mr. S. Roberts, Vice-President, "Note on Normals and the Surface of Centres of an Algebraical Surface."

Mr. M. Jenkins (Hon. Sec.), "Proof of the proposition that a number which divides the product of two numbers, and is prime to one of them, will divide the others."

In the course of a discussion on Mr. Hayward's paper, Mr. Merrifield remarked that if a cone of revolution has its vertex on the surface of a sphere, and passes through the centre of a sphere, one of the projections of the intersection is a common parabola; and that the projection by lines parallel to the axis of the cone on a plane at right angles to it is Pascal's Limaçon, reducing to the cardioid when the angle at the vertex of the cone is a right angle.

The following presents were received:—

"On the Contact of Surfaces" (read before the Royal Society, Feb. 22nd, 1872); and "On some Recent Generalizations of Algebra," by Mr. W. Spottiswoode, F.R.S., Vice-President: from the Author.

"A Complete Practical Treatise on the Nature and Use of Logarithms, and on Plane Trigonometry, with Logarithmic and Trigonometrical Tables," by James Elliot (5th edition), with Key (3rd edition): from the Publishers.

"Monatsbericht," Nov. and Dec. 1872.

"Annali di Matematica," serie ii^a., tom. v., fasc. 3°., Nov. 1872.

"Bulletin de la Société Mathématique de France," publié par les Sécrétaires, No. 1: from the Society.

"Journal of London Institution," No. 19.

"Bulletin des Sciences Mathématiques et Astronomiques," Feb. 1873.

"Papers relating to the Transit of Venus in 1874, prepared under the direction of the Commission authorised by Congress, and published by authority of the Hon. Sec. of the Navy;" Part ii.; New York, 1872. In duplicate.

"Reports on Observations of Encke's Comet during its return in 1871," by Asaph Hall and Wm. Harkness; Washington, 1872.

"On the Right Ascensions of the Equatorial Fundamental Stars, and the corrections necessary to reduce the Right Ascensions of different Catalogues to a mean homogeneous system," by Simon Newcomb: from U. S. Royal Observatory, Washington.

"Bulletins de l'Académie Royale des Sciences, des Lettres et des Beaux Arts de Belgique," 2^e série, tomes 31—34.

Ditto, Annuaires de 1872 et 1873: Centième Anniversaire de Fondation, tomes i°. and ii°.; also, Tables de Mortalité, Notices extraites de l'Annuaire de l'Observatoire pour 1873; Notice sur Babbage: from M. A. Quetelet.

Salmon's "Higher Plane Curves" (2nd edition) : from the Author.

"Crelle," 75 Band, 4es Heft, Feb. 1873.

"Journal of the Institute of Actuaries and Assurance Magazine," No. xc., Jan. 1873.

"Bulletin des Sciences Mathématiques et Astronomiques," tome troisième, Dec. 1872 ; tome quatrième, Mars 1873.

"Extrait du Bulletin des Sciences Mathématiques et Astronomiques" (from tom. iv. 1873).

Prof. J. Clerk Maxwell on "Electricity and Magnetism" (2 vols.) : from the Author.

On an Extension of the term "Area" to any closed Circuit in Space.

By R. B. Hayward, M.A.

[*Read March 13th*, 1873.]

De Morgan has shown (*Cam. and Dub. Math. Journ.*, vol. v.) that the term "Area" may be applied in a perfectly definite sense to any closed circuit in one plane, however *autotomic* its character ; and that it may be regarded as having an algebraical sign determined by the direction in which the circuit is supposed to be described.

I propose to show that there is, for any closed circuit in space, a plane area determined by the circuit itself without any extrinsic elements, which is definite as well in the *aspect* of its plane as in magnitude ; and which, since it reduces to the area of the circuit in De Morgan's sense when the circuit is plane, may properly be termed the area of the circuit in the general case.

In this sense, area is no longer a mere magnitude, or a magnitude affected only with the positive or negative sign, but a magnitude affected with direction ; in other words, it is a *vector*, not simply a *scalar*.

Conceive a point to describe continuously a given closed circuit in space of three dimensions. Then its projection on any given plane will describe a closed circuit in that plane, the area of which will have a definite sign depending on the convention which is made as to the positive direction of revolution on the positive aspect of the plane. Let the positive aspects of the rectangular coordinate planes of yz, zx, xy be those which are turned towards the positive sides of the axes of x, y, z respectively, then the projections of the given circuit on the coordinate planes will have areas determinate in sign as well as in magnitude. Let them be denoted by A_x, A_y, A_z respectively. Then, if the direction-cosines of a given plane, that is, the cosines of the angles between its

positive aspect and the positive aspects of the coordinate planes respectively, be denoted by l, m, n, the area of the projection of the given circuit on that plane will be given, both in sign and magnitude, by the expression $lA_x+mA_y+nA_z$.

This expression is well known for a plane circuit, and it will be easily seen to be true for any circuit whatever. For, if PQ be an element of the circuit, and O any fixed point, the expression will be true for the projections of the plane element POQ; and therefore, for the algebraical sums of the projections of all such elements, and these determine the areas of the projections of the complete circuit.

Now, if $A=\sqrt{A_x^2+A_y^2+A_z^2}$, the expression $lA_x+mA_y+nA_z$ is a maximum, and is equal to A, when $l=\frac{A_x}{A}$, $m=\frac{A_y}{A}$, $n=\frac{A_z}{A}$. Hence, on a plane, whose aspect is determined by these values of l, m, n, the area of the projection (A) of the given circuit will be a maximum; and if θ be the angle between the positive aspect of a given plane and that of the plane of A, it will be readily seen that $A\cos\theta$ is the area of the projection of the circuit on that plane. Thus the projection of the area A* on any plane has an area equal to that of the projection of the circuit itself. Farther, it is obvious that, if the circuit be plane, A is the area of the circuit itself in the ordinary sense. It is therefore a natural extension of the term to speak of this maximum area A as *the* area of the circuit, in the general case of any circuit whatever in space.

Adopting this more general sense of the term area, it is plain that areas are vector magnitudes subject to the laws of vector addition and resolution, like finite lines, forces, couples, velocities, &c.

It is worth noting that on any plane perpendicular to that of the area of a circuit, the area of its projection will be zero, so that on all such planes the projection of the circuit must be either a finite arc of a curve, or an autotomic circuit, in which the positive and negative "primary parts" of the area are equal. Conversely, if *two* planes can be found on which the areas of the projections of a closed circuit are zero, on *every* plane parallel to their common section, the area of its projection will also be zero, and the plane perpendicular to both of them will be the plane of the area of the circuit.

Let me conclude with a few illustrations of the use of this extension of the term area.

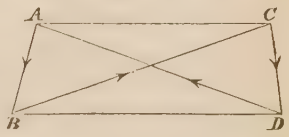
It is well known that if a set of forces, acting along the sides of a plane polygon, taken in order, be represented in magnitude by those

* It is to be observed that the projections of the area A and of the circuit are not in general *identical* or *identically equal*, but only equal in area.

sides, their resultant will be a couple represented in sign and magnitude by double the area of the polygon. With the extended sense of the term area, it is easily seen that the same proposition is true for any polygon in space, the aspect of the area defining that of the couple.

Again, if a homogeneous incompressible fluid move in a tubular channel of small section, whose axis is a closed curve, and which it completely fills, it is plain that, as the centre of inertia of the fluid is at rest, the total momentum of the fluid is reducible to an angular momentum, which may easily be shown to be in the plane of "the area of the curve," and proportional to that area in intensity.

Lastly, to take an example from pure geometry, the area of a skew quadrilateral is the area of its projection on a plane parallel to both its diagonals. For, on any plane perpendicular to this, its diagonals will be projected into two parallel lines as AC, BD, and the quadrilateral into the plane autotomic quadrilateral ABCD whose area is evidently zero, so that on a plane parallel to both diagonals the projection is a maximum. Hence the area of a skew quadrilateral is the same as that of a plane quadrilateral with diagonals equal to those of the former, and inclined to one another at the same angle; and we may state generally that the area of *any* quadrilateral, skew or plane, is half that of the parallelogram contained by its diagonals. The same result might have been obtained by the vector-addition of the areas of the two triangles, into which the quadrilateral is divided by one of its diagonals.



On the Evaluation of a Class of Definite Integrals involving Circular Functions in the Numerator, and Powers of the Variable only in the Denominator. *By* J. W. L. Glaisher, B.A., F.R.A.S., Fellow of Trinity College, Cambridge.

[*Read March 13th*, 1873.]

The main purpose of the present paper will be best understood by commencing with an example. Suppose it were required to find the value of $\int_0^\infty \frac{\sin ax \sin bx}{x^2}\,dx$, which is clearly finite, then the most obvious mode of procedure would be to replace $\sin ax \sin bx$ by

the equivalent expression $\frac{1}{2}\{\cos(a-b)x - \cos(a+b)x\}$, so as to obtain

$$\frac{1}{2}\cdot\int_0^\infty \frac{\cos(a-b)x}{x^{\frac{3}{2}}}\,dx - \frac{1}{2}\cdot\int_0^\infty \frac{\cos(a+b)x}{x^{\frac{3}{2}}}\,dx \quad \ldots\ldots\ldots(1).$$

Substituting, then, from the known formula

$$\int_0^\infty \frac{\cos bx}{x^n}\,dx = \frac{\pi}{2\Gamma(n)}\,\frac{b^{n-1}}{\cos\frac{n\pi}{2}} \quad \ldots\ldots\ldots\ldots\ldots(2),$$

we should obtain as the result

$$\frac{\sqrt{\pi}}{\sqrt{2}}\left\{(a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

(a being supposed $>b$).

The value (3) is correct, although the manner in which it has been obtained is not sound; as, both the integrals in (1) being really infinite in value, we have no justification for replacing them by the evaluations in (3), which are derived from (2), a formula only true when the integral involved is finite.

It will be proved that the mode that has been pursued does, however, always give correct results; and a corresponding rule will be investigated for the case when the right-hand side of (2) becomes infinite (viz., when n is an odd integer), and for other similar cases of failure. For the sake both of convenience and greater generality, an exponential factor will be introduced under the integral sign throughout, and the integrals considered will be of the form $\int^\infty e^{-ax}\,\frac{f(x)}{x^n}\,dx$, $f(x)$ being composed of sines and cosines of x.

Before proceeding to the necessary developments, it is worth noticing, in the above simple example, how the accuracy of the result can be verified. This can be done in two ways: (i.) by integration by parts,

$$\int_0^\infty \frac{\sin ax\,\sin bx}{x^{\frac{3}{2}}}\,dx$$

$$= -2\left[\frac{\sin ax\,\sin bx}{x^{\frac{1}{2}}}\right]_0^\infty - \int_0^\infty \{(a-b)\sin(a-b)x - (a+b)\sin(a+b)x\}\,\frac{dx}{x^{\frac{1}{2}}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}}\left\{(a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}}\right\},$$

as follows at once from the formula

$$\int_0^\infty \frac{\sin bx}{x^n}\,dx = \frac{\pi}{2\Gamma(n)}\,\frac{b^{n-1}}{\sin\frac{n\pi}{2}},$$

the application of which is free from uncertainty, as both the integrals are finite.

Or (ii.) we may find the value of $\int_0^\infty e^{-cx} \frac{\sin ax \sin bx}{x^{\frac{3}{2}}} dx$ as follows, and then put $c=0$:

$$\int_0^\infty e^{-cx} \frac{\sin ax \sin bx}{x^{\frac{3}{2}}} dx = \frac{1}{2} \cdot \int_0^\infty e^{-cx} \{\cos(a-b)x - \cos(a+b)x\} \frac{dx}{x^{\frac{3}{2}}}$$

$$= \frac{1}{2} \cdot \int_0^\infty dx \cdot \frac{e^{-cx}}{x^{\frac{3}{2}}} \left\{1 - \frac{\alpha^2 x^2}{2!} + \frac{\alpha^4 x^4}{4!} - \ldots\ldots - 1 + \frac{\beta^2 x^2}{2!} - \frac{\beta^4 x^4}{4!} + \ldots\ldots\right\},$$

writing α and β for $(a-b)$ and $(a+b)$ for brevity, and $n!$ for $1.2 \ldots n$,

$$= \frac{1}{2} \left\{ - \frac{\Gamma(\frac{3}{2})}{2!} \frac{\alpha^2}{c^{\frac{3}{2}}} + \frac{\Gamma(\frac{7}{2})}{4!} \frac{\alpha^4}{c^{\frac{7}{2}}} - \ldots\ldots + \frac{\Gamma(\frac{3}{2})}{2!} \frac{\beta^2}{c^{\frac{3}{2}}} - \frac{\Gamma(\frac{7}{2})}{4!} \frac{\beta^4}{c^{\frac{7}{2}}} + \ldots\ldots \right\}$$

$$= \frac{\sqrt{\pi}}{2} c^{\frac{1}{2}} \left\{ - \left(1 + \frac{\alpha i}{c}\right)^{\frac{1}{2}} - \left(1 - \frac{\alpha i}{c}\right)^{\frac{1}{2}} + \left(1 + \frac{\beta i}{c}\right)^{\frac{1}{2}} + \left(1 - \frac{\beta i}{c}\right)^{\frac{1}{2}} \right\}$$

$$= \frac{\sqrt{\pi}}{2} \left\{ \beta^{\frac{1}{2}} \left(\frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\right) - \alpha^{\frac{1}{2}} \left(\frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\right) \right\} \quad \begin{matrix}\text{(when } c \text{ is put equal} \\ \text{to zero)}\end{matrix}$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}} \left\{ (a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}} \right\}.$$

The general cases can usually be treated in the same way; but it is clearly advantageous to be able to replace the long and tedious calculations that would then be necessary by a direct rule.

To resume, then, the general investigation, it is to be noticed that, for

$$\int_0^\infty e^{-ax} \frac{f(x)}{x^n} dx$$

to be of finite value, $f(x)$, when expanded in ascending powers of x, must contain no power of x less than x^{n-1}; this is evident, for $\int_0^\infty \frac{e^{-ax}}{x^r} dx$ is infinite, if r lie between 1 and ∞ inclusive:

First, suppose n fractional, and let m be the integer next below n.

Let $f(x)$, a function of $\sin x$ and $\cos x$, be expressed as a series of sines and cosines of multiple arcs, thus:

$$f(x) = A_1 \cos b_1 x + A_2 \cos b_2 x + \ldots + A_1' \sin b_1' x + A_2' \sin b_2' x + \ldots;$$

then we must have $\Sigma A b = 0$, $\Sigma A b^2 = 0$, $\Sigma A b^3 = 0$, $\ldots$ $\Sigma A b^{m-1} = 0$; but $\Sigma A b^m$, and terms involving higher powers of b need not vanish [of course if $f(x)$ consists of cosines alone, there are no terms involving odd powers of b, and if $f(x)$ consists of sines alone, there are no terms involving even powers of b, so that in these cases some of the conditions are superfluous].

If r be intermediate to 0 and 1, it is known that

$$\int_0^\infty e^{-ax}\frac{\cos bx}{x^r}\,dx = \Gamma(1-r)(a^2+b^2)^{\frac{1}{2}(r-1)}\cos\left\{(r-1)\tan^{-1}\frac{b}{a}\right\}$$

$$= \frac{\pi}{\Gamma(r)\sin r\pi}(a^2+b^2)^{\frac{1}{2}(r-1)}\cos\left\{(r-1)\tan^{-1}\frac{b}{a}\right\},$$

$$\int_0^\infty e^{-ax}\frac{\sin bx}{x^r}\,dx = -\frac{\pi}{\Gamma(r)\sin r\pi}(a^2+b^2)^{\frac{1}{2}(r-1)}\sin\left\{(r-1)\tan^{-1}\frac{b}{a}\right\}.$$

Suppose $n=m+r$, and let m be an integer, and r as above, then since

$$\int_0^\infty e^{-ax}\frac{f(x)}{x^r}\,dx = \frac{\pi}{\Gamma(r)\sin r\pi}\Sigma\left[A(a^2+b^2)^{\frac{1}{2}(r-1)}\cos\left\{(r-1)\tan^{-1}\frac{b}{a}\right\}\right.$$

$$\left.-A'(a^2+b'^2)^{\frac{1}{2}(r-1)}\sin\left\{(r-1)\tan^{-1}\frac{b'}{a}\right\}\right] \quad \ldots\ldots\ldots\ldots (4),$$

integrate m times with regard to a between the limits ∞ and a, and it will be shown that there results

$$\int_0^\infty e^{-ax}\frac{f(x)}{x^n}\,dx = \frac{\pi}{\Gamma(n)\sin n\pi}\Sigma\left[A(a^2+b^2)^{\frac{1}{2}(n-1)}\cos\left\{(n-1)\tan^{-1}\frac{b}{a}\right\}\right.$$

$$\left.-A'(a^2+b'^2)^{\frac{1}{2}(n-1)}\sin\left\{(n-1)\tan^{-1}\frac{b'}{a}\right\}\right] \quad \ldots\ldots\ldots\ldots (5).$$

To demonstrate this, it is necessary to establish two results, viz. (i.) that the indefinite integral with regard to a of

$$\frac{1}{\Gamma(p)\sin p\pi}(a^2+b^2)^{\frac{1}{2}(p-1)}\cos\left\{(p-1)\tan^{-1}\frac{b}{a}\right\},$$

is $$-\frac{1}{\Gamma(p+1)\sin(p+1)\pi}(a^2+b^2)^{\frac{1}{2}p}\cos\left\{p\tan^{-1}\frac{b}{a}\right\},$$

and that a similar formula is true for the corresponding expression involving the sine; (ii.) that at the limit ∞ the expression on the right-hand side of (5) always vanishes after each of the integrations.

The truth of the first result is evident on differentiation, and the second may be proved as follows:—It will be found to be sufficient to show that the right-hand side of (5) vanishes as it stands when $a=\infty$, as this includes the cases of the inferior integrations.

By De Moivre's theorem,

$$(a^2+b^2)^{\frac{1}{2}n}\cos\left(n\tan^{-1}\frac{b}{a}\right) = a^n\left\{1-\frac{n(n-1)}{2!}\frac{b^2}{a^2}\right.$$

$$\left.+\frac{n(n-1)\ldots(n-4)}{4!}\frac{b^4}{a^4}-\ldots\ldots\right\},$$

and

$$(a^2+b^2)^{\frac{1}{2}n}\sin\left(n\tan^{-1}\frac{b}{a}\right) = a^n\left\{n\frac{b}{a}-\frac{n(n-1)(n-2)}{3!}\frac{b^3}{a^3}+\ldots\right\};$$

whence the right-hand side of (5), leaving out the constant factor, is seen to be

$$\Sigma\left[\mathrm{A}a^{n-1}\left\{1-\frac{(n-1)(n-2)}{2!}\frac{b^2}{a^2}+\ldots\ldots\right\}\right.$$
$$\left.-\mathrm{A}'a^{n-1}\left\{n\frac{b'}{a}-\frac{(n-1)(n-2)(n-3)}{3!}\frac{b'^3}{a^3}+\ldots\ldots\right\}\right.$$

but in virtue of the relations $\Sigma\mathrm{A}=0$, $\Sigma\mathrm{A}b=0$, ... $\Sigma\mathrm{A}b^{m-1}$, &c., all the terms involving a raised to a positive power vanish (or do not occur at all), and the first term that can remain is of the form

$$\mathrm{P}a^{n-1}\cdot\frac{b^m}{a^m}=\mathrm{P}a^{m+r-1}\cdot\frac{b^m}{a^m}=\mathrm{P}\frac{b^m}{a^{1-r}},$$

which, with all the succeeding terms, vanishes for $a=\infty$.

The same reasoning clearly shows that all the expressions obtained by the previous integrations vanish also for $a=\infty$.

The case of $n=$ an integer, presents rather more difficulty.

In order that $\int_0^\infty e^{-ax}\frac{f(x)}{x^n}\,dx$ may then be finite, it is necessary that all terms up to and including x^{n-1} in the expansion of $f(x)$ should vanish, so that we have $\Sigma\mathrm{A}=0$, $\Sigma\mathrm{A}b=0$, ..., $\Sigma\mathrm{A}b^{n-1}=0$ (subject to the condition that if a term in b^s does not occur in $f(x)$, the condition $\Sigma\mathrm{A}b^s=0$ need not be satisfied); but neither $\Sigma\mathrm{A}b^n$, nor terms involving higher powers, need vanish.

We start with the well known formulæ

$$\int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}=\frac{1}{2}\left\{\frac{1}{a-bi}+\frac{1}{a+bi}\right\},$$
$$\int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}=\frac{1}{2i}\left\{\frac{1}{a-bi}-\frac{1}{a+bi}\right\};$$

so that, in order to perform the integrations that will be required, it is clearly necessary to find the nth integral of $\log x$. This can be obtained as follows:

By a well known theorem, if u_m denote the mth integral of u, then

$$m!\,.\,u_{m+1}=x^m\int u\,dx-mx^{m-1}\int xu\,dx+\frac{m(m-1)}{2!}\int x^2u\,dx\ldots$$
$$\ldots+(-)^m\int x^mu\,dx,$$

and
$$\int x^m\log x\,dx=\frac{x^{m+1}}{m+1}\left\{\log x-\frac{1}{m+1}\right\};$$

whence the $(n+1)$th integral of $\log x$

$$=\frac{x^{n+1}}{n!}\log x\left\{1-n\cdot\frac{1}{2}+\frac{n(n-1)}{2!}\cdot\frac{1}{3}-\ldots\ldots\pm\frac{1}{n+1}\right\}$$
$$-\frac{x^{n+1}}{n!}\left\{1-n\cdot\frac{1}{2^2}+\frac{n(n-1)}{2!}\cdot\frac{1}{3^2}-\ldots\ldots\pm\frac{1}{(n+1)^2}\right\}$$
$$=\frac{x^{n+1}}{(n+1)!}\log x-\frac{x^{n+1}}{n!}\psi(n),$$

putting for the first series its equivalent, viz. $-\frac{1}{n+1}$, and denoting the second by $\psi(n)$.

The nth integral of $\frac{1}{2}\left\{\frac{1}{a-bi}+\frac{1}{a+bi}\right\}$ with regard to a is therefore equal to one-half the real part of

$$\frac{\rho^{n-1}}{(n-1)!}\{\cos(n-1)\theta - i\sin(n-1)\theta\}(\log\rho-\theta i)$$
$$-\frac{\rho^{n-1}}{(n-2)!}\psi(n-2)\{\cos(n-1)\theta - i\sin(n-1)\theta\}$$
$$+\text{a similar expression with } -i \text{ written for } i;$$

viz. is equal to $\frac{\rho^{n-1}}{(n-1)!}\{\log\rho\cos(n-1)\theta-\theta\sin(n-1)\theta\}$
$$-\frac{\psi(n-2)}{(n-2)!}\rho^{n-1}\cos(n-1)\theta,$$

where $\rho=(a^2+b^2)^{\frac{1}{2}}$ and $\theta=\tan^{-1}\frac{b}{a}$.

Similarly the nth integral of $\frac{1}{2i}\left(\frac{1}{a-bi}-\frac{1}{a+bi}\right)$ is

$$-\frac{\rho^{n-1}}{(n-1)!}\{\log\rho\sin(n-1)\theta+\theta\cos(n-1)\theta\}$$
$$+\frac{\psi(n-2)}{(n-2)!}\rho^{n-1}\sin(n-1)\theta.$$

We thus have

$$\int_0^\infty e^{-ax}\frac{f(x)}{x^n}\,dx=(-)^n.\Sigma\left[\frac{A\rho^{n-1}}{(n-1)!}\{\log\rho\cos(n-1)\theta-\theta\sin(n-1)\theta\}\right.$$
$$-A\frac{\psi(n-2)}{(n-2)!}\rho^{n-1}\cos(n-1)\theta-A'\frac{\rho'^{n-1}}{(n-1)!}\{\log\rho'\sin(n-1)\theta'$$
$$\left.+\theta'\cos(n-1)\theta'\}+A'\frac{\psi(n-2)}{(n-2)!}\rho'^{n-1}\sin(n-1)\theta'\right]\ldots\ldots\ldots\ldots\ldots(6);$$

ρ' being $=(a^2+b'^2)^{\frac{1}{2}}$ and $\theta'=\tan^{-1}\frac{b'}{a}$; it will now be demonstrated, as before, that the right-hand side of (6) vanishes when $a=\infty$.

Since $\log\rho=\frac{1}{2}\log(a^2+b^2)=\log a+\frac{b^2}{a^2}-\frac{1}{2}\frac{b^4}{a^4}+\ldots\ldots$,

it is clear that the expression

$$a^{n-1}\left(1+\alpha_1\frac{b}{a}+\alpha_2\frac{b^2}{a^2}+\ldots\ldots\right)\left(\beta\log a+\beta_1\frac{b}{a}+\beta_2\frac{b^2}{a^2}+\ldots\ldots\right)$$
$$\times\left(1+\gamma_1\frac{b}{a}+\gamma_2\frac{b^2}{a^2}+\ldots\ldots\right)$$

$\alpha_1\alpha_2\ldots$, $\beta\beta_1\ldots$, $\gamma_1\gamma_2\ldots$ being coefficients, any of which may be zero, includes all the cases that can arise by the expansion of terms in (6); and in virtue of the relations $\Sigma A=0$, $\Sigma Ab=0$, $\ldots$, $\Sigma Ab'^{-1}=0$, the

only terms that remain are of the form $\frac{b^n}{a}$, $\frac{b^{n+1}}{a^2}$, $\frac{b^{n+2}}{a^3}$, ..., or $\frac{b^n}{a}\log a$, $\frac{b^n}{a^2}\log a$, ..., all of which vanish when a is infinite.

The formulæ will assume a somewhat less confused appearance if we write $\phi(n)$ for $n\psi(n-1)$, so that

$$\phi(n) = n - \frac{n(n-1)}{2!}\frac{1}{2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{3} \ldots\ldots \pm \frac{1}{n};$$

$\frac{\psi(n-2)}{(n-2)!}$ then becomes $\frac{\phi(n-1)}{(n-1)!}$.

We have therefore the general rule that whenever $\int_0^\infty e^{-ax}\frac{f(x)}{x^n}\,dx$ is finite, its value may be obtained by expressing $f(x)$ as a series of sines and cosines of multiple arcs, and substituting for the separate integrals that occur from the formulæ,

$$\int_0^\infty e^{-ax}\frac{\cos bx}{x^n}\,dx = \frac{\pi}{\Gamma(n)\sin n\pi}\rho^{n-1}\cos(n-1)\theta \quad \ldots\ldots (7),$$

$$\int_0^\infty e^{-ax}\frac{\sin bx}{x^n}\,dx = -\frac{\pi}{\Gamma(n)\sin n\pi}\rho^{n-1}\sin(n-1)\theta \ldots\ldots (8),$$

when n is fractional; and

$$\int_0^\infty e^{-ax}\frac{\cos bx}{x^n}\,dx = (-)^n\frac{\rho^{n-1}}{(n-1)!}\{\log\rho\cos(n-1)\theta - \theta\sin(n-1)\theta - \phi(n-1)\cos(n-1)\theta\} \ldots\ldots (9),$$

$$\int_0^\infty e^{-ax}\frac{\sin bx}{x^n}\,dx = (-)^{n-1}\frac{\rho^{n-1}}{(n-1)!}\{\log\rho\sin(n-1)\theta + \theta\cos(n-1)\theta - \phi(n-1)\sin(n-1)\theta\} \ldots\ldots (10),$$

n being an integer, $\rho = (a^2+b^2)^{\frac{1}{2}}$, $\tan\theta = \frac{b}{a}$,

and $\phi(n-1) = n-1-\frac{(n-1)(n-2)}{2!}\cdot\frac{1}{2}+\ldots$, as above.

The most important applications of these formulæ take place when $a=0$, and they then become much simpler. Thus, in the case of an integral of the form $\int_0^\infty\frac{f(x)}{x^n}\,dx$, we may use, just as if they were always true, the formulæ

$$\int_0^\infty\frac{\cos bx}{x^n}\,dx = \frac{\pi b^{n-1}}{2\Gamma(n)\cos\frac{n\pi}{2}} \quad \ldots\ldots (11),$$

$$\int_0^\infty\frac{\sin bx}{x^n}\,dx = \frac{\pi b^{n-1}}{2\Gamma(n)\sin\frac{n\pi}{2}} \quad \ldots\ldots (12),$$

when n is fractional; these follow from (7) and (8), since when $a=0$, $\theta=\frac{\pi}{2}$ and $\rho=b$.

When n is an integer (positive), it is necessary to separate the cases of n even or odd.

$$\int_0^\infty \frac{\cos bx}{x^n}\,dx = (-)^{\frac{1}{2}n}\frac{b^{n-1}}{(n-1)!}\frac{\pi}{2}, \text{ if } n \text{ be even} \quad \ldots\ldots\ldots\ldots (13),$$

$$= (-)^{\frac{1}{2}(n+1)}\frac{b^{n-1}}{(n-1)!}\{\log b - \phi(n-1)\}, \text{ if } n \text{ be odd} \quad \ldots (14),$$

$$\int_0^\infty \frac{\sin bx}{x^n}\,dx = (-)^{\frac{1}{2}n}\frac{b^{n-1}}{(n-1)!}\{\log b - \phi(n-1)\}, \text{ if } n \text{ be even} \quad \ldots\ldots (15),$$

$$= (-)^{\frac{1}{2}(n-1)}\frac{b^{n-1}}{(n-1)!}\frac{\pi}{2}, \text{ if } n \text{ be odd} \quad \ldots\ldots\ldots\ldots (16).$$

But two of these formulæ admit of still further simplification, as it will be shown that the terms involving $\phi(n-1)$ will always vanish, and may therefore be omitted.

To prove this, we need only remark, that $f(x)$, when expressed as a series of sines and cosines of multiple arcs, consists of two portions, viz., an even portion

$$f_2(x) = A_0 + A_1 \cos b_1 x + A_2 \cos b_2 x \ldots\ldots + A_r \cos b_r x,$$

which does not change sign with x, and an odd portion

$$f_1(x) = A'_1 \sin b'_1 x + A'_2 \sin b'_2 x \ldots\ldots + A'_s \sin b'_s x,$$

which does change sign with x; and it is evident that for $\int_0^\infty \frac{f(x)}{x^n}\,dx$ to be finite, we must have $\int_0^\infty \frac{f_2(x)}{x^n}\,dx$ and $\int_0^\infty \frac{f_1(x)}{x^n}\,dx$ each finite also, as $f_2(x)$ contains only even powers of x, and $f_1(x)$ only odd powers. We may therefore, without loss of generality, consider separately the cases of $f(x)$ wholly even or wholly odd (and perhaps it would have been better to have done so throughout).

(i.) If $f(x)$ be even, and n be even, then x^{n-1} being an odd power does not occur in $f(x)$, so that, although we must have $\Sigma A b^{n-2}$ zero, there is no reason why $\Sigma A b^{n-1}$ should be zero, so that (13) needs no alteration; but if n be odd, then x^{n-1} is an even power, and we must have $\Sigma A b^{n-1} = 0$, else the integral would be infinite, thus in (14) the term involving $\phi(n-1)$ may be omitted.

(ii.) If $f(x)$ be odd, and n be even, x^{n-1} is odd, and we must have $\Sigma A' b'^{n-1} = 0$, so that, in (15), the term involving $\phi(n-1)$ may be omitted; but if n be odd, there is no necessity for the vanishing of $\Sigma A' b'^{n-1}$, and the integral is as written in (16).

The formulæ of substitution are therefore

$$\int_0^\infty \frac{\cos bx}{x^n}\,dx = (-)^{\frac{1}{2}n}\,\frac{b^{n-1}}{(n-1)!}\,\frac{\pi}{2} \text{ if } n \text{ be even} \quad \ldots\ldots\ldots\ldots (17),$$

$$= (-)^{\frac{1}{2}(n+1)}\,\frac{b^{n-1}}{(n-1)!}\log b, \text{ if } n \text{ be odd} \quad \ldots\ldots (18),$$

$$\int_0^\infty \frac{\sin bx}{x^n}\,dx = (-)^{\frac{1}{2}n}\,\frac{b^{n-1}}{(n-1)!}\log b, \text{ if } n \text{ be even} \quad \ldots\ldots\ldots (19),$$

$$= (-)^{\frac{1}{2}(n-1)}\,\frac{b^{n-1}}{(n-1)!}\,\frac{\pi}{2}, \text{ if } n \text{ be odd} \quad \ldots\ldots\ldots (20).$$

For example, since $\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3)$, we have, from (18),

$$\int_0^\infty \frac{\sin^4 x}{x^3}\,dx = \frac{1}{8\,.\,2}\left\{4^2\log 4 - 4\,.\,2^2\log 2\right\} = \log 2,$$

and from (17), $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^2}\,dx = -\frac{1}{8\,.\,1}\left\{4 - 4\,.\,2\right\}\frac{\pi}{2} = \frac{\pi}{4} \quad \ldots\ldots\ldots (21).$

These can be verified as follows:—

By integration by parts,

$$\int \frac{\cos bx}{x^3}\,dx = -\frac{1}{2}\,\frac{\cos bx}{x^2} + \frac{b\sin bx}{2x} - \frac{b^2}{2}\int \frac{\cos bx}{x}\,dx,$$

whence

$$\int_0^\infty \frac{\sin^4 x}{x^3}\,dx = -\frac{1}{8\,.\,2}\left\{4^2\int_0^\infty \frac{\cos 4x}{x}\,dx - 4\,.\,2^2\int_0^\infty \frac{\cos 2x}{x}\,dx\right\} \quad \ldots (22).$$

Now $\displaystyle\int_\infty^x \frac{\cos x}{x}\,dx$ is the cosine-integral, usually written $\mathrm{Ci}\,x$, and we have the known formulæ

$$\mathrm{Ci}\,x = \gamma + \tfrac{1}{4}\log(x^4) - \frac{1}{2}\,\frac{x^2}{2!} + \frac{1}{4}\,\frac{x^4}{4!} - \frac{1}{6}\,\frac{x^6}{6!} + \ldots\ldots,$$

where γ is Euler's constant ·5772156..., and

$$\mathrm{Ci}\,x = \sin x\left\{\frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \ldots\right\} - \cos x\left\{\frac{1}{x^2} - \frac{3!}{x^4} + \frac{5!}{x^6} - \ldots\right\},$$

the former showing (what, however, is clear from inspection of the integral) that $\mathrm{Ci}\,\infty = 0$; thus, in (22), taking the lower limit of the integral, an evanescent quantity ϵ, instead of 0, we have, since

$$\int_\infty^x \frac{\cos nx}{x}\,dx = \mathrm{Ci}\,nx,$$

$$\int_0^\infty \frac{\sin^4 x}{x^3}\,dx = \mathrm{Ci}\,4\epsilon - \mathrm{Ci}\,2\epsilon$$

$$= \gamma + \log 4\epsilon - \frac{1}{2}\,\frac{4^2\epsilon^2}{2!} + \ldots\ldots - \left(\gamma + \log 2\epsilon - \frac{1}{2}\,\frac{2^2\epsilon^2}{2!} + \ldots\ldots\right)$$

$$= \log 2, \text{ when } \epsilon = 0.$$

The verification of (21) by integration by parts presents no difficulty, the integral being reduced to dependence on $\int_0^\infty \frac{\sin ax}{x}\,dx$, which, when a is positive, $=\frac{\pi}{2}$. I may here remark that throughout, in applying the formulæ (17)—(20), we must make b positive before substituting it in b^{n-1}, *e.g.* in (17), we must substitute $(-)^{\frac{1}{2}n}\frac{2^{n-1}}{(n-1,!}$ for $\int_0^\infty \frac{\cos 2x}{x^n}\,dx$ and not $(-)^{\frac{1}{2}n}\frac{(-2)^{n-1}}{(n-1)\,!}$.

The general formulæ can be proved by integration by parts by means of the cosine-integral exactly as in the example above; for, n being odd,

$$\begin{aligned}\int_0^\infty \frac{\Sigma A\cos bx}{x^n}\,dx &= \text{terms that vanish between the limits}\\ &\qquad \pm\frac{1}{(n-1)\,!}\int_0^\infty \frac{\Sigma A b^{n-1}\cos bx}{x}\,dx\\ &= \mp\frac{1}{(n-1)\,!}\,\Sigma A b^{n-1}\,\mathrm{Ci}\,(b\epsilon)\\ &= \mp\frac{1}{(n-1)\,!}\,\Sigma A b^{n-1}\left\{\gamma+\log b\epsilon-\frac{1}{2}\cdot\frac{(b\epsilon)^2}{2\,!}+\ldots\ldots\right\}\\ &= \mp\frac{1}{(n-1)\,!}\,\Sigma A b^{n-1}\left\{\gamma+\log b+\log\epsilon-\frac{1}{2}\cdot\frac{(b\epsilon)^2}{2\,!}+\ldots\ldots\right\}\\ &= \mp\frac{\Sigma A b^{n-1}\log b}{(n-1)\,!},\end{aligned}$$

since $\Sigma A b^{n-1}=0$ absolutely, and we take $\epsilon=0$ ultimately.

There is one special case that deserves notice, viz., when $n=1$ and $f(x)$ is an even function; for should $f(x)$, when expanded as a series of cosines, involve a constant term, the value of the integral is infinite, as is seen by making $b=0$ in (18). The same thing appears by integration by parts; thus, for example,

$$\begin{aligned}\int_0^\infty \frac{\sin^2 x}{x}\,dx &= \frac{1}{2}\cdot\int_0^\infty \frac{1-\cos 2x}{x}\,dx\\ &= \frac{1}{2}\left\{\log\infty-\log\epsilon+\gamma+\log 2\epsilon-\frac{1}{2}\,\frac{(2\epsilon)^2}{2\,!}+\ldots\right\}\\ &= \infty.\end{aligned}$$

In the same way we see that $\int_0^\infty \frac{\sin^m x}{x}\,dx$ is infinite whenever m is an even integer, though its value is finite for m odd. The reason for this is clear, for the whole area of the curve $y=\frac{\sin^m x}{x}$ (m even) between $x=0$ and ∞ is positive, while there are positive and negative portions, when n is odd, which help to cancel one another. The infinite term occurs

only when $n=1$, for $b^{n-1}\log b$ vanishes with b in every other case. If we evaluate the integral by integration by parts, when $n>1$ we have to perform at least one differentiation upon the series of cosines before the integral is reduced to depend on cosine-integrals, and the constant term is thus removed.

The investigations contained in this communication were suggested by, and are in part an extension of, a paper by Leslie Ellis, in vol. iii. of the Cambridge Mathematical Journal.* The formulæ (17) and (20) are there obtained and applied to several very interesting questions. Ellis did not, however, discuss the cases when $f(x)$ and x^n were not both even or both odd; nor did he consider (except in one special instance) what the formulæ became when n was fractional, or when the factor e^{-ax} was introduced under the integral sign. A desire to extend the investigation so as to include these cases led me to examine Ellis's paper carefully, and I found that, although his results were correct, and the principle involved was so similar to that used above as to suggest it at once, the method employed was not rigorous. He has, in fact, considered the integrals of the terms that arise by expanding $f(x)$ in sines or cosines separately, and has thus obtained his results by the method which it is one of the main objects of this communication to justify. Starting with the integral

$$\int_0^\infty e^{-ax}\frac{\sin rx}{x}\,dx=\tan^{-1}\frac{r}{a},$$

and integrating $(n-1)$ series with regard to r, there results

$$\int_0^\infty e^{-ax}\frac{\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}rx}{x^n}\,dx=\pm\frac{r^{n-1}}{(n-1)!}\tan^{-1}\frac{r}{a}+\alpha\mathrm{F}(r\alpha)+\mathrm{C}r^{n-2}+\mathrm{C}_1r^{n-4}+\&c. \quad\ldots\ldots\ldots(23);$$

the sine or cosine being taken on the left-hand side according as r is odd or even, and $\mathrm{F}(r\alpha)$ being a function that does not become infinite when $\alpha=0$. In virtue of the conditions $\Sigma\mathrm{A}a^{n-2}=0$, $\Sigma\mathrm{A}a^{n-4}=0$, ... (which are equivalent to the $\Sigma\mathrm{A}b^{n-1}$, &c., in the notation of this paper) by putting $a_1, a_2, \ldots$ for r in (23), multiplying by $\mathrm{A}_1, \mathrm{A}_2, \ldots$ and adding, there results

$$\int_0^\infty e^{-ax}\frac{f(x)}{x^n}\,dx=\pm\Sigma\frac{\mathrm{A}a^{n-1}}{(n-1)!}\tan^{-1}\frac{a}{a}+a\Sigma\mathrm{A}\mathrm{F}(a\alpha),$$

which, taking $\alpha=0$, gives

$$\int_0\frac{f(x)}{x^n}\,dx=\pm\frac{\mathrm{A}a^{n-1}}{(n-1)!}\frac{\pi}{2}.$$

It thus appears that the course of the investigation requires us to assign a finite value to $\int_0^\infty e^{-ax}\frac{\cos rx}{x^n}\,dx$ and $\int_0^\infty e^{-ax}\frac{\sin rx}{x^n}\,dx$, while in

* Evaluation of Certain Definite Integrals, p. 185.

reality they are infinite, so that the method cannot be regarded as rigorous.

The concluding paragraph in Ellis's paper is,—"The method of this paper leads to some elegant results when applied to the definite integral $\int_0^\infty e^{-ax^2}\,dx$, but it is enough to point out this application, which involves no difficulty whatever." I do not see what application is here referred to, unless integrals of the form $\int_0^\infty \frac{e^{-ax^2}}{x^n}\,dx$ or $\int_0^\infty \frac{\Sigma A e^{-ax^2}}{x^n}\,dx$ are meant; but the former is infinite for $n=1$, and the discussion of the latter is of little value, as $\Sigma A e^{-ax^2}$, subject to the required conditions for the integral to be finite, includes no functions of any interest.

In connexion with Ellis's paper I may refer to one by Bronwin in vol. xxiv., p. 491, of the Philosophical Magazine (1844), in which several integrals are evaluated, to which Ellis's results, and those given in this communication, are at once applicable.

Note on Normals and the Surface of Centres of an Algebraical Surface. By Samuel Roberts, M.A.

[*Read March 13th*, 1873.]

I.

There are several methods by which we can determine, in ordinary cases, the number of normals to a surface which can be drawn from an arbitrary point. We may, for instance, consider a system of concentric spheres, and find the number of these which touch the given surface.

It must be assumed, however, that this has no special relation to the plane at infinity, or the imaginary circle there.

Let the order of the surface, its class, and the degree of the tangent cone be m, n, a. Then, since one sphere of the system can be drawn through any point ($\mu=1$), one sphere touches any plane ($\nu=1$), and one sphere touches any line ($\rho=1$), we have, by a formula of De Jonquières

$$\mu n+\nu m+\rho a = n+m+a.$$

For a surface, order m, without singularities,

$$n = m(m-1)^2, \quad a = m(m-1), \quad m+n+a = m^3-m^2+m.$$

2. If, however, the surface is specially related to the plane at infinity, the above formula fails. We will now endeavour to determine the number sought by finding how many normals can be drawn to the surface from a point in the plane at infinity. In general, the normals to a surface at points in the plane at infinity lie altogether in that plane. We have therefore altogether at infinity $m+a$ normals, and of course there are n finite parallel normals, so that we get, as before,

$$m+n+a$$

Suppose now, as a first speciality, that the surface has in the plane at infinity certain nodal and cuspidal curves, these not being specially related to the imaginary circle there, and that the surface has certain nodes also infinitely distant. The formula is not affected by such an arrangement of the singularities *quà* arrangement. For, having allowed for the effect of these singularities in estimating the class and therefore the number of finite normals, we must not allow for their effect again in estimating the number of normals at infinity. In other words, we must count as multiple normals the lines through singular points of the section at infinity. The formula therefore remains unchanged.

3. Let us now suppose that the surface has p contacts with the plane at infinity. We can still draw from any point $m+n+a$ normals or quasi normals. The quasi normals are p lines directed towards the points of contact with the plane at infinity. Every finite line is analytically normal to the plane at infinity, and therefore all lines directed towards the supposed points of contact are normal to the surface; but we must evidently exclude these as improper normals. Hence the formula now is

$$m+n+a-p.$$

It will be observed that here we shall have p additional double points on the section at infinity. But if we reduce the number of the finite normals by p, it is necessary to reckon as double normals at infinity lines through these nodes.

4. If the section at infinity has a double contact with the imaginary circle, the section is of a circular nature. Any right line meets the circle normally, and therefore any right line through a point of contact with the circle is normal to the section. But these normals have to be rejected. Hence, for $2q$ such separate contacts, the formula is

$$m+n+a-2q.$$

When, however, these points of contact become singular points, as when the surface is one of revolution, we must proceed by finding the number of finite normals similarly directed, namely n, if n is the class of an axial section. The section at infinity, consisting of say m quasi circles for a surface of order $2m$, gives us $2m$ normals at infinity, and the number of normals is $n+2m$. The surface may, however, be of odd order, $2m+1$, we then have at infinity only m quasi circles, but there is a line to which we can draw one normal, so that we get for the number of normals $n+2m+1$.

5. We have next to consider the case in which the surface contains the imaginary circle. If a surface contains the circle in the order r, this

in itself implies $2(m-2r)$ singular points on the section, which may arise partly in one way, as from a multiple curve, and partly in another, as by points of contact at infinity. It becomes necessary to regard the specialities of each case in this respect. The formula is of the form

$$n+\mathrm{P},$$

where P depends on the nature of the singularities of the non-circular part of the section, and n is the number of finite normals.

Thus, in the case of a quartic surface on which the circle is nodal, the number of normals through a point is n, the class of the surface. When a cubic surface passes through the circle, so that the plane at infinity contains a line of the surface, this plane is doubly tangential, and at infinity we can draw one normal to the line. Hence the number of normals through a point is

$$n-2+1=n-1.$$

For Dupin's parabolic cyclide this number is 3, and for his general cyclide the number is 4.

II.

6. The order and class of the surface of centres of a surface of the order m without singular points and curves have been determined by M. Darboux (Comptes Rendus, 1870), and M. Löthar Mareks (Math. Annal., Clebsch and Neumann, B. v. H. 1). The order in this case is $2m(m-1)(2m-1)$, and the class $2(m^3-m^2-m)$.

I have sought to determine the order and class in the more general case where the surface possesses ordinary singularities, as nodal and cuspidal curves, nodes, &c. The assumption is made, however, that the primitive surface is not specially related to the plane at infinity, *i.e.*, that it does not touch that plane or pass through the imaginary circle, &c. It is evident that such singularities materially affect the numbers sought, as in the analogous cases of plane space.

In the first place, consider the series of normals to the surface parallel to a fixed plane. These normals will be those which can be drawn to the surface along the curve of contact of an enveloping cylinder whose axis is perpendicular to the fixed plane. The projections, therefore, of the normals in question in that plane form the series of normals to the section of the cylinder by the plane. We may say then that, to an eye placed at a very great distance, the axis of the cylinder being the axis of vision, the normals apparently envelope the evolute of the plane section.

The apparently successive normals, drawn as mentioned, do not in general intersect. The apparent envelope is due to apparent intersections.

But it is important to remark that the points at infinity of the evo-

lute correspond to actual intersections of normals, and moreover of successive normals, and are therefore on the surface of centres. To show this, consider first of all the normals at finite points of the curve of contact, whose projections intersect successively at infinitely distant points. Those normals must be coincident, or rather approximate, along their whole length, for normals with an apparent interval cannot be regarded as successive; and the projections are normal to the plane section of the cylinder at points of inflection which, as we know, correspond to parabolic points on the curve of contact. Hence the points at infinity now in question are the infinitely distant extremities of binormals of the curve of contact, which, as their name indicates, are normal to two successive elements of the curve, being perpendicular to the osculating planes. The points are therefore due to parabolic points, one of the principal radii of curvature being infinite at these points. They are simple points on the line at infinity determined as the intersection with the plane at infinity of the fixed plane. Their number is equal to the number of inflections on the plane section of the cylinder.

7. Besides the points thus determined on the line at infinity determined by the fixed plane, we have to consider the cusps of the evolute, which count each three times on that line which is their common tangent. These cusps are due to the points at infinity of the plane section of the cylinder, and these correspond to the points at infinity of the curve of contact.

The normals at these points are, by the doctrines of plane geometry, cuspidal tangents of the evolute, and their correspondents *must* in like manner be successive twice over with regard to the surface.

Let us consider the generators of the cylinder lying at infinity. These will be tangents to the corresponding section of the surface at the points where the curve of contact meets the section, and pass through the infinitely distant points of the plane section of the cylinder. The cusps spoken of are harmonic conjugates to those points with regard to the pair of circular points lying on the same line at infinity. In other words, the cusps are on the polar reciprocal of the section of the surface at infinity taken with regard to the imaginary circle.

We may infer, then, by shifting our fixed plane, that the locus of these cusps is the reciprocal of the section at infinity of the surface with respect to the imaginary circle, and that it counts three times on the corresponding section of the surface of centres, and is, in fact, a cuspidal curve thereof, the plane at infinity being tangential all along it.

8. Further, however, the points on the line at infinity with which we are dealing, which correspond to the section of the surface at infinity, must be reckoned. The normals to the surface along this section lie

altogether at infinity, and envelope a quasi-evolute. The number of the points in question is therefore the order of this evolute.

9. Writing then—

Order of plane section of the primitive... ...	m
Class	a
Number of its cusps	c
„ of its inflexions...	i
Class of tangent cone	n
Number of its cuspidal edges	κ
„ of its stationary planes	ι

we have for the order of the surface of centres

$$3n+\kappa(=3a+\iota)+3a+c(=3m+i),$$

a form bearing a remarkable analogy to the corresponding one for plane space.

10. To determine the class of the surface of centres, I make use of a theorem due, I believe, to M. Klein, and employed by the before-named authors.

If a ray system is of the order m' and class n', *i.e.*, m' lines meet in any point and n' lines lie in any plane, the difference of the order and class of the focal surface is

$$\text{order}-\text{class}=2(m'-n').$$

For such a system consisting of normals to a surface, the focal surface is the surface of centres, and

$$m'=m+n+a, \quad n'=a, \quad \text{and therefore } m'-n'=m+n.$$

As regards the formula $n'=a$, observe that the points belonging to the normals which lie in a given plane are the intersections of the given plane by the curve of contact with the surface of the cylinder enveloped by the tangent planes which cut at right angles the given plane, *i.e.*, which pass through the point at infinity on any perpendicular to the given plane; the number of normals is thus equal to the order of the curve of contact, viz., it is $=a$.

We get, then, for the required number

$$\begin{aligned}&3(n+a)+c+\kappa-2(m+n)\\&=n+3a+c+\kappa-2m\\&=6a+c+\iota-2m-2n,\end{aligned}$$

which for a surface of order m without singularities is

$$=2(m^3-m^2-m),$$

agreeing with Darboux.

11. Using accented letters for the corresponding characteristics of

the reciprocal surface taken with regard to a centre of reciprocation not specially related to the surface, we may write

$$c-\kappa' = 3(m-a)$$
$$c'-\kappa = 3(n-a);$$

whence
$$3(n+a)+c+\kappa = 3(m+a)+c'+\kappa,$$
$$n+3a+c+\kappa-2m = m+3a+c'+\kappa'-2n;$$

that is, the order and class of the surface of centres of a surface and of that of its general reciprocal are the same.

What we require, then, to enable us to determine the order and class of the surface of centres of a surface possessing ordinary singularities as a nodal and a cuspidal curve, nodes or singularities which can be estimated in terms of nodes, but not specially related to the plane at infinity, is to know

Order	m
Class	n
Rank	a
Order of cuspidal curve	c
Class of spinode torse	c'

The case of singularities at infinity, and especially in connection with the imaginary circle, requires separate discussion.

April 10th, 1873.

Dr. HIRST, F.R.S., President, in the Chair.

Mr. W. D. Niven, M.A., Fellow of Trinity College, Cambridge, was proposed for election; and Prof. Wolstenholme was admitted into the Society.

Prof. Clifford made a few remarks in correction of a statement he had made at the previous meeting, during the discussion on Mr. Hayward's paper.

Mr. J. W. L. Glaisher then proceeded to read his paper "On the calculation of the value of the Theoretical Unit-Angle to a great number of decimal places."

The following papers were discussed (in the absence of the authors) by Messrs. T. Cotterill, Clifford, Merrifield, and the President:—

"On Systems of Porismatic Equations, Algebraical and Trigonometrical;" "Note on Epicycloids and Hypocycloids;" "Locus of Point of Concourse of Perpendicular Tangents to a Cardioid;" and "Elliptic Motion under Acceleration constant in direction:" Prof. Wolstenholme.

"On the Theory of a System of Electrified Conductors;" "On the Focal Lines of a Refracted Pencil:" Prof. J. Clerk Maxwell.

The following presents were received:—

"Sur les trajectoires des points d'une droite mobile dans l'espace," and "Démonstration géométrique d'une proposition due à M. Bernard," by M. Mannheim: from the Author.

On the Calculation of the Value of the Theoretical Unit-Angle to a Great Number of Decimal-Places. By J. W. L. Glaisher, B.A., Fellow of Trinity College, Cambridge.

[*Read April 10th*, 1873.]

The number of degrees in the theoretical unit of angular measure (viz., the angle the length of whose arc is equal to the radius) is $\frac{180}{\pi}$, so that merely by a division this angle can be determined to about as many decimals of a degree as there are decimals of π known. More than a year ago I had need of the value of the unit-angle to a much greater degree of accuracy than it was given to in any place that I was acquainted with, and to make certain of having an abundant number of decimals, I had the division performed, taking as the divisor the first fifty-four figures of π (this being a good point to break off at, as the fifty-fifth figure is a cipher), and shortly afterwards I had the same work performed by another computer, nearly independently. Having in the course of the year examined a great number of tables, collections of constants, &c., and never having met with the angle in question given to even a moderate number of places, I thought it would be desirable to communicate the result of the calculation to the Society. I have accordingly examined one of the calculations myself, and have also had the division performed a third time, so that the present results are submitted with confidence.

The theoretical unit-angle to 52 decimals of a degree is

57° · 29577	95130	82320	87679	81548
14105	17033	24054	72466	56432
15 . . .				

which, expressed in degrees, minutes, and seconds, is, to 47 decimals of a second,

57° 17′ 44″·80624	70963	55156	47335	73307
78613	19665	97008	79631	55 . . .

The same, expressed in grades or centesimal degrees (400 to the circumference), is, to 51 decimals of a grade

63ᵍ·66197	72367	58134	30755	35053
49005	74181	37838	58296	18257
9 . . .				

Although a division of 50 figures by 50 figures to 50 figures quotient is neither a very elaborate nor a very laborious piece of work, still it is sufficiently troublesome to render the publication of the result valuable, as the difficulty of attainment of a numerical quantity is measured, not by the time required for the performance of the arithmetical work, but by this amount plus the time required to ensure certainty that what has been done is *free from error*.

It is a matter of surprise that the unit-angle is not more generally regarded as one of the mathematical constants, along with π, e, γ, &c., for it is one with which the mathematical computer is much concerned, chiefly in semi-convergent series of the form

$$\cos x \left(\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} \ldots\right) + \sin x \left(\frac{B_1}{x} + \frac{B_2}{x^2} + \frac{B_3}{x^3} + \ldots\right).$$

In tabulating a function from such a series (very frequently the only one available when x is large), the process is to multiply the unit-angle by x, reduce to degrees, minutes, and seconds, and then use the ordinary trigonometrical canon. It thus appears that there is some reason for giving the unit-angle to a fair number of decimal places, as it is likely in practice to be required after multiplication by large quantities; thus for $x = 1{,}000{,}000$ the angle is 57295779°·513082..., and leaving out multiples of 360°, and reducing to minutes and seconds, sin x is equal to the sine of 339° 30′ 47″·096; and the log sine of this angle (the sine being treated as positive) is taken out from the ordinary tables as 9·5440600.

I had written so far, and, as I thought, completed this communication, when I remembered to have seen the value of $\frac{1}{\pi}$ (from which the unit-angle can of course be easily deduced) given (together with other constants) to a great many places in an early volume of Grunert's "Archiv." On examination, I found the paper in question, which is by Dr. G. Paucker, in vol. i., pp. 9—11 (1841), where $\frac{1}{\pi}$ is given to 140 decimal places.

Taking this value of $\frac{1}{\pi}$, we have, for the number of grades,

$63^g \cdot$	66197	72367	58134	30755	35053
	49005	74481	37838	58296	18257
	94990	66937	62355	87190	53690
	61403	60455	21106	50123	43824
	29137	09070	31832	14757	16473
	84458	31460	55 ... ;		

for the number of degrees,

$57^\circ \cdot$	29577	95130	82320	87679	81548
	14105	17033	24054	72466	56432
	15491	60243	86120	28471	48321
	55263	24409	68995	85111	09441
	86223	38163	28648	93281	44826
	46012	48314	49 ... ;		

for the number of degrees, minutes, and seconds,

$57^\circ\ 17'\ 44'' \cdot$	80624	70963	55156	47335	73307
	78613	19665	97008	79631	55757
	69768	77900	33024	97339	57589
	47678	74883	85063	99939	90704
	04173	87831	36158	13213	75256
	44939	32194 ... ;			

agreeing exactly with my own calculations, as far as the latter extend. But I must here point out that the last four (and perhaps the last fourteen) figures of the value of π used by Paucker are erroneous, so that about the last four figures of the three results last written are necessarily erroneous, and perhaps as many as the last fourteen figures. The cause of this uncertainty is that Paucker says that he took Vega's value of π ("Hier folgt diese Zahl auf 140 Stellen, nach dem Vega-schen Werthe von π berechnet"), but gives no further information.

Now Vega gave two values of π, both to 140 places, the one correct to 126 places, and the other to 136; and there is nothing to indicate distinctly which of the two was the value that Paucker used.

The first was published in the 'Nova Acta Petropolitana" for 1790 (t. ix., p. 41), and the last twenty figures (viz., those from the 121st to the 140th both inclusive) are

09384 44767 21386 11733

There are also three more figures added, viz. 138, but these are printed with bars through them, to indicate that they are not to be relied upon; so that the value may be described as extending to 140 places.

Vega's second value was published in his "Thesaurus Logarithmorum Completus," fol. Leipzig, 1794 (p. 633), where the same twenty figures appear as

09384 46095 50582 26136,

the correct figures being

09384 46095 50582 23172.

It is much to be regretted that Vega, in giving his second value, did not call attention to the fact that he had previously published one that was erroneous; and this omission is very surprising, when it is remembered how much Vega did for the detection of errors in the table of logarithms of numbers, and in the cause of accuracy generally.

The formula he used in his first calculation was equivalent to

$$\pi = 4 \tan^{-1}\frac{1}{7} + 8 \tan^{-1}\frac{1}{3},$$

while the second was derived from

$$\pi = 20 \tan^{-1}\frac{1}{7} + 8 \tan^{-1}\frac{3}{79};$$

and in his "Thesaurus" he gives the values of the series depending on $\tan^{-1}\frac{1}{7}$ and $\tan^{-1}\frac{3}{79}$ to more than 140 places, while that depending on $\tan^{-1}\frac{1}{3}$ is only given to 128; and he remarks that a verification to 126 places is thus obtained. It appears therefore that in the first calculation an error must have been made in the 127th place (or thereabouts) of $\tan^{-1}\frac{1}{3}$, and it is a pity he did not allude to his former calculation, and state definitely that such was the case.

It is unfortunate, too, that although Paucker has given the values of $\sqrt{\pi}$ to 140 places, yet the only two quantities, viz. $\frac{1}{4}\pi$ and $\frac{1}{6}\pi$, from which the value of π he used could be easily determined, are only given by him to 51 places. I am inclined to think, for several reasons, that it was the second of Vega's values that Paucker referred to; and if so, only the last four figures of the quantities written above, and deduced from it, are necessarily erroneous.

Paucker remarks that Euler gave the value ("Introd. in Anal. Infin.," Lausannæ, 1743, tom. i., § 198, p. 160) to 36 places, but that the 9 in the 25th place should be a 5; and this is so. In Lambert's "Supplementa Tabularum" (Olisipone, 1798, p. 139) $\frac{1}{\pi}$ is given correctly to 20 places; and on the same page the value of the unit-angle is given

as $57°.17^{i}.44^{ii}.48^{iii}.22^{iv}.29^{v}.21^{vi}$. This is not quite correct in the last figure, as the sixths and sevenths are $22^{vi}.22^{vii}$....

It will have been seen that the present triple calculation of the angle, confirmed by Paucker, leaves not a shadow of a doubt of the accuracy of the values of the unit-angle here given to as many places as the former extends to.

On Systems of Porismatic Equations, Algebraical and Trigonometrical. By Prof. Wolstenholme.

[*Read April 10th*, 1873.]

The system of algebraical equations

$$\left.\begin{aligned}
&\frac{a}{yz}+byz+c+a'(y+z)+b'\left(\frac{1}{y}+\frac{1}{z}\right)+c'\left(\frac{y}{z}+\frac{z}{y}\right)=0\\
&\frac{a}{zx}+bzx+c+a'(z+x)+b'\left(\frac{1}{z}+\frac{1}{x}\right)+c'\left(\frac{z}{x}+\frac{x}{z}\right)=0\\
&\frac{a}{xy}+bxy+c+a'(x+y)+b'\left(\frac{1}{x}+\frac{1}{y}\right)+c'\left(\frac{x}{y}+\frac{y}{x}\right)=0
\end{aligned}\right\}\text{(A)},$$

where x, y, z are unequal quantities, is porismatic; that is, there is either no solution at all; or if a certain relation between the coefficients be satisfied, there is an infinite number of solutions, any one of the equations being then deducible from the other two.

The same proposition is true for a system of any number of such equations, as also of course for the system of trigonometrical equations of which the type is

$$a\cos\beta\cos\gamma+b\sin\beta\sin\gamma+c$$
$$+a'(\sin\beta+\sin\gamma)+b'(\cos\beta+\cos\gamma)+c'\sin(\beta+\gamma).$$

I propose to investigate these propositions directly for systems of three, four, and five algebraical equations, and for systems of three and four trigonometrical equations.

I. Suppose we have the system (A). The second and third equations prove that y, z are the two roots of the quadratic equation in u,

$$u^2(bx^2+a'x+c')+u(a'x^2+cx+b')+(c'x^2+b'x+a)=0,$$

so that we have

$$\frac{1}{bx^2+a'x+c'}=\frac{-(y+z)}{a'x^2+cx+b'}=\frac{yz}{c'x^2+b'x+a}\quad\ldots\ldots\ldots\ldots\ (1);$$

or $bx^2+a'x+c'=\lambda$, $a'x^2+cx+b'=-\lambda(y+z)$, $c'x^2+b'x+a=\lambda yz$.

If we denote the determinant $\begin{vmatrix} a, & c', & b' \\ c', & b, & a' \\ b', & a', & c \end{vmatrix}$ by $-\mathrm{N}$, we have

$$\mathrm{N}x^2 = \lambda\{\mathrm{B}-\mathrm{A}'(y+z)+\mathrm{C}'yz\}, \quad \mathrm{N}x = \lambda\{\mathrm{A}'-\mathrm{C}(y+z)+\mathrm{B}'yz\},$$
$$\mathrm{N} = \lambda\{\mathrm{C}'-\mathrm{B}'(y+z)+\mathrm{A}yz\},$$

A, B, C, A′, B′, C′ denoting as usual a'^2-bc, ... $aa'-b'c'$, ... reciprocal to a, b, c, a', b', c'.

We have then the equation in yz,

$$\{\mathrm{B}'yz-\mathrm{C}(y+z)+\mathrm{A}'\}^2 = \{\mathrm{B}-\mathrm{A}'(y+z)+\mathrm{C}'yz\}\{\mathrm{C}'-\mathrm{B}'(y+z)+\mathrm{A}yz\};$$

or,

$$\frac{\mathrm{A}'^2-\mathrm{BC}'}{yz}+(\mathrm{B}'^2-\mathrm{AC}')yz+(2\mathrm{C}^2-\mathrm{C}'^2-\mathrm{AB})+(\mathrm{B}'\mathrm{C}'+\mathrm{AA}'-2\mathrm{B}'\mathrm{C})(y+z)$$
$$+(\mathrm{C}'\mathrm{A}'+\mathrm{BB}'-2\mathrm{A}'\mathrm{C})\left(\frac{1}{y}+\frac{1}{z}\right)+(\mathrm{C}^2-\mathrm{A}'\mathrm{B}')\left(\frac{y}{z}+\frac{z}{y}\right)=0.$$

Now
$$\frac{\mathrm{A}'^2-\mathrm{BC}}{a} = \frac{\mathrm{B}'^2-\mathrm{CA}}{b} = \frac{\mathrm{C}'^2-\mathrm{AB}}{c}$$
$$= \frac{\mathrm{AA}'-\mathrm{B}'\mathrm{C}'}{a'} = \frac{\mathrm{BB}'-\mathrm{C}'\mathrm{A}'}{b'} = \frac{\mathrm{CC}'-\mathrm{A}'\mathrm{B}'}{c'} = \mathrm{N},$$

so that the above equation may be written

$$\mathrm{N}\left\{\frac{a}{yz}+byz+c+a'(y+z)+b'\left(\frac{1}{y}+\frac{1}{z}\right)+c'\left(\frac{y}{z}+\frac{z}{y}\right)\right\}$$
$$+(\mathrm{C}-\mathrm{C}')\left\{\frac{\mathrm{B}}{yz}+\mathrm{A}yz+2(\mathrm{C}-\mathrm{C}')+2\mathrm{B}'(y+z)\right.$$
$$\left.+2\mathrm{A}'\left(\frac{1}{y}+\frac{1}{z}\right)+\mathrm{C}\left(\frac{y}{z}+\frac{z}{y}\right)\right\}=0.$$

It therefore appears that if $\mathrm{C}=\mathrm{C}'$, any one of the three equations (A) is deducible from the other two. Moreover, substituting from (1) in the first of the equations (A), we obtain, on reduction,

$$(\mathrm{C}-\mathrm{C}')\left(\frac{a}{x^2}+bx^2+c+2a'x+2\frac{b'}{x}+2c'\right)=0,$$

or, if $\mathrm{C}-\mathrm{C}'$ be not 0, we have for x exactly the equation we should have by putting y or $z=x$ in the equations (A). Assuming, then, x, y, z to be unequal, two of the roots of the above quartic in x will be y and z, and rejecting these we shall have only a quadratic for x; and since, by exactly the same process, we should come to the same equation in y or z; *i. e.*, the system cannot be satisfied by unequal values of x, y, z unless $\mathrm{C}=\mathrm{C}'$. If this condition be satisfied, an infinite number of solutions exist, any one of the three equations being deducible from the other two.

II. Let there be a system of four such equations in x_1x_2, x_2x_3, x_3x_4, x_4x_1 respectively. We shall have as before from the first pair,

$$\frac{1}{bx_2^2+a'x_2+c'} = \frac{-(x_1+x_3)}{a'x_2^2+cx_2+b'} = \frac{x_1x_3}{c'x_2^2+b'x_2+a};$$

and also from the last pair

$$\frac{1}{bx_4^2+a'x_4+c'} = \frac{-(x_1+x_3)}{a'x_4^2+cx_4+b'} = \frac{x_1x_3}{c'x_4^2+b'x_4+a};$$

whence $$\frac{bx_2^2+a'x_2+c'}{bx_4^2+a'x_4+c'} = \frac{a'x_2^2+cx_2+b'}{a'x_4^2+cx_4+b'} = \frac{c'x_2^2+b'x_2+a}{c'x_4^2+b'x_4+a}.$$

These give us $(cc'-a'b')\,x_2x_4+(b'c'-aa')(x_4+x_2)+b'^2-ac=0,$

or $$C'x_2x_4-A'(x_2+x_4)+B=0,$$

and similarly, $$B'x_2x_4-C(x_2+x_4)+A'=0,$$

$$Ax_2x_4-B'(x_2+x_4)+C'=0,$$

which lead to the determinant $\begin{vmatrix} A, & B', & C' \\ B', & C, & A' \\ C', & A', & B \end{vmatrix} = 0$, or $N^2=0$.

The condition for the coexistence of four such equations is therefore

$$aa'^2+bb'^2+cc'^2-abc-2a'b'c'=0.$$

The relation between x_2x_4 may by means of this condition be written in the symmetrical form $$\frac{x_2x_4}{A'}-\frac{x_2+x_4}{C'}+\frac{1}{B'}=0,$$

each of the three equations between x_2, x_4 being equivalent to this. The same relation also holds between x_1x_3, so that the solutions give a system in involution, x_1x_3 being conjugate, as also x_2x_4.

III. If we have a system of five such equations between x_1x_2, x_2x_3, x_3x_4, x_4x_5, and x_5x_1, we shall obtain, as in I., the equation

$$\frac{a_1}{x_1x_3}+b_1x_1x_3+c_1+a_1'(x_1+x_3)+b_1'\left(\frac{1}{x_1}+\frac{1}{x_3}\right)+c_1'\left(\frac{x_1}{x_3}+\frac{x_3}{x_1}\right)=0,$$

and similar equations between x_2x_4, x_3x_5, x_4x_1, and x_5x_2, where

$$a_1=aN+BM, \quad b_1=bN+AM, \quad c_1=cN+2(C+C')M,$$
$$a_1'=a'N-2B'M, \quad b_1'=b'N-2A'M, \quad c_1'=c'N+CM,$$

where M denotes $C-C'$.

But if we treat this system in exactly the same way, we shall be led to another system of equations in x_1x_2, x_2x_3, ... as at first, the coefficients a_2, b_2, c_2, a_2', b_2', c_2' being formed from a_1, b_1, c_1, a_1', b_1', c_1' in the same way as these from a, b, c, a', b', c'. This system must coincide with the original system (*equal* values not being admitted as solutions); or

$$\frac{a_2}{a}=\frac{b_2}{b}=\frac{c_2}{c}=\frac{a_2'}{a'}=\frac{b_2'}{b'}=\frac{c_2'}{c'}.$$

Of course these are all satisfied, if one condition only be satisfied; we will take the first $a_2b-ab_2=0$.

Now $a_2=a_1N_1+B_1M_1=a_1N_1+M_1(b_1'^2-a_1c_1)$

$=(aN+BM)N_1+(b_1'^2-a_1c_1)M_1$.

Also $b_1'^2-a_1c_1=(b'N-2A'M)^2-(aN+BM)\{cN+2(C+C')M\}$

$=N^2(b'^2-ac)-MN\{4b'A'+cB+2a(C+C')\}$

$+2M^2\{2A'^2-B(C+C')\}$.

But $b'^2-ac=B$,

$2A'^2-B(C+C')=2(A'^2-BC)+B(C-C')=2aN+BM$.

Hence $ba_2-ab_2\equiv MN_1(Bb-Aa)$

$+M_1[N^2(Bb-Aa)-MN\{4(bb'A'-aa'B')+c(Bb-Aa)\}$

$+2M^3(Bb-Aa)]$.

But $Bb-Aa\equiv bb'^2-aa'^2$, $bb'A'-aa'B'\equiv -c'(bb'^2-aa'^2)$, so that

$ba_2-ab_2=(bb'^2-aa'^2)[MN_1+M_1\{N^2-MN(c-4c')+2M^3\}]$,

and the required condition manifestly is

$MN_1+M_1\{N^2-MN(c-4c')+2M^3\}=0$,

or, calling $c-4c'=L$,

$MN_1+M_1(N^2-LMN+2M^3)=0$.

Again $M_1=C_1-C_1'=c_1'^2-a_1b_1-c_1c_1'+a_1'b_1'$

$\equiv(c'N+CM)^2-(aN+BM)(bN+AM)$

$-\{cN+2(C+C')M\}(c'N+CM)+(a'N-2B'M)(b'N-2A'M)$,

in which the coefficient of N^2 is $C-C'$ or M,

" " MN is

$2Cc'-Aa-Bb-Cc-2(C+C')c'-2A'a'-2B'b'$,

or $-(Aa+Bb+Cc+2A'a'+2B'b'+2C'c')$,

or $-3N$;

and the coefficient of M^2 is

$C^2-AB-2C(C+C')+4A'B'$,

or $C'^2-AB-4(CC'-A'B')-(C-C')^2\equiv cN-4c'N-M^2\equiv LN-M^2$.

Hence $M_1=-2MN^2+LM^2N-M^4$.

Finally $N_1=(aN+BM)(a'N-2B'M)^2+(bN+AM)(b'N-2A'M)^2$

$+\{cN+2(C+C')M\}(c'N+CM)^2$

$-(aN+BM)(bN+AM)[cN+2(C+C')M]$

$-2(a'N-2B'M)(b'N-2A'M)(c'N+CM)$.

In this expression the coefficient of N^3 is N; that of N^2M is

$$\begin{aligned} &A(b'^2-ac)+B(a'^2-bc)+2(C+C')(c'^2-ab)+2C(cc'-a'b') \\ &\qquad -4A'(bb'-c'a')-4B'(aa'-b'c') \\ &\equiv 2AB+2C(C+C')+2CC'-8A'B' \\ &\equiv 2(AB-C'^2)+8(CC'-A'B')+2(C-C')^2 \\ &\equiv -2(c-4c')N+2M^2 \equiv -2LN+2M^2. \end{aligned}$$

The coefficient of NM^2 is

$$\begin{aligned} &a[4B'^2-2A(C+C')]+b[4A'^2-2B(C+C')]+c(C^2-AB) \\ &\quad +a'(-4BB'+4CA')+b'(-4AA'+4CB')+c'[4C(C+C')-8A'B'] \\ &\equiv 2a[2(B'^2-AC)+A(C-C')]+2b[2(A'^2-BC)+B(C-C')] \\ &\quad +c(C'^2-AB+C^2-C'^2)+4a'[A'(C-C')+C'A'-BB') \\ &\quad +4b'[B'(C-C')-(B'C'-AA')]+4c'[C(C-C')+2(CC'-A'B')] \\ &\equiv 2a(2bN+AM)+2b(2aN+BM)+c[cN+(C+C')M] \\ &\quad +4a'(-b'N+A'M)+4b'(-a'N+B'M)+4c'(2c'N+CM) \\ &\equiv N(8ab+c^2-8a'b'+8c'^2) \\ &\quad +M[2Aa+2Bb+(C+C')c+4A'a'+4B'b'+4Cc'] \\ &\equiv N[8ab-8c'^2+8a'b'-8cc'+(c-4c')^2] \\ &\quad +M[2(Aa+Bb+Cc+2A'a'+2B'b'+2C'c')-(c-4c')(C-C')] \\ &\equiv -8MN+L^2N+6MN+LM^2N \\ &\equiv -2MN+L^2N-LM^2. \end{aligned}$$

Lastly, the coefficient of M^3 is

$$\begin{aligned} &4BB'^2+4AA'^2+2(C+C')C^2-2AB(C+C')-8A'B'C \\ &\quad \equiv 4(AA'^2+BB'^2+CC'^2-ABC-2A'B'C') \\ &\qquad +2(C-C')(C^2+2CC'+AB-4A'B'), \end{aligned}$$

and $C^2+2CC'+AB-4A'B' = (C-C')^2+4(CC'-A'B')-C'^2+AB$

$$\equiv M^2+4c'N-cN \equiv M^2-NL,$$

or the coefficient of M^3 is $4N^2-2LMN+2M^3$.

The complete expression of N_1 is then

$$\begin{aligned} &N^4+2MN^2(M^2-LN)-M^2N(2MN+LM^2-L^2N) \\ &\qquad +M^3(4N^2-2LMN+2M^3), \end{aligned}$$

or is $N^4+4M^3N^2+2M^6-LMN(2N^2+3M^3)+L^2M^2N^2$;

and the condition for the subsistence of the system of five equations is

$$\begin{aligned} &M\{N^4+4M^3N^2+2N^6-LMN(2N^2+3M^3)+L^2M^2N^2\} \\ &\qquad = M(2N^2-LMN+M^3)(N^2-LMN+2M^3), \end{aligned}$$

or $$MN^2(N^2+M^3-LMN)=0,$$

of which M, N are irrelevant (though easily interpretable), the true condition being $M^3 - LMN + N^2 = 0$.

The method of proceeding in such equations is now manifest, but the labour clearly increases fast. The geometrical interpretation of the equations will show us that all such conditions will involve only the three functions L, M, N of the coefficients, and we will afterwards determine the connexion of these with the invariants of a system of two conics.

Now consider the system of trigonometrical equations whose type is

$$a\cos\beta\cos\gamma + b\sin\beta\sin\gamma - c + a'\sqrt{-1}(\sin\beta+\sin\gamma) + b'\sqrt{-1}(\cos\beta+\cos\gamma) + c'\sin(\beta+\gamma) = 0,$$

the notation being slightly altered for reasons which will be seen when the geometrical meaning is considered.

First, if we have a system of three in α, β, γ, we see that β, γ are the two values of θ which satisfy the equation

$$a\cos\alpha\cos\theta + b\sin\alpha\sin\theta - c + a'\sqrt{-1}(\sin\alpha+\sin\theta) + b'\sqrt{-1}(\cos\alpha+\cos\theta) + c'\sin(\alpha+\theta) = 0,$$

so that
$$\frac{\cos\frac{\beta+\gamma}{2}}{a\cos\alpha + c'\sin\alpha + b'\sqrt{-1}} = \frac{\sin\frac{\beta+\gamma}{2}}{c'\cos\alpha + b\sin\alpha + a'\sqrt{-1}} = \frac{\cos\frac{\beta-\gamma}{2}}{b'\sqrt{-1}\cos\alpha + a'\sqrt{-1}\sin\alpha - c} \quad \ldots\ldots\ldots(2).$$

Hence
$$\tan\frac{\beta+\gamma}{2} = \frac{c'\cos\alpha + b\sin\alpha + a'\sqrt{-1}}{a\cos\alpha + c'\sin\alpha + b'\sqrt{-1}} \equiv \frac{p}{q},$$

and so
$$\tan\frac{\gamma+\alpha}{2} = \frac{c'\cos\beta + b\sin\beta + a'\sqrt{-1}}{a\cos\beta + c'\sin\beta + b'\sqrt{-1}} \equiv \frac{p'}{q'};$$

whence
$$\tan\frac{\alpha-\beta}{2} = \frac{pq'-p'q}{pp'+qq'},$$

of which the numerator is

$$(c'^2-ab)\sin(\alpha-\beta) - \sqrt{-1}(bb'-a'c')(\sin\alpha-\sin\beta) - \sqrt{-1}(aa'-b'c')(\cos\beta-\cos\alpha),$$

and the denominator

$$(c'^2+a^2)\cos\alpha\cos\beta + (c'^2+b^2)\sin\alpha\sin\beta - (a'^2+b'^2) + \sqrt{-1}(b'c'+a'b)(\sin\alpha+\sin\beta) + \sqrt{-1}(c'a'+ab')(\cos\alpha+\cos\beta) + c'(a+b)\sin(\alpha+\beta),$$

or $(c'^2+a^2)\cos\alpha\cos\beta+(c'^2+b^2)\sin\alpha\sin\beta-(a'^2+b'^2)$
$+\sqrt{-1}\,(b'c'+a'b)(\sin\alpha+\sin\beta)+\sqrt{-1}\,(c'a'+ab')(\cos\alpha+\cos\beta)$
$+c'(a+b)\sin(\alpha+\beta)$
$=(c'^2-ab)\{1+\cos(\alpha-\beta)\}-\sqrt{-1}\,(bb'-a'c')(\cos\alpha+\cos\beta)$
$-\sqrt{-1}\,(aa'-b'c')(\sin\alpha+\sin\beta),$

or $(a+b)\{a\cos\alpha\cos\beta+b\sin\alpha\sin\beta-c+\sqrt{-1}\,a'(\sin\alpha+\sin\beta)$
$+\sqrt{-1}\,b'(\cos\alpha+\cos\beta)+c'\sin(\alpha+\beta)\}$
$=a'^2-bc+b'^2-ca+c'^2-ab.$

Hence the condition required is

$$a'^2+b'^2+c'^2-bc-ca-ab=0.$$

Unless this condition be satisfied, the system cannot be satisfied except by two of the three α, β, γ being equal; while if this condition be satisfied, any one equation is deducible from the other two.

If by means of equations (2) we eliminate α, we obtain an equation between β, γ, which coincides with the given equation when, and only when, $bc+ca+ab=a'^2+b'^2+c'^2$; also, if we substitute from (2) in the given equation between β, γ, the resulting equation is (as in the algebraic case)

$$(bc+ca+ab-a'^2-b'^2-c'^2)$$
$$\times(a\cos^2\alpha+b\sin^2\alpha-c+2a'\sqrt{-1}\cos\alpha+2b'\sqrt{-1}\sin\alpha+c'\sin 2\alpha)=0,$$

and the same reasoning applies.

Next, if there be a system of four such equations between $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, and $\delta\alpha$ respectively, we shall have, as before,

$$\frac{\cos\frac{\alpha+\gamma}{2}}{a\cos\beta+c'\sin\beta+b'\sqrt{-1}}=\frac{\sin\frac{\alpha+\gamma}{2}}{c'\cos\beta+b\sin\beta+a'\sqrt{-1}}$$
$$=\frac{\cos\frac{\beta-\gamma}{2}}{b'\sqrt{-1}\cos\beta+a'\sqrt{-1}\sin\beta-c},$$

and also $$\frac{\cos\frac{\alpha+\gamma}{2}}{a\cos\delta+\ldots}=\frac{\sin\frac{\alpha+\gamma}{2}}{c'\cos\delta+\ldots}=\frac{\cos\frac{\beta-\gamma}{2}}{b'\sqrt{-1}\cos\delta+\ldots},$$

whence $$\frac{a\cos\beta+c'\sin\beta+b'\sqrt{-1}}{a\cos\delta+\ldots}=\frac{c'\cos\beta+b\sin\beta+a'\sqrt{-1}}{c'\cos\delta+\ldots}$$
$$=\frac{b'\sqrt{-1}\cos\beta+a'\sqrt{-1}\sin\beta-c}{b'\sqrt{-1}\cos\delta+\ldots};$$

and taking the first pair, we get, on dividing out the extraneous factor $\sin\frac{\beta-\delta}{2}$,

$$(c'^2-ab)\cos\frac{\beta-\delta}{2}-\sqrt{-1}\,(bb'-c'a')\cos\frac{\beta+\delta}{2}$$
$$-\sqrt{-1}\,(aa'-b'c')\sin\frac{\beta+\delta}{2}=0,$$

or $$\mathrm{C}\cos\frac{\beta-\delta}{2}-\sqrt{-1}\,\mathrm{B}'\cos\frac{\beta+\delta}{2}-\sqrt{-1}\,\mathrm{A}'\sin\frac{\beta+\delta}{2}=0,$$

and similarly, from the others,

$$\mathrm{A}'\cos\frac{\beta-\delta}{2}-\sqrt{-1}\,\mathrm{C}'\cos\frac{\beta+\delta}{2}-\sqrt{-1}\,\mathrm{B}\sin\frac{\beta+\delta}{2}=0,$$

$$\mathrm{B}'\cos\frac{\beta-\delta}{2}-\sqrt{-1}\,\mathrm{A}\cos\frac{\beta+\delta}{2}-\sqrt{-1}\,\mathrm{C}'\sin\frac{\beta+\delta}{2}=0;$$

for which to exist simultaneously, we must have

$$\begin{vmatrix}\mathrm{C}, & \mathrm{B}', & \mathrm{A}'\\ \mathrm{B}', & \mathrm{A}, & \mathrm{C}'\\ \mathrm{A}', & \mathrm{C}', & \mathrm{B}\end{vmatrix}=0, \quad \text{or} \quad (aa'^2+bb'^2+cc'^2-abc-2a'b'c')^2=0.$$

If this condition be satisfied, of the three equations in β, δ each becomes

$$\frac{1}{\mathrm{A}'}\cos\frac{\beta+\delta}{2}+\frac{1}{\mathrm{B}'}\sin\frac{\beta+\delta}{2}+\frac{\sqrt{-1}}{\mathrm{C}'}\cos\frac{\beta-\delta}{2}=0,$$

and the same relation holds between α, γ.

Now consider the conics U, V, whose equations are

$$x^2+y^2+z^2=0, \quad \text{and} \quad lx^2+my^2+nz^2+2l'yz+2m'zx+2n'xy=0;$$

any point on U is $\cos\alpha,\ \sin\alpha,\ \sqrt{-1}$, and the intersection of the tangents at two such points β, γ is $\cos\frac{\beta+\gamma}{2},\ \sin\frac{\beta+\gamma}{2},\ \sqrt{-1}\cos\frac{\beta-\gamma}{2}$, and the condition that this may lie on V is, on reduction,

$$(l-m-n)\cos\beta\cos\gamma+(-l+m-n)\sin\beta\sin\gamma+l+m-n$$
$$+2l'\sqrt{-1}\,(\sin\beta+\sin\gamma)+2m'\sqrt{-1}\,(\cos\beta+\cos\gamma)$$
$$+2n'\sin(\beta+\gamma)=0.$$

This also expresses that the two points are conjugate to each other with respect to a third conic (W) whose equation is

$$(l-m-n)x^2+(-l+m-n)y^2+(-l-m+n)z^2$$
$$+4l'yz+4m'zx+4n'xy=0,$$

or $\Theta\mathrm{U}-2\Delta\mathrm{V}=0$, if $\Delta, \Theta, \Theta', \Delta'$ be the fundamental invariants of the system U, V.

Hence any system of such equations can be geometrically interpreted, with reference to any two of the three covariant conics U, V, W.

Also, putting $2\cos\alpha = x_1 + \frac{1}{x_1}$, &c., the trigonometrical equation between α, β leads to the algebraical equation

$$\frac{l-m+2n'\sqrt{-1}}{x_1x_2} + (l-m-2n'\sqrt{-1})\,x_1x_2 - 2(l+m-n)$$
$$+2(l'+m'\sqrt{-1})(x_1+x_2) + 2(m'\sqrt{-1}-l')\left(\frac{1}{x_1}+\frac{1}{x_2}\right) + n\left(\frac{x_1}{x_2}+\frac{x_2}{x_1}\right) = 0;$$

and if we make this coincide with our algebraic equation,

$$a = l-m+2n'\sqrt{-1}, \quad b = l-m-2n'\sqrt{-1}, \quad c = -2(l+m-n),$$
$$a' = 2(l'+m'\sqrt{-1}), \quad b' = 2(m'\sqrt{-1}-l'), \quad c' = +n;$$

also the invariants of the two conics are

$$\Delta = 1, \quad \Theta = l+m+n, \quad \Theta' = mn+nl+lm-l'^2-m'^2-n'^2,$$
$$\Delta' = lmn+2l'm'n'-ll'^2-mm'^2-nn'^2.$$

Now $$\mathrm{L} = c-4c' = -2(l+m+n) = -2\Theta,$$

$$\mathrm{M} = c'^2-ab-cc'+a'b' = n^2-(l-m)^2-4n'^2+2n(l+m-n)-4(l'^2+m'^2)$$
$$\equiv -(l+m+n)^2+4(mn+nl+lm-l'^2-m'^2-n'^2) = -\Theta^2+4\Delta\Theta',$$

and $$\mathrm{N} = aa'^2+bb'^2+cc'^2-abc-2a'b'c'$$
$$\equiv 4(l-m+2n'\sqrt{-1})(l'+m'\sqrt{-1})^2+4(l-m-2n'\sqrt{-1})(l'-m'\sqrt{-1})^2$$
$$+2n(l+m-n)^2-2(l+m-n)[(l-m)^2+4n'^2]-8n(l'^2+m'^2)$$
$$\equiv 8(l-m)(l'^2-m'^2)-32l'm'n'-8n(l'^2+m'^2)$$
$$+2(l+m-n)[n^2-(l-m)^2-4n'^2]$$
$$\equiv 8[(l-m)(l'^2-m'^2)+(l+m)(l'^2+m'^2)-4l'm'n'+2nn'^2]$$
$$-8(l+m+n)(l'^2+m'^2+n'^2)+2(l+m-n)(l-m+n)(-l+m+n)$$
$$\equiv 16(ll'^2+mm'^2+nn'^2-lmn-2l'm'n')$$
$$+8(l+m+n)(mn+nl+lm-l'^2-m'^2-n'^2)+16lmn$$
$$-8(l+m+n)(mn+nl+lm)+2(-l+m+n)(l-m+n)(l+m-n)$$
$$\equiv -16\Delta'+8\Theta\Theta'-2\Theta^3,$$

or $$= -16\Delta^2\Delta'+8\Theta\Theta'\Delta-2\Theta^3,$$

making it homogeneous.

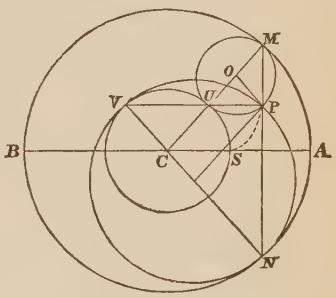
We have thus expressed our three fundamental functions of the coefficients in terms of the invariants of the two conics U, V. The conditions for the coexistence of six, or any other number, of such equations can be expressed in terms of Δ, Θ, Θ', Δ', and therefore in terms of the L, M, N already used.

On Epicycloids and Hypocycloids. *By* Prof. Wolstenholme.

[*Read April 10th*, 1873.]

The following method of considering epicycloids (and of course hypocycloids) seems to show more directly the relation between their twin modes of generation than the ordinary one of considering them primarily as loci. It also directly determines the class number of any such curve; and, from the examples given, it would seem also to lend itself to the discussion of the more important properties at least as easily.

Let the two points M, N describe the same circle, whose centre is C, with velocities in the ratio $m : n$, m and n being whole numbers prime to each other; AB a diameter of the circle such that when M is at A, N is at B; $\angle ACM = m\theta$, $\angle BCN = n\theta$. Then, if M′N′ be the consecutive position of MN, and P the point of ultimate intersection of MN, M′N′,



$$MM' : NN' = MP : PN;$$

but $$MM' : NN' = m : n,$$

therefore $$\frac{MP}{PN} = \frac{m}{n}.$$

Describe a circle touching the prime circle at M and passing through P, take O its centre, and let it meet CM in U; then

$$\frac{MO}{MC} = \frac{MP}{MN} = \frac{m}{m+n}, \quad \text{or} \quad \frac{MU}{MC} = \frac{2m}{m+n};$$

therefore $$\frac{CU}{CM} = \frac{n-m}{n+m}.$$

(The figure has assumed n to be $> m$.) Describe another circle with centre C and radius CU, which is a fixed quantity, and let this circle meet CA in S; then

$$\angle MCN = \pi - n\theta + m\theta,$$

therefore $$\angle CMN = \frac{n-m}{2}\theta,$$

therefore $$\angle UOP = (n-m)\theta, \quad \text{and} \quad \angle UCS = m\theta,$$

therefore $$\angle UOP : \angle UCS = n-m : m = CU : UO;$$

whence the arc PU is equal to the arc SU, and the locus of P is an epicycloid to which MN is a tangent and PU a normal at P. In exactly

the same way, if we describe a circle through P touching the prime circle at N, this circle will be of fixed magnitude, its radius being to CA as $n : m+n$, and it will touch the same fixed circle in V, the point of concourse of PU, NC; the arc SV will be equal to the arc PV, and we get the second mode of generation of the epicycloid.

Moreover, since the angular velocity of U is that of M, and the angular velocity of V is that of N, we see that UV will touch a similar epicycloid, whose dimensions are to that of the original as $n-m : n+m$; or since UV is the normal at P, the evolute of any epicycloid is a similar epicycloid, whose linear dimensions are to that of the original as $n-m : n+m$.

If M and N move in opposite senses, we must put $-m$ for m, P will lie in NM produced, and we shall have the case of the hypocycloid, which does not need any separate discussion.

To find the *class* of the curve, consider any point L of the prime circle, and when M first arrives at L, let N_1, P_1 be the positions of N and P; then when M next arrives at L, N will have described an arc measured by $\frac{n}{m} . 2\pi$, and is at N_2 suppose, P being at P_2, and so on, until M is at L for the mth time, when N will have described an arc $\frac{m-1}{m} . 2n\pi$, and N_m, P_m are the positions of N and P; hence there will be m tangents drawn from L as a position of M, the m points N_1, N_2, ... N_m will be the corners of a regular m-gon inscribed in the prime circle, and therefore P_1, P_2, ... P_m will be the corners of a regular m-gon inscribed in the first moving circle when L is its point of contact.

In exactly the same way, we shall have n tangents drawn from L as an N point, and their points of contact will be the corners of a regular n-gon inscribed in the second moving circle when its point of contact with the prime circle is L.

There can be no other tangents from L, and the class number of the curve is thus $m+n$ both for the epicycloids and hypocycloids.

The prime circle is the circle through the vertices of the epicycloid; and we see that, if at any point of this circle we draw the two circles touching this circle and the concentric circle through the cusps of the epicycloid, these two circles will pass through the points of contact of all the tangents which can be drawn from that point, and these points of contact will be the corners of two regular polygons.

For a cardioid $n : m = 2 : 1$, and for a three-cusped hypocycloid $n : m = 2 : -1$, explaining somewhat the close relationship between these curves, each of which is of the fourth degree and third class, and each a projection of the other.

It follows also from the properties already proved, that if an epicycloid be generated by a circle of radius ma rolling without one of

radius $(n-m)a$, m, n being whole numbers prime to each other, and if in the moving circle we inscribe a regular m-gon one of whose corners is the describing point, all the other corners will move in the same epicycloid, and the whole epicycloid will be completely generated by these different points in one complete revolution about the fixed circle. The same epicycloid may also be generated by the n corners of a regular n-gon inscribed in a circle of radius na rolling on the same fixed circle of radius $(n-m)a$, the contact being in this case internal.

So for hypocycloids, if a circle of radius ma, or of radius na, roll within a fixed circle of radius $(m+n)a$, the whole hypocycloid will be traced out in going once round the fixed circle by the corners of a regular m-gon, or a regular n-gon, inscribed in the moving circle. A singular case is $m=1$, $n=2$, where a circle of radius 2 rolls within one of radius 3; for if AB be a diameter of the moving circle, A, B trace out the same three-cusped hypocycloid to which AB is also a tangent throughout the motion.

It will, I think, be found in all cases quite as easy to prove any properties of special curves from this mode of generation as from the usual one. I append two or three examples.

For a cardioid, $n=2m$; and when M is at L, N is at N_1, such that $\angle BCN_1 = 2\angle ACL$, therefore

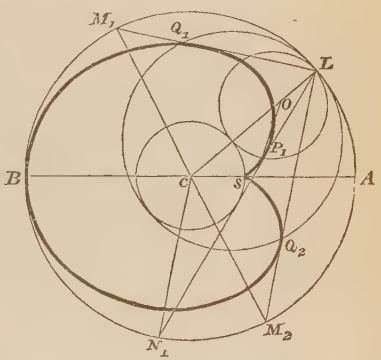
$$\angle ACL = \angle BLN_1;$$

and when N is at L, we shall have M at either M_1 or M_2, where

$$\angle ACM_1 = \tfrac{1}{2}(\pi + ACL),$$

and M_2 is at the other end of the diameter through M_1, since M describes a half circumference while N describes the whole. Hence, if

$LP_1 = \frac{1}{3}LN_1$, and $\frac{LQ_1}{LM_1} = \frac{LQ_2}{LM_2} = \frac{2}{3}$,



P_1, Q_1, Q_2 will be the points of contact of the tangents from L. Since $\angle ACM_1 = \frac{1}{2}(\pi + ACL)$, CM_1 is at right angles to BL, and therefore parallel to AL; and since $\frac{AS}{AC} = \frac{LQ_1}{LM_1}$, SQ_1 is parallel to AL or CM_1, so also SQ_2 is parallel to CM_2; whence Q_1SQ_2 is one straight line, and the tangents drawn at the ends of a chord through the cusp are at right angles to each other, and intersect on a fixed circle.

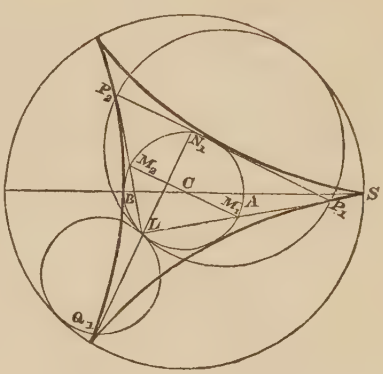
Since $\angle BCN_1 = 2\angle ACL = 2\angle BAN_1$, $\angle ACL = \angle CAN_1$, or AN_1 is parallel to CL; whence SP_1 is parallel to either, since $\frac{CS}{CA} = \frac{LP_1}{LN_1}$.

(This is really only the same theorem as proved before, viz., SQ_1 parallel

to CM_1.) Hence, if the tangent at P_1 meet the axis in T, we have

$$\angle P_1TA = 3\angle ABL = \tfrac{3}{2}\angle TSP_1, \quad \text{or} \quad \angle SP_1T = \tfrac{1}{2}\angle P_1ST.$$

For the three-cusped hypocycloid, $n = -2m$; and M starting from A, and N starting from B, as before, S will be the cusp, where $BS = 2AS$; and if $\angle BCL = 2\angle ACM_1$, then when N first gets to L, M will be at M_1; and when N next gets to L, M will be at M_2, the other end of the diameter M_1CM_2; and LM_1, LM_2 will be two tangents at right angles to each other, the points of contact being P_1, P_2, where $LP_1 = 2LM_1$, $LP_2 = 2LM_2$, therefore $P_1P_2 = 2M_1M_2$, or is a constant quantity. When M arrives at L, N will be at N_1, such that



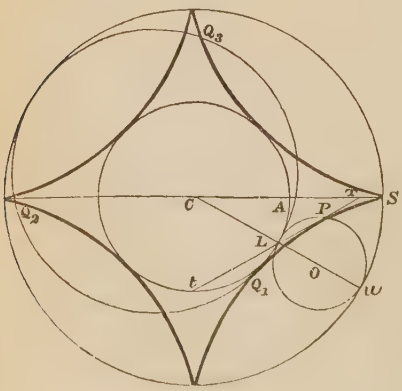
$$\angle BCN_1 = 2\angle BCL = 4\angle ACM_1,$$

therefore

$$\angle M_2CN_1 = 3\angle ACM_1 = \angle M_2CL,$$

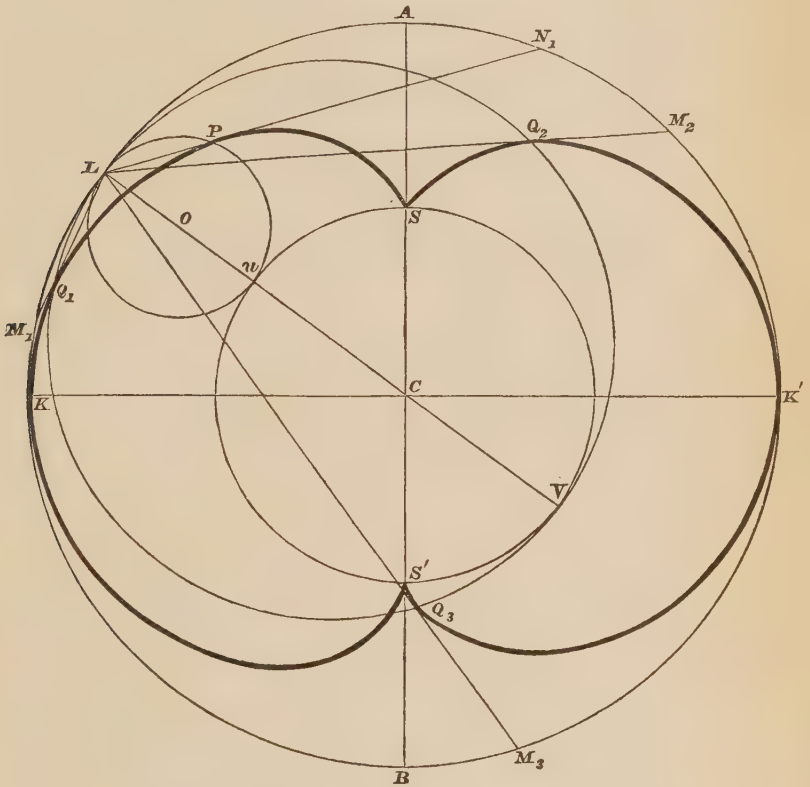
or LN_1 is perpendicular to M_1M_2; hence P_2, N_1, P_1 lie in one straight line at right angles to LN_1, which is therefore itself a tangent to the hypocycloid. Hence, if two tangents to this hypocycloid be at right angles, the straight line joining their points of contact is also a tangent to the hypocycloid. Also, of the three tangents drawn to the hypocycloid from any point on the inscribed circle, two are at right angles and the third is perpendicular to the chord of contact of the other two. These are the best known properties peculiar to this hypocycloid.

Similarly, the properties of the four-cusped hypocycloid, for which $n = 3$, $m = -1$, may be discussed.



The radius of the prime circle being 2, that of the fixed circle (through the cusps) will be 4, and of the two moving circles 1 and 3. If L be a point on the prime circle, LP, LQ_1, LQ_2, LQ_3 the four tangents; Q_1, Q_2, Q_3 are corners of an equilateral triangle. In this case $\angle UOP = 4\angle UCS = 2\angle OLP$, therefore $\angle OLP = 2\angle LCS$; hence CL, LP are equally inclined to CS; and if LP meet CS and the diameter at right angles to it in

T, t, Tt = 2CL = CS, a well known property. This would follow at once from our fundamental idea; for if AB be a diameter, and M, N be originally at A, B, $\angle$ BCN = 3 $\angle$ ACM, therefore each of the angles CMN, CNM = 2MCA, the same result as above.



In a two-cusped hypocycloid, m=1, n=3; S, S′ the two cusps; K, K′ the two vertices; LN_1, LM_1, LM_2, LM_3 the four tangents from a point on the prime circle; P, Q_1, Q_2, Q_3 the points of contact; then Q_1, Q_2, Q_3 are the corners of an equilateral triangle.

If $KCL=\theta$,

$$KCM_1=\frac{\theta}{3},\quad KCM_2=\frac{2\pi+\theta}{3},\quad KCM_3=\frac{4\pi+\theta}{3},\quad KCN_1=3\theta;$$

and if ρ, ρ_1, ρ_2, ρ_3 be the radii of curvature at P, Q_1, Q_2, Q_3,

$$\rho=\tfrac{3}{2}PU,\quad \rho_1=\tfrac{1}{2}VQ_1,\quad \&c.;$$

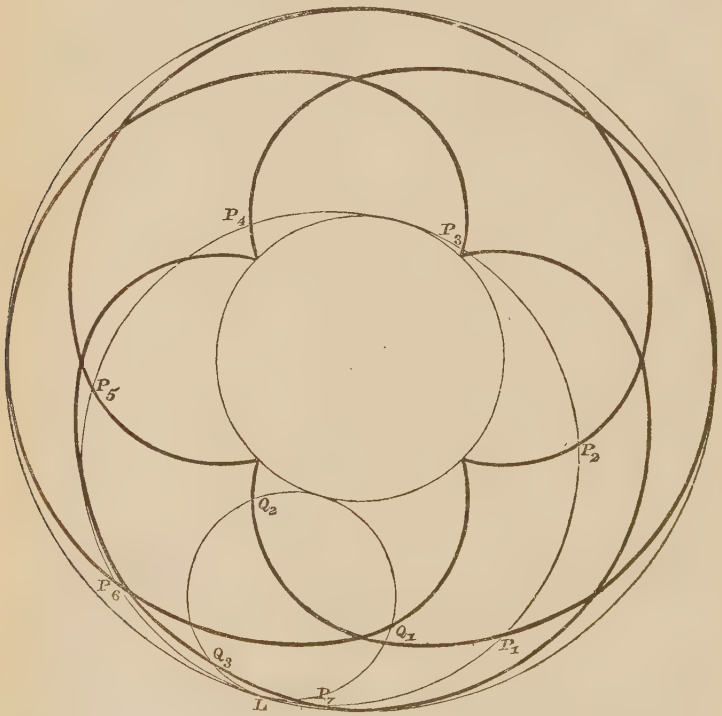
and $\angle CLP=\frac{\pi}{2}-\theta$, therefore $PU=UL\cos\theta=CU\cos\theta$;

$$\angle \mathrm{CLQ_1} = \frac{\pi}{2} - \frac{\theta}{3}, \quad \text{therefore} \quad \mathrm{VQ_1} = \mathrm{VL}\cos\frac{\theta}{3} = 3\,\mathrm{CU}\cos\frac{\theta}{3}$$

therefore $\rho = \frac{3}{2}\,\mathrm{CU}\cos\theta, \quad \rho_1 = \frac{3}{2}\,\mathrm{CU}\cos\frac{\theta}{3}$, &c.,

the four radii of curvature being as $\cos\theta : \cos\frac{\theta}{3} : \cos\frac{2\pi+\theta}{3} : \cos\frac{4\pi+\theta}{3}$.

Hence, of the radii of curvature at Q_1, Q_2, Q_3, the greatest is equal to the sum of the other two, and their product varies as the radius of curvature at P.



If $n=7$, $m=3$, the radii of the two circles by which the epicycloid is described will be 4, 3, or 4, 7; and if L be any point on the prime circle (radius 10), the tangents will be $\mathrm{LP_1}$, $\mathrm{LP_2}$, $\mathrm{LP_3}$ ... $\mathrm{LP_7}$, where $\mathrm{P_1}$, $\mathrm{P_2}$ ... $\mathrm{P_7}$ are corners of a regular heptagon in a circle of radius 7; and $\mathrm{LQ_1}$, $\mathrm{LQ_2}$, $\mathrm{LQ_3}$ where $\mathrm{Q_1}$, $\mathrm{Q_2}$, $\mathrm{Q_3}$ are corners of an equilateral triangle inscribed in a circle of radius 3.

Hence we see that, if a circle of radius 3 roll on a circle of radius 4, the corners of an equilateral triangle inscribed in the moving circle will trace out the same epicycloid, and in one complete revolution round

the fixed curve, the three points will between them have traced out the complete epicycloid.

Since any point on the general epicycloid is determined by equations of the form $x = m\cos n\theta + n\cos m\theta$, $y = m\sin n\theta + n\sin m\theta$, we get the order of the curve at once by considering the number of points lying on any line $px+qy=1$; and if we put $\cos\theta = \frac{1-z^2}{1+z^2}$, $\sin\theta = \frac{2z}{1+z^2}$, the resulting equation in z is of the degree $2n$, $(n>m)$. The curves being obviously unicursal, the deficiency is 0; and to account for this it becomes necessary to ascertain the nature of the circular points.

By taking $x+y\sqrt{-1} \equiv \mathrm{X}$, $x-y\sqrt{-1} = \mathrm{Y}$, $1 = \mathrm{Z}$, I find that at the point (0, 1, 0) the form of the curve is $\mathrm{Z}^{m+n} = \mathrm{Y}^m\mathrm{X}^n$, $(n>m)$; and that for the hypocycloids we get the correct result by putting $-m$ for m, so that the line at ∞ is a double tangent, n-fold at each point. The circular points are therefore multiple points of the nth degree, but their effect in reducing the class is still greater on account of the tangents all coinciding. There are, in general, also impossible ordinary double points, and I have not yet completed the theory of these. The class and order of the hypocycloids were determined by Mr. Cotterill in the "Educational Times," for Nov. 1865.

On the Locus of the Point of Concourse of Perpendicular Tangents to a Cardioid. By Prof. Wolstenholme, M.A.

[*Read April 10th*, 1873.]

If P be a point generating a cardioid by the rolling of a circle on an equal circle, O the centre of the fixed circle, U the point of contact, S the cusp, then PU is the normal at P; and if $\angle \mathrm{SOU} = \theta$ and a be the radius of either circle, the equation of the tangent at P is

$$x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} = 3a\sin\frac{\theta}{2},$$

or, writing $\pi - 2\phi$ for θ, and b for $3a$,

$$-x\cos 3\phi + y\sin 3\phi = b\cos\phi.$$

In order to get a tangent at right angles to this, we must put $\frac{\pi}{6}+\phi$, $\frac{\pi}{2}+\phi$, or $\frac{5\pi}{6}+\phi$ for ϕ, the subsequent values giving no new tangent, the cardioid being of the third class.

The second of these gives the corresponding tangent

$$x\sin 3\phi + y\cos 3\phi = b\sin\phi,$$

and the locus of the intersection of these perpendicular tangents is the circle $x^2+y^2=b^2$.

These are tangents at the ends of a chord through the cusp, and this property is easily proved geometrically. It has, I think, often been stated that this is the locus of the intersection of tangents to the cardioid at right angles to each other, whereas it is only a part of the locus. I proceed to find the locus of the other intersections.

We have the two equations

$$-x\cos 3\phi + y\sin 3\phi = b\cos\phi,$$

$$x\sin 3\phi + y\cos 3\phi = b\cos\left(\phi+\frac{\pi}{6}\right);$$

from which to eliminate ϕ.

These give

$$\begin{aligned} 2x &= b\left\{\sin\left(4\phi+\frac{\pi}{6}\right)+\sin\left(2\phi-\frac{\pi}{6}\right)-\cos 2\phi-\cos 4\phi\right\} \\ &\equiv b\left\{\sin\left(4\phi-\frac{\pi}{6}\right)+\sqrt{3}\sin\left(2\phi-\frac{\pi}{3}\right)\right\} \\ &\equiv b\left\{\cos\left(4\phi-\frac{2\pi}{3}\right)+\sqrt{3}\sin\left(2\phi-\frac{\pi}{3}\right)\right\} \\ &\equiv b\left(\cos 2z+\sqrt{3}\sin z\right); \quad \text{if} \quad z\equiv 2\phi-\frac{\pi}{3}. \end{aligned}$$

But $$\begin{aligned} 2(x^2+y^2) &= b^2\left\{2+\cos 2\varphi+\cos\left(2\phi+\frac{\pi}{3}\right)\right\} \\ &\equiv 2b^2+b^2\sqrt{3}\cos\left(2\phi+\frac{\pi}{6}\right) \equiv 2b^2-b^2\sqrt{3}\sin z, \end{aligned}$$

and $$2x-b = -2b\sin^2 z + b\sqrt{3}\sin z,$$

whence we have immediately the equation of the locus

$$8(x^2+y^2-b^2)^2+6b^2(x^2+y^2-b^2)+3b^3(2x-b)=0.$$

If we had taken $\frac{5\pi}{6}$ instead of $\frac{\pi}{6}$, the only change in the equations would have been changing the sign of $\sqrt{3}$ where it occurred, and the equation obtained must therefore have been the same. The curve represented by this equation is a bicircular quartic; we seek for a node with a view to move the origin to that point. This proves to be the point $2x=b$, $y=0$; and the equation, referred to the new origin, is

$$(4x^2+4y^2+4bx-3b^2)^2+3b^2(4x^2+4y^2+4bx-3b^2)+12b^3x=0,$$

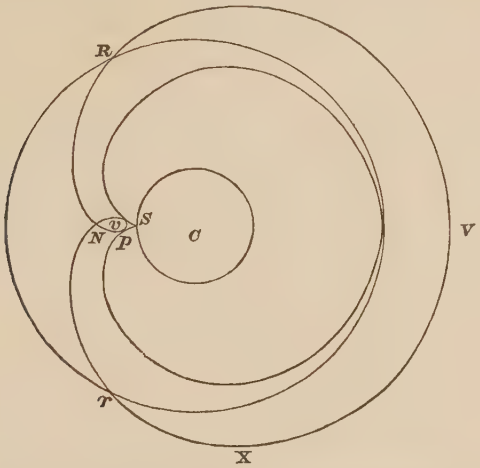
or $$4(x^2+y^2)^2+(x^2+y^2)(8bx-3b^2)+4b^2x^2=0,$$

or, in polar coordinates,

$$4r^2+8br\cos\theta-3b^2+4b^2\cos^2\theta=0,$$

or $$r+b\cos\theta = \pm b\cos\frac{\pi}{6}.$$

The curve represented by the equation is the same whether we take the upper or lower sign, and is the limaçon represented in the figure with the node at N and the vertex at V.



S being the cusp, C the centre of the fixed circle, if we draw the tangents parallel and perpendicular to the axis, geometrical considerations show that the node must be the point N where the double tangent meets the axis. Also the points R, r, where the double tangent meets the tangents parallel to the axis, must belong to both parts of the locus. The points where the tangent at the vertex of the cardioid meets those parallel to the axis belong to the limaçon alone, since the chord of contact does not pass through the cusp, and the vertex V of the limaçon is the intersection of tangents inclined at $\frac{1}{4}\pi$ to the axis. It is also obvious that the limaçon will have double contact with the cardioid at the points where the normal to the cardioid is also a tangent.

The centre of inversion of the limaçon is the point dividing SN in the ratio 5 : 3; the vertex v of the inner loop divides SN in the ratio $\sqrt{3}-1 : 3$, or 11 : 45 very nearly. The points of contact P, p of the limaçon and cardioid are at distances from S and N of $\frac{b}{21}$ and $\frac{b\sqrt{3}}{14}$ respectively; the tangents at P, p divide SC in the ratio 1 : 9; the sines of the angles NSP, SNP are $\frac{3}{14}\sqrt{3}$ and $\frac{1}{7}$ respectively, and the angle NPS is $\frac{5}{6}\pi$.

The limaçon is itself an epitrochoid generated by a circle of radius $\frac{b\sqrt{3}}{4}$ rolling on an equal circle whose centre is C, the generating point being at a distance $\frac{1}{2}b$ from the centre of the moving circle.

Of course, all these properties can be translated into properties of any three-cusped quartic. The equation of the cardioid may be written

$\frac{1}{\sqrt{X}}+\frac{1}{\sqrt{Y}}+\frac{1}{\sqrt{Z}}=0$, if, with our original meaning for x, y, a, we write $X=x+iy+a$, $Y=x-iy+a$, $Z=a$. We may, for instance, take the ordinary three-cusped hypocycloid, whose equation is the same if we now make $X=x-y\sqrt{3}+a$, $Y=x+y\sqrt{3}+a$, $Z=a-2x$. With the equation in this form, if A, B, C be the cusps, we may state the results as follows:

If through C we draw any straight line, the tangents at the two points where it meets the curve again will divide AB harmonically, and the locus of their point of intersection is the conic $(X-Z)(Y-Z)=9Z^2$, which will be found to reduce to $x^2+y^2=9a^2$, as it ought to do.

I make the locus of the point of intersection of other tangents which divide AB harmonically to be

$$8(X-Z)^2(Y-Z)^2-9Z^2(X-Z)(Y-Z)-81XYZ^2=0,$$

but I have not tested this by an independent investigation.

In the same way, from well-known properties of the tricusped hypocycloid, I obtain the theorems:

If a tangent to the quartic $X^{-\frac{1}{2}}+Y^{-\frac{1}{2}}+Z^{-\frac{1}{2}}=0$ meet the curve again in P, Q, the tangents at P, Q divide the points I, J on the absolute (determined by $X+Y+Z=0$, $X^{-1}+Y^{-1}+Z^{-1}=0$) harmonically, and their point of intersection lies on the conic

$$(Y+Z-8X)^{\frac{1}{2}}+(Z+X-8Y)^{\frac{1}{2}}+(X+Y-8Z)^{\frac{1}{2}}=0;$$

also the third tangent from their point of intersection divides IJ harmonically to PQ. These points I, J are the points of contact of the double tangent, and are always unreal when the three cusps are real, but real if two cusps be impossible. The conic has, of course, triple contact with the quartic, and intersects it in the points I, J, which accounts for all their common points.

The figure attempts to give a fairly accurate representation of the complete locus of the intersection of perpendicular tangents to a cardioid.

Note.—I have since discovered that the locus of the point of concourse of tangents to any epicycloid inclined at a constant angle is an epitrochoid for which the radii of the fixed and moving circles are respectively $\frac{a(a+2b)}{a+b}\frac{\sin\frac{a+b}{a+2b}\alpha}{\sin\alpha}$, and $\frac{b(a+2b)}{a+b}\frac{\sin\frac{a+b}{a+2b}\alpha}{\sin\alpha}$; and the distance of the describing point from the centre of the moving circle is $(a+2b)\frac{\sin\frac{b}{a+2b}\alpha}{\sin\alpha}$; a, b being the radii of the two circles for the epicycloid, and α the constant angle which must always be taken as the angle through which one of the tangents would turn in passing to the position of the other always in contact with the curve.

On Elliptic Motion under Acceleration Constant in Direction.

By Prof. Wolstenholme, M.A.

[*Read April 10th*, 1873.]

The following method of exhibiting the motion of a point which describes an ellipse under acceleration in a fixed direction may be of some interest.

Suppose (Fig. 1) A, C two points fixed in a lamina, and moving along two fixed straight lines in the same plane with the lamina; draw AS, CS at right angles to these lines, so that S is the centre of instantaneous rotation. The circle OASC, in which the chord AC, of fixed length, subtends the fixed angle AOC, will be fixed in the lamina, and may be supposed rigidly connected with it; also OS, its diameter, will be of constant length, so that S will describe a circle fixed in space, of twice the radius of the former circle; and since S is instantaneously at rest, the motion of the lamina will be completely represented by supposing the circle AOCS, fixed in the lamina, to roll within a circle of twice the radius fixed in space. It is well known that every point rigidly connected with the circle—that is, every point fixed in the lamina—will describe an ellipse with its centre at O, which ellipse degenerates into a limited straight line for all points on the moving circle. Now draw OB at right angles to OA, meeting the circle in B; then, AB being a diameter of the circle, is of constant length, and the angle BAC, being equal to BOC, is also known, or B is a fixed point in the lamina. We may therefore, without loss of generality, take the two fixed straight lines to be at right angles to each other.

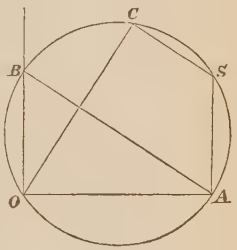


Fig. 1.

Suppose, then (Fig. 2), A, B to be two fixed points of the lamina moving on the two fixed straight lines OA, OB, which are at right angles to each other; C to be any other point fixed in the lamina; and let the circle ABC meet OA, OB again in A′, B′. Join CA′, CB′, draw BM, AN perpendicular to A′C, B′C respectively, and join OM, ON; then, because A, N, B′, O lie on a circle

$$\angle AON = \angle AB'N = \angle AB'C = \angle ABC = \angle AA'C,$$

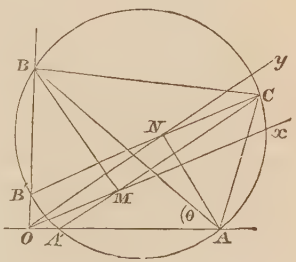


Fig. 2.

or ON is parallel to A′C, and is in a given direction such that, when B is at O, BC lies along ON.

Similarly, OM is parallel to B′C, and in a given direction such that, when A is at O, AC lies along OM.

Take OM, ON as axes of x, y, and refer C to them; $OM = x$, $MC = ON = y$; the triangles BOM, BAC are similar, and $\frac{OM}{OB} = \frac{AC}{AB}$ and $\frac{ON}{OA} = \frac{BC}{AB}$; or if $AB = a$ and $\angle BAO = \theta$, $x = AC \sin\theta$, $y = BC \cos\theta$, and the locus of C is an ellipse having a pair of conjugate diameters along Ox, Oy, the lengths of these diameters being 2AC, 2BC. The area of the ellipse is then $2\pi . \Delta ABC . \cot ACB$, and the sum of the squares on its semi-axes is $AC^2 + BC^2$.

Also, since x bears a constant ratio to OB and y to OA, the accelerations of C parallel to Ox, Oy will be proportional to the accelerations of B, A respectively, and we shall have the case of acceleration in a constant direction, if we make the acceleration of one of the points (A suppose) zero, or if we suppose A to move with uniform velocity u. For the description of the whole ellipse by C, the motion of A must be suddenly reversed when B arrives at O, requiring an infinite acceleration or an impulse; but at all points except the two critical points the acceleration of A will be zero, and therefore the acceleration of C will be parallel to Ox, and proportional to that of B.

To find the law of acceleration if $AB = a$, we have

$$u = a \sin\theta \frac{d\theta}{dt}, \quad \text{therefore} \quad a \frac{d^2\theta}{dt^2} = -\frac{u^2}{a} \frac{\cos\theta}{\sin^3\theta},$$

and the acceleration of B is

$$a \cos\theta \frac{d^2\theta}{dt^2} - a \sin\theta \left(\frac{d\theta}{dt}\right)^2 \equiv -\frac{u^2}{a \sin^3\theta}.$$

The acceleration of C will then be equal to $\frac{u^2 . AC^4}{AB^2 . CN^3}$; also the component velocity of C parallel to Oy will be $u . \frac{BC}{AB}$; or, if we call this V, the whole acceleration of C will be $\frac{V^2 . AC^4}{BC^2 . CN^3}$, and will be in direction CN.

Of course, the motion cannot exactly be exhibited at the critical points, where the velocities of B and C would each be infinite, and the motion of A discontinuous; but for the rest of the motion it would not be difficult to devise mechanism which should give A a uniform velocity and exhibit this kind of motion. If we take C on AB, OA, OB will be the directions of the axes of its locus.

When the angular velocity of the lamina is uniform $= \omega$, then, if D

be the centre of AB, D describes a circle about O under acceleration $\omega^2 . DO$ along DO. Also C describes, relative to D, a circle about D with acceleration $\omega^2 . CD$ along CD. Hence the absolute acceleration of C is $\omega^2 . CO$ along CO, proving that the path of a particle under a central force varying as the distance is an ellipse with its centre at the centre of force.

Of course, AB envelopes a four-cusped hypocycloid; and any straight line fixed in the lamina, which we may suppose parallel to AB, will envelope a parallel to this, and this envelope will therefore have the same evolute, or will be an involute of a certain four-cusped hypocycloid, the radius of whose fixed circle is AB.

That AB envelopes such an hypocycloid, admits of very simple geometrical proof as follows (Fig. 3) :—

From S, the centre of rotation, let fall SP perpendicular to AB; take C the centre of AB, Q of CS, and describe the circle CPS; then the angle PQS = twice the angle PCS = four times the angle AOS; hence, if aS be the arc described by S, since B was at O, arc aS = arc PS, and the locus of P is a four-cusped hypocycloid to which AB is a tangent at P, and generated by a circle of radius $\frac{AB}{4}$ rolling on a fixed circle of radius AB. Its evolute is then a similar hypocycloid, which will be of twice the dimensions (since the central distance of the cusps is twice that of the vertices), and whose fixed circle will therefore be of radius 2AB. This will be the evolute of the envelope of any straight line of the lamina parallel to AB.

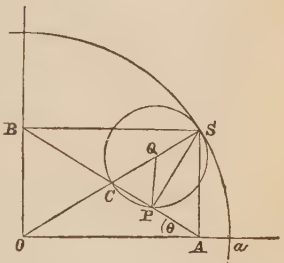


Fig. 3.

The path of any point of the lamina whose coordinates, referred to CA (Fig. 3), and a straight line through C at right angles to it, are (X, Y), will be found from the equations

$$\left.\begin{aligned} x &= (a+X)\cos\theta + Y\sin\theta \\ y &= (a-X)\sin\theta + Y\cos\theta \end{aligned}\right\}, \quad \text{where } AB = 2a.$$

The equation of the path is then

$$[x(a-X)-yY]^2+[xY-y(a+X)]^2 = (a^2-X^2-Y^2)^2,$$

or $x^2[Y^2+(a-X)^2]+y^2[Y^2+(a+X)^2]-4aYxy = (a^2-X^2-Y^2)^2$;

and the equations for its foci are

$$\frac{\xi^2-\eta^2}{-4aX} = \frac{\xi\eta}{-2aY} = \frac{(a^2-X^2-Y^2)^2}{4a^2Y^2-(a^2+X^2+Y^2)^2+4a^2X^2} = -1;$$

that is, $$\xi^2-\eta^2 = 4aX, \quad \xi\eta = 2aY.$$

Hence the locus of the foci of the paths of all points lying in a straight line of the lamina is a rectangular hyperbola, and that of all points lying in a circle about C is a circle about O. If to every point in the lamina be marked off the foci of its path, and the points be uniformly distributed in the lamina, the density of the distribution of the foci will vary as the square of the distance from O, being equal to that of the lamina along the circle which is the locus of S. To any curve in the lamina corresponds a curve in the plane; and since $\frac{dX}{d\xi} = \frac{dY}{d\eta}$, $\frac{dX}{d\eta} = -\frac{dY}{d\xi}$, any angle in one figure will be equal to the corresponding angle in the other. This transformation is however, I believe, well known.

On the Theory of a System of Electrified Conductors, and other Physical Theories involving Homogeneous Quadratic Functions. *By* Prof. J. Clerk Maxwell.

[*Read April 10th*, 1873.]

The theory of homogeneous functions of the second degree is useful in several parts of natural philosophy.

The general form of such a function may be written

$$V = \tfrac{1}{2}(A_{11}x_1^2 + 2A_{12}x_1x_2 + \&c.) \quad \ldots\ldots\ldots\ldots (1),$$

or more concisely

$$V_x = \tfrac{1}{2}\sum_{r=1}^{r=n}\sum_{s=1}^{s=n}(A_{rs}x_rx_s) \quad \ldots\ldots\ldots\ldots (1^*),$$

in which each term consists of a product of two out of n variables $x_1, \ldots x_n$, which may or may not be different, and of a coefficient A_{rs} belonging to that pair of variables.

Differentiating V with respect to each of the n variables in succession, we get n new quantities $\xi_1, \ldots \xi_n$ of the form

$$\frac{dV}{dx_r} = \xi_r = \sum_{s=1}^{s=n}(A_{rs}x_s) \quad \ldots\ldots\ldots\ldots (2).$$

Multiplying each ξ by its corresponding x, and taking half the sum of the products, we obtain a second expression for V,

$$V = \tfrac{1}{2}\sum_{r=1}^{r=n}(x_r\xi_r) \quad \ldots\ldots\ldots\ldots (3).$$

Since each of the n quantities ξ is a linear function of the n variables x, we may, by solving the n equations of the form (2), obtain expressions for each x in terms of the ξ's,

$$x_r = \sum_{r=1}^{r=n}(a_r\xi_r) \quad \ldots\ldots\ldots\ldots (4);$$

and by substituting these values of the x's in (3), we obtain a third expression for V,

$$V_\xi = \tfrac{1}{2} \sum_{s=1}^{s=n} \sum_{r=1}^{r=n} (\alpha_{sr} \xi_s \xi_r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The first and last of these expressions for V are distinguished by the suffixes x and ξ, in order to show that the first is expressed in terms of the variables x and the last in terms of the variables ξ. The coefficients A and α may be functions of any number, m, of variables. Let y_t be one of these variables.

Since V_x, V, and V_ξ are three different expressions for the same quantity,

$$V_x + V_\xi = 2V = \sum_{r=1}^{r=n} (x_r \xi_r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

If we now suppose the three sets of variables x, ξ, and y to vary in any consistent manner, and remember that V_x is a function of the x's and y's, V_ξ of the ξ's and y's, and V of the x's and ξ's, we find

$$\sum_{r=1}^{r=n} \left\{ \left(\frac{dV_x}{dx_r} - \xi_r \right) \delta x_r \right\} + \sum_{s=1}^{s=n} \left\{ \left(\frac{dV_\xi}{d\xi_s} - x_s \right) \delta \xi_s \right\} + \sum_{t=1}^{t=m} \left\{ \left(\frac{dV_x}{dy_t} + \frac{dV_\xi}{dy_t} \right) \delta y_t \right\} = 0 \quad \ldots (7).$$

The three sets of variations δx, $\delta \xi$, and δy are not independent of each other, for if the variations δy and either of the other sets be given, the other set may be determined. Hence we cannot immediately deduce any definite results from this equation. But we know, from equation (2), that the coefficients of the variations δx vanish of themselves, and the remaining variations $\delta \xi$ and δy are all independent of each other, so that we may equate the coefficient of each of them to zero. We thus obtain two sets of equations,

$$x_s = \frac{dV_\xi}{d\xi_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and

$$\frac{dV_x}{dy_t} + \frac{dV_\xi}{dy_t} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

In this purely algebraical theory of quadratic functions, of the two sets of variables x and ξ, either may be taken as the primary set. In the physical applications of the theory, however, the variables form two classes which are not interchangeable.

Thus, in kinetics, the variables are the components of velocity and those of momentum; in the theory of elasticity, they are stresses and strains; in electromagnetism, they are electric currents and the "electrotonic state" of circuits; and in electrostatics, they are the potentials and the charges of conductors.

Now, if a change takes place in the configuration of the system, by which the energy (whether potential or kinetic) of the system is diminished, while the momenta, the strains, the electrotonic states, or the charges remain constant, the displacement will be aided by a force of such a magnitude that the work done by the force during the displacement is equal to the *diminution* of the energy of the system.

For the momenta, the strains, the electrotonic states, and the charges do not require for their maintenance the application of energy from without the system.

The same change of configuration, if effected under the condition that the velocities, stresses, potentials, or electric currents remain constant, would, by equation (9), *increase* the energy of the system by exactly the same amount as it was diminished in the former case. The work done by the system during the displacement will be the same as before, so that energy must in this case be supplied from without to an amount double of this work, in order to satisfy these new conditions.

For it is only by external compulsion that the velocities of the system can be maintained constant when the configuration changes. Work must be done on an elastic system to keep the stress constant while the strain varies. The currents in a conducting circuit tend to vary when the electrotonic state of the circuit varies, and can only be kept constant by battery power. The same agency must be employed to maintain the potential of a conductor constant during the displacement of the system of conductors.

Hence, when momenta, strains, electrotonic states, or charges are maintained constant, the internal forces of the system tend to produce displacements which would *diminish* the energy of the system.

If, on the other hand, the velocities, stresses, electric currents, or potentials are maintained constant, the internal forces tend to produce displacements which would *increase* the energy of the system.

This distinction between the two sets of physical quantities is of great importance in the theory of classification of such quantities. The characteristic of the first set is inherent persistence. Any persistence which we may observe in the second set is only apparent, and arises from the second set being functions of the first set when the configuration of the system either does not vary or is not such as to cause a variation of the coefficients A and a.

On the Focal Lines of a Refracted Pencil. By Prof. J. Clerk Maxwell.

[*Read April 10th*, 1873.]

Hamilton has shown, by means of his Characteristic Function, that in whatever manner a pencil may be refracted, the rays always pass through two focal lines, the planes through which and the axis of the pencil are at right angles to each other. The same method leads to the following geometrical construction for finding the focal lines of any thin pencil after refraction through a curved surface, the focal lines of the incident pencil and the nature of the curvature of the surface being given.

1. The characteristic function of a thin pencil whose axis coincides with the axis of z, and whose rays pass through focal lines in the planes of xz and yz at distances a and b from the origin respectively, is for points near the origin

$$\mathrm{V} = \mathrm{K} + \mu \left(z - \frac{x^2}{2a} - \frac{y^2}{2b} \right) \quad \ldots\ldots\ldots\ldots (1).$$

If we turn this system of rays round the axis of z, through an angle ϕ reckoned from x towards y, the expression for V becomes

$$\mathrm{V} = \mathrm{K} + \mu \left(z - \frac{x^2}{2\mathrm{A}} - \frac{y^2}{2\mathrm{B}} - \frac{xy}{2\mathrm{C}} \right) \quad \ldots\ldots\ldots\ldots (2),$$

where

$$\left.\begin{aligned} \frac{1}{\mathrm{A}} &= \frac{1}{a}\cos^2\phi + \frac{1}{b}\sin^2\phi \\ \frac{1}{\mathrm{B}} &= \frac{1}{a}\sin^2\phi + \frac{1}{b}\cos^2\phi \\ \frac{1}{\mathrm{C}} &= \left(\frac{1}{a} - \frac{1}{b}\right)\sin 2\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3).$$

If we now turn the system round the axis of y, through an angle θ from z towards x, we find

$$\mathrm{V} = \mathrm{K} + \mu \left\{ z\cos\theta + x\sin\theta - \frac{x^2\cos^2\theta}{2\mathrm{A}} - \frac{y^2}{2\mathrm{B}} - \frac{xy\cos\theta}{2\mathrm{C}} - \frac{(2x\cos\theta + z\sin\theta)\, z\sin\theta}{2\mathrm{A}} - \frac{yz\sin\theta}{2\mathrm{C}} \right\} \quad \ldots\ldots (4).$$

2. Now consider two portions of a pencil, close to the origin but in different media, whose indices of refraction are μ_1, μ_2 respectively, and let the coefficients belonging to these portions be distinguished by corresponding suffixes ($_1$ and $_2$).

Let the media be separated by the surface whose equation is

$$z = \frac{x^2}{2A} + \frac{y^2}{2B} + \frac{xy}{2C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Then, since the characteristic function is continuous at this surface, we have the condition $V_1 - V_2 = 0$ when z has the value given in (5).

Substituting this value of z in the expressions for V_1 and V_2, and neglecting terms of the third degree in x and y, we obtain

$$\left(\frac{x^2}{2A} + \frac{y^2}{2B} + \frac{xy}{2C}\right)(\mu_1 \cos\theta_1 - \mu_2 \cos\theta_2) + x(\mu_1 \sin\theta_1 - \mu_2 \sin\theta_2)$$

$$-\mu_1\left(\frac{x^2\cos^2\theta_1}{2A_1} + \frac{y^2}{2B_1} + \frac{xy\cos\theta_1}{2C_1}\right) + \mu_2\left(\frac{x^2\cos^2\theta_2}{2A_2} + \frac{y^2}{2B_2} + \frac{xy\cos\theta_2}{2C_2}\right) = 0$$

$\ldots\ldots\ldots (6).$

That the term in x may vanish, we must have

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

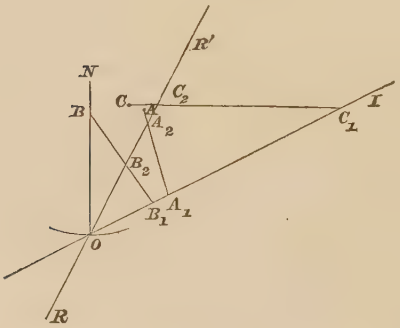
the ordinary law of refraction.

Equating to zero the coefficients of x^2, y^2, and xy, we find

$$\frac{\cos^2\theta_1}{A_1 \sin\theta_1} - \frac{\cos^2\theta_2}{A_2 \sin\theta_2} = \frac{1}{A}(\cot\theta_1 - \cot\theta_2) \quad \ldots\ldots\ldots (8),$$

$$\frac{1}{B_1 \sin\theta_1} - \frac{1}{B_2 \sin\theta_2} = \frac{1}{B}(\cot\theta_1 - \cot\theta_2) \quad \ldots\ldots\ldots (9),$$

$$\frac{\cot\theta_1}{C_1} - \frac{\cot\theta_2}{C_2} = \frac{1}{C}(\cot\theta_1 - \cot\theta_2) \quad \ldots\ldots\ldots (10).$$



3. These relations of the quantities A, B, C may be found by the following construction:—

Let IO be the incident and OR the refracted ray, and let ON be the normal to the surface; $NOI = \theta_1$, $NOR' = \theta_2$.

Find the points A, B, C, whose coordinates are

for A, $\quad x = A\frac{\cos^2\theta_1 - \cos^2\theta_2}{\cot\theta_1 - \cot\theta_2}, \quad z = A\frac{\cos^2\theta_1\cot\theta_1 - \cos^2\theta_2\cot\theta_2}{\cot\theta_1 - \cot\theta_2}$

$\ldots\ldots (11);$

for B, $x = 0$, $z = \mathrm{B}$ .....(12);

for C, $x = \mathrm{C}\dfrac{\cos\theta_1 - \cos\theta_2}{\cot\theta_1 - \cot\theta_2}$, $z = \mathrm{C}\dfrac{\cos\theta_1 \cot\theta_1 - \cos\theta_2 \cot\theta_2}{\cot\theta_1 - \cot\theta_2}$ ..... (13).

The positions of these points depend only on the form of the surface and on the directions of the axes of the incident and refracted pencil. The point B is absolutely fixed, being the centre of curvature of a normal section of the surface perpendicular to the plane of refraction.

Let OA_1, OB_1, OC_1 be the values of A_1, B_1, C_1 for the incident ray. To find the corresponding quantities for the refracted ray, draw the straight lines AA_1, BB_1, CC_1 intersecting the refracted ray in A_2, B_2, C_2. OA_2, OB_2, and OC_2 are the required values of A_2, B_2, C_2.

When any of these quantities becomes infinite, the line must be drawn in a given direction. For A_1, B_1, or C_1 infinite, it must be parallel to the incident ray. For A_2, B_2, or C_2 infinite, it must be parallel to the refracted ray. For B infinite, it must be parallel to the normal; for A infinite, it must make with the normal an angle whose

tangent is $\dfrac{\cos^2\theta_1 - \cos^2\theta_2}{\cos^2\theta_1 \cot\theta - \cos^2\theta_2 \cot\theta_2}$ .....(14);

and for C infinite the tangent of the angle must be

$$\frac{\cos\theta_1 - \cos\theta_2}{\cos\theta \cot\theta_1 - \cos\theta_2 \cot\theta_2} \quad \text{..... (15).}$$

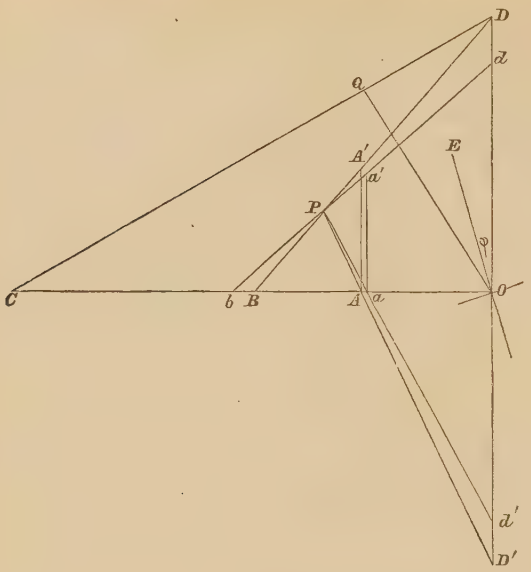
If the plane of refraction cuts the surface along a line of curvature, $\mathrm{C} = \infty$, and if one of the focal lines of the incident pencil is in the plane of refraction, $C_1 = \infty$. The points A_1, B_1 then coincide with the focal lines of the incident pencil, and A_2, B_2 with those of the refracted pencil.

That A_1 may coincide with B_1, and A_2 with B_2, the line joining both pairs of points must be on the line AB. There is therefore one, and only one, point on the axis of the incident pencil from which a pencil may diverge so that, after refraction, it still diverges from or converges to a single point.

4. When the quantity C has a finite value, the plane of refraction is not a plane of symmetry, and we have to deduce the quantities a, b, ϕ from A, B, C. The following construction enables us to pass from either of these systems to the other:—

Let OA, OB, OC be the values of A, B, C.

Draw AA′ perpendicular and equal to AO. Join BA′, and produce to D, a point on the perpendicular to OA through O. Cut off OD′ equal to OD, but in the opposite direction. Join D′A, and produce to P, where D′A meets BD.



Join CD, and draw OQ perpendicular to CD from O.

Make Od, Od' each equal to OQ. Draw dP, $d'P$ cutting OA in b and a. Bisect the angle DOQ by OE.

Then $Oa = a$, $Ob = b$, and $DOE = \phi$, the angle which the first focal line makes with the plane of xz.

If a, b, and ϕ are given, the construction is easily reversed, thus:

Let $Oa = a$, $Ob = b$, and $DOE = \phi$.

Draw aa' perpendicular and equal to aO. Draw ba' cutting OD, the perpendicular to aO from O in d. Cut off Od' equal and opposite to Od. Draw $d'a$ cutting bd in P.

Draw $OQ = Od$ so that the angle $DOE = OEQ$.

Draw CQD perpendicular to OQ, cutting Oa in C and Od in D.

Make OD′ equal and opposite to OD. Draw DP, D′P cutting Oa in B and A.

Then $OA = A$, $OB = B$, and $OC = C$.

6. If therefore the given data be the radii of curvature of the refracting surface and the angle, ϕ, which the plane of incidence makes with the principal section whose curvature is a, we may determine A, B, C for the refracting surface.

Then from a_1 and b_1, the distances of the focal lines of the incident pencil, and ϕ_1, the angle which a_1 makes with the plane of incidence, we must find A_1, B_1, C_1 for the incident pencil.

From these data, by § 4, we must determine A_2, B_2, C_2 for the refracted pencil, and from these a_2, b_2, and ϕ_2.

I have not been able to obtain any simpler construction for the general case of a refracted pencil.

7. For a pencil after passing through any series of surfaces the construction is necessarily more complex, as ten constants are involved in the general term of the second degree of the characteristic function, which is of the form

$$V = \tfrac{1}{2}a_1x_1^2 + \tfrac{1}{2}b_1y_1^2 + c_1x_1y_1 + \tfrac{1}{2}a_2x_2^2 + \tfrac{1}{2}b_2y_2^2 + c_2x_2y_2 + px_1x_2 + qx_1y_2 + ry_1x_2 + sy_1y_2;$$

and if $$V_1 = \tfrac{1}{2}A_1x_1^2 + \tfrac{1}{2}B_1y_1^2 + C_1x_1y_1,$$

and $$V_2 = \tfrac{1}{2}A_2x_2^2 + \tfrac{1}{2}B_2y_2^2 + C_2x_2y_2,$$

and we put $$D = (A_1+a_1)(B_1+b_1) - (C_1+c_1)^2,$$

$$(A_2-a_2)D + p^2(B_1+b_1) + r^2(A_1+a_1) - 2pr(C_1+c_1) = 0,$$

$$(B_2-b_2)D + q^2(B_1+b_1) + s^2(A_1+a_1) - 2qs(C_1+c_1) = 0,$$

$$(C_2-c_2)D + pq(B_1+b_1) + rs(A_1+a_1) - (ps+qr)(C_1+c_1) = 0.$$

These equations enable us to determine A_2, B_2, C_2 when A_1, B_1, C_1 are given.

Here we must observe that the quantities A, B, C, &c., do not represent lines, as in the first part of this paper, but the reciprocals of lines.

May 8th, 1873.

Dr. HIRST, F.R.S., President, in the Chair.

Mr. W. D. Niven, M.A., Fellow of Trinity College, Cambridge, was elected a Member of the Society; and Mr. Robert Rawson, Head Master of Portsmouth Dockyard School, was proposed for election.

The following papers were read:—

"On an Application of the Theory of Unicursal Curves" (extract from letter of M. Hermite to Prof. Cayley).

"Plan of a Curve-tracing Apparatus," and "On a Rational Quintic Correspondence of Two Points in a Plane:" by Prof. Cayley. (A paper on "Bicursal Curves," by Prof. Cayley, was taken as read.)

"Note on the Plückerian Characteristics of Epi- and Hypo-trochoids and allied Curves:" by Mr. S. Roberts.

In connexion with this last paper, Mr. J. W. L. Glaisher suggested that Mr. Perigal's word "Bicircloids" should be adopted, as a convenient term to include both Epi- and Hypo-cycloids; such curves being

capable of generation by the uniform motion of a point round the circumference of a circle, the centre of which itself uniformly describes a circle, whence the two circular movements implied in the name. Similarly Tricircloids, &c., might be defined.

The President, Prof. Clifford, and Mr. Cotterill made remarks upon the papers.

The following presents were received :—

"An easy Rule for Filling-up all Magic Squares :" by S. M. Drach, F.R.A.S. : from the Author.

"Journal of London Institution," No. 20.

"Extrait des procès verbaux des séances" (Société des Sciences Physiques et Naturelles de Bordeaux), 1869.

"Bulletin des Sciences Mathématiques et Astronomiques" (Avril, Mai), 1873.

Memoirs, Notes, &c. by the late Prof. De Morgan, presented by Mrs. De Morgan ; viz. :—

"On a point connected with the dispute between Keil and Leibnitz about the Invention of Fluxions" (Phil. Trans., pt. ii. for 1846).

From the Cambridge Phil. Soc. Trans. :—

"Methods of Integrating Partial Differential Equations" (vol. viii., pt. v.).

"On some points of the Integral Calculus" (vol. ix., pt. ii.).

"On the Singular Points of Curves, and on Newton's Method of Coordinated Exponents" (vol. ix., pt iv.).

"A Proof of the Existence of a Root in every Algebraic Equation, with an Examination and Extension of Cauchy's Theorem on Imaginary Roots, and Remarks on the Proofs of the Existence of Roots given by Argand and by Mourey" (vol. x., pt. i.).

"On the question, What is the Solution of a Differential Equation?" "On the General Principles of which the Composition or Aggregation of Forces is a consequence;" "On the Theory of Errors of Observation;" "On the Syllogism, No. IV., and on the Logic of Relations, with Appendix on Syllogisms of Transposed Quantity ;" "On the Syllogism, No. V., and on the various points of the Onymatic System" (5 parts of vol. x., pt. ii.).

"On the Early History of the Signs + and — ;" "Theorem relating to Neutral Series ;" "On Infinity ;" and "On the Sign of Equality" (3 parts of vol. xi., pt. i.).

"On the Root of any Function ;" and "On Neutral Series," No. ii. (vol. xi., pt. ii.).

From the Cambridge and Dublin Math. Journal :—

"Suggestion on the Integration of Rational Fractions" (Nov. 1848) ; "On the Connexion of Involute and Evolute in Space" (Nov. 1851) ; "On Partial Differential Equations of the First Order" (Feb. 1852).

"On Some Points in the History of Arithmetic" (Companion to the Almanac for 1851).

From the Philosophical Magazine:—

"On the Authorship of the Account of the *Commercium Epistolicum*" (June, 1852); "On the Early History of Infinitesimals in England" (Nov. 1852); "On Indirect Demonstration" (Dec. 1852).

"The English Cyclopædia," Article "Tables," 1861.

On an Application of the Theory of Unicursal Curves. By M. Hermite. (*Extract from a Letter to Prof. Cayley.*)

[*Read May 8th*, 1873.]

Prof. Cayley communicated to the Society a letter, dated 28th March, 1873, which he had received from M. Hermite. In connexion with some investigations on elliptic functions which Prof. Cayley is engaged with, M. Hermite calls attention to the question of determining all the quantities

$$\operatorname{sinam}\frac{4m\mathrm{K}+4m'\imath\mathrm{K}'}{n}$$

in terms of the $n+1$ roots of the modular equation $\mathrm{F}(u, v) = 0$, without, as said Jacobi, the resolution of any equation. Is it necessary, for this purpose, to make use of the singular equations indicated by Abel between the quantities $\operatorname{sinam}\frac{l}{n}(4m\mathrm{K}+4m'i''\mathrm{K}')$ for $l=1, 2 \ldots$ $\ldots \overline{n-1}$ and the nth roots of unity?

And after referring to a remark on the employment of the theory of unicursal curves in his "Cours d'Analyse de l'Ecole Polytechnique," and noticing that it is not only in the commencement of the Integral Calculus that these find an application, M. Hermite proceeds as follows:—I have remarked that they give rise to a method of integration of equations of the form $\mathrm{F}\left(\frac{du}{x}, u\right) = 0$, treated of by MM. Briot and Bouquet, in the Journal de l'Ecole Polytechnique. Suppose, in fact, that the question is to determine the integral when it is an algebraic function of the independent variable. The question is easily resolved in all the cases where the number which determines the nature of this function is $=0$; that is, if it is possible to take rationally $u=\phi(t)$, $x=\psi(t)$. In fact, from this hypothesis, it follows that $\frac{du}{dx}, = \frac{\phi'(t)}{\psi'(t)}$, is also rational in t; wherefore it is necessary (although

not sufficient) that, assuming $\frac{du}{dx} = v$, the curve $F(v, u) = 0$ should be unicursal. Deriving, then, from this relation the rational expressions $v = \Phi(t)$, $u = \phi(t)$, we obtain

$$dx = \frac{du}{v} = \frac{\phi'(t)}{\Phi(t)}\,dt, \text{ and thence } x = \int \frac{\phi'(t)}{\Phi(t)}\,dt.$$

But this integral can always be obtained rationally, and, in the case where the logarithms disappear, gives the value of x in the assumed form.

In the case where u is of the form $u = \phi(\tan x)$, ϕ being rational; making $\tan x = t$, we obtain $\frac{du}{dx} = \phi'(t)(1+t^2)$; therefore the equation $F(v, u)$ must give a unicursal curve; and a solution of this form presents itself when the integral $x = \int \frac{\varphi'(t)}{\Phi(t)}\,dt$ reduces itself to $x = \tan^{-1} t$.

Again, lastly, assuming

$$u = \phi\left(\text{sinam } x,\ \frac{d.\,\text{sinam } x}{dx}\right),$$

ϕ denoting a rational function of the sine-amplitude, and its derived function (this being the hypothesis of MM. Briot and Bouquet); it is clear that, writing sinam $x = t$, the derivative $\frac{du}{dx}$ as well as u must be a rational function of t and of the radical $\sqrt{(1-t^2)(1-k^2t^2)}$. Consequently, the equation $F(v, u) = 0$ denotes a curve of the species (deficiency) 1. Thus the example X of these authors,

$$v^5 + (u^2-1)\,v^4 - au^2(u^2-1)^4 = 0, \qquad \left(v \text{ denoting } \frac{du}{dx}\right),$$

$\left(\text{where } a = \frac{4^4}{5^5}\right)$, on writing $v = (u^2-1)\,t$, gives

$$u^2 = \frac{t^5+t^4}{t^5+t^4-a} = \frac{t^5+t^4}{\left(t+\frac{4}{5}\right)^2\left(t^3-\frac{3}{5}t^2+\frac{8}{5^2}t-\frac{4^2}{5^3}\right)}.$$

If, then, $T = (t+1)\left(t^3-\frac{3}{5}t^2+\frac{8}{5^2}t-\frac{4^2}{5^3}\right)$, we have

$$u = \frac{t^2(t+1)}{t+\frac{4}{5}}\,\frac{1}{\sqrt{T}}, \text{ then } v = \frac{4^4}{5^5}\cdot\frac{t}{\left(t+\frac{4}{5}\right)^2.\,T},$$

whence $dx = \frac{du}{v} = -\frac{5}{2}\frac{dt}{\sqrt{T}}$, whence $x = -\frac{5}{2}\int\frac{dt}{\sqrt{T}}$.

Consequently the equation is integrable by elliptic functions.

The other examples are contained in the type

$$v^3 + 3Pv^2 + 4Q = 0, \quad \text{(with the condition } P^3 + Q = R^2),$$

P, Q, R being integral functions of u of the degrees 2, 6, 3. But this equation may be written

$$(v+2P)^2(v-P) = -4(P^3+Q), = -4R^2,$$

and on writing $v+2P = -\frac{2w}{R}$ becomes simply $w^3 - 3Pw - 2R = 0$.

And this transformed equation being of the degree 3 in w, u, these two quantities, and consequently also v, u, can be expressed as rational functions of t and of an elliptic radical.

The equation $F\left(\frac{d^2u}{dx^2}, u\right)$ gives rise to similar substitutions.

Plan of a Curve-tracing Apparatus. *By* Prof. Cayley.

[*Read May 8th*, 1873.]

I have devised a curve-tracing apparatus on the following plan:—

Imagine two planes Π, Π' moving in the same horizontal plane, and above the two planes respectively the two points P, P′ moving in the same or a parallel plane.

To fix the ideas, suppose that the two planes each move according to a law (that is, let the motion of each of them depend on a single variable parameter; for instance, they may each of them rotate about an axis); but let the motion of the two points be free.

Suppose, next, that the planes are connected together in such manner that the motion of one of them determines the motion of the other (*e. g.*, by a train of wheel-work); and that the two points are also connected together in such manner that the motion of one of them determines the motion of the other (*e. g.*, by a pentagraph; or by a slotted rod, the slot of which works on an axle, so as to allow the rod to move lengthways as well as rotate).

Suppose, finally, that one of the points, say P′, is attached to a point of the plane Π'; then the plane Π being set in motion, this determines the motion of Π', consequently of P′, consequently of P; and the moving point P, or say the pencil P, will describe on the moving plane Π a curve, the nature of which will of course depend on the nature of the motions of Π, Π', and on that of the connexion between these planes and of the connexions between the points P and P′.

I propose to describe the apparatus as nearly as I have actually constructed it. (See sketch-plan Fig. 1, and side-elevation Fig. 2.)

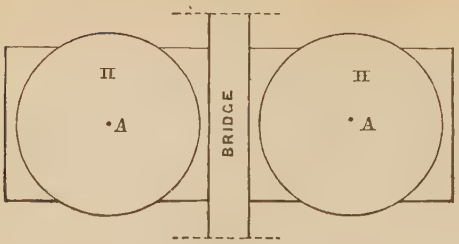


Fig. 1.

The framework of the instrument consists of two longitudinal bars (B) each about three feet long, one inch thick, and three inches broad, supported edgewise at a distance of about eighteen inches on the cross pieces C, C; and half way between them, supported by the same cross pieces, is an axis carrying at each extremity two mitre wheels. The bars B support three cross-pieces D, D, D, and between these are the moveable cross-pieces E, E carrying the axes A, A with the attached mitre wheels, and circular disks X, X. Each of these wheels separately may be placed (as in the figure) out of gear with the two vertical wheels, or it can (by moving the cross-piece E) be brought into gear with either of these wheels. Each axis A passes through a circular disk H, capable of rotating about it, so that it may be fixed in any position, and serving as the bearing for the plane X.

The disks X, X may be regarded as being themselves the planes Π, Π (say these planes are rigidly attached to X, X respectively); or we may, in any other manner, move the plane Π by means of the disk X; for instance, X may carry a spur-wheel gearing into a spur-wheel on the under surface of Π, and thereby communicating a rotation of different velocity to the plane Π; or the connexion may be as in the ordinary oval chuck. In any such case, the disk H (which, for this reason, was made to project beyond X) serves as a support for the plane Π, and the apparatus connected therewith; and observe that the angular position of such apparatus, and therefore of the path of any point of Π, may be varied at pleasure by moving the disk H through any angle.

Fig. 2.

Detached altogether from the rest of the instrument, or what is better, supported on longitudinal pieces carried by the cross pieces C, C

we have a bridge (see fig. 1) capable of adjustment as regards height, and serving as a support for the pentagraph-apparatus, or other connection of the one plane Π with the pencil which works upon the other plane Π.

It is hardly necessary to remark that in the simple form of the instrument where the disks X, X are themselves the planes Π, Π, then putting the mitre wheel of one plane X in gear with either of the corresponding vertical wheels, and making the plane rotate, the other plane X will rotate with the same angular velocity, in the same or the opposite direction, or it will remain at rest, according as its mitre wheel is in gear with one or the other of the corresponding vertical wheels, or is out of gear with each of them.

On Bicursal Curves. *By* Prof. Cayley.

[*Read May 8th*, 1873.]

A curve of deficiency 1 may be termed bicursal: there is some distinction according as the order is even or odd, and to fix the ideas I take it to be even.

A bicursal curve of the order n contains

$$\tfrac{1}{2}n(n+3)-\{\tfrac{1}{2}(n-1)(n-2)-1\}, = 3n \text{ constants};$$

hence, if the order is $= 2n$, the number of constants is $= 6n$; such a curve is normally represented by a system of equations

$$(x, y, z) = (1, \theta)^n + (1, \theta)^{n-2}\sqrt{\Theta},$$

where Θ is a quartic function, which may be taken to be of the form $(1-\theta^2)(1-k^2\theta^2)$, or otherwise to depend on a single constant; viz., (x, y, z) are proportional to n-thic functions involving such a radical: since in the values of (x, y, z) one constant divides out, the number of constants is $3(n+1+\overline{n-1})-1+1, = 6n$, as it should be.

But the curve of the order $2n$ may be abnormally or improperly represented by a system of equations

$$(x, y, z) = (1, \theta)^{n+k} + (1, \theta)^{n+k-2}\sqrt{\Theta},$$

viz., these equations, instead of representing a curve of the order $2n+2k$, will represent a curve of the order $2n$, provided only there exist $2k$ common values of θ for which each of the three functions vanish. The passage to a normal representation is effected by finding θ' a function of θ, $\sqrt{\Theta}$ (viz., an irrational function of θ) such that the foregoing equations become

$$(x, y, z) = (1, \theta')^n + (1, \theta')^{n-2}\sqrt{\Theta'};$$

it is shown that such a transformation is possible, and a mode of effecting it, derived from a theorem of Hermite's in relation to the Jacobian H, Θ functions, is given in Clebsch's Memoir "Ueber diejenigen Curven, deren Coordinaten sich als elliptische Functionen eines Parameters darstellen lassen," Clebsch-Crelle, t. 64 (1865), pp. 210—270. The demonstration is a very interesting one, and I reproduce it at the end of this paper. I remark, in passing, that the analogous reduction in the case of unicursal curves is self-evident; the equations

$$(x, y, z) = (1, \theta)^{n+k}$$

will represent a curve, not of the order $n+k$, but of the order n, provided there exist k common values of θ for which the three functions vanish; in fact, the three functions have then a common factor of the order k, and omitting this, the form is $(x, y, z) = (1, \theta)^n$.

Returning to the curves of deficiency 1, we see that a curve of the order $2m+2n$ contains $6(m+n)$ constants, and is normally represented by a system of equations

$$(x, y, z) = (1, \theta)^{m+n} + (1, \theta)^{m+n-2}\sqrt{\Theta}.$$

Such a curve may be otherwise represented: viz., we may derive it by a rational transformation from the curve (D=1) of the order 4 (binodal quartic), the equation of which is

$$(1, u)^2\,(1, v)^2 = 0;$$

viz., the coordinates are here connected by a quadriquadric equation; and we then express x, y, z in terms of these by a system of equations

$$(x, y, z) = (1, u)^m\,(1, v)^n.$$

It is, however, to be observed that the form of these functions is not determinate: each of them may be altered by adding to it a term $\{(1, u)^{m-2}\,(1, v)^{n-2}\}\,\{(1, u)^2\,(1, v)^2\}$, where the second factor is that belonging to the quadriquadric transformation, and the first factor is arbitrary. Using the arbitrary function to simplify the form, the real number of constants is reduced to $(m+1)\,(n+1) - (m-1)\,(n-1)$, $= 2(m+n)$; or the three functions contain together $6(m+n)$ constants, one of which divides out. The quadriquadric equation, dividing out one constant, contains eight constants; but reducing by linear transformations on u, v respectively, the number of constants is $8-6$, $=2$. Hence, in the system of equations, the whole number of constants is $6(m+n)-1+2$, $= 6(m+n)+1$; viz., this is greater by unity than the number of constants in the curve (D=1) of the order $2(m+n)$. The explanation of the excess is that the same curve of the order $2(m+n)$ may be derived from the different quartic curves $(1, u)^2\,(1, v)^2 = 0$; this will be further examined.

The transition from the one form to the other is not immediately obvious: in fact, if from the quadriquadric equation $(1, u)^2\,(1, v)^2 = 0$

(say this is $A+2Bv+Cv^2=0$, where A, B, C are quadric functions of u) we determine v; this gives $Cv=-B+\sqrt{B^2-AC}$, $=-B+\sqrt{\Omega}$ suppose; and then substituting in the equations $(x, y, z)=(1, u)^m(1, v)^n$, we find $(x, y, z)=(1, u)^m(C, -B+\sqrt{\Omega})^n$;

viz., we have (x, y, z) proportional to functions of u, involving the quartic radical $\sqrt{\Omega}$; but these functions are of the order (not $m+n$, but) $m+2n$.

In particular, if $n=m$, then, instead of functions of the order $2n$, we have functions of the order $3n$. The reduction in this last case to the form where the order is $2n$ can be effected without difficulty, but in the general case where m and n are unequal, I do not know how it is to be effected except by the general process explained in Clebsch's Memoir.

We may, in fact, by linear transformations on u, v, reduce the quadri-quadric relation to

$$\begin{matrix} 1 & & +u^2 \\ & +2buv & \\ +v^2 & & +cu^2v^2 \end{matrix} = 0,$$

that is, $1+u^2+2buv+v^2+cu^2v^2=0$;

or putting herein $u+v=p$, $uv=q$, the relation is

$$1+p^2-2q+2bq+cq^2=0,$$

that is, $p^2=-1+(2-2b)q-cq^2$,

$$p^2-4q=-1+(-2-2b)q-cq^2;$$

viz., extracting the square roots,

$$u+v=\sqrt{Q}, \quad u-v=\sqrt{Q'},$$

if for shortness $Q=-1+(2-2b)q-cq^2$,

$$Q'=-1+(-2-2b)q-cq^2;$$

we may then rationalise one of the radicals, for instance, Q; viz., writing

$$-c\{-1+(2-2b)q-cq^2\}=\{cq-(1-b)\}^2+c-(1-b)^2,$$

then, if $cq-(1-b)=\sqrt{c-(1-b)^2}\cdot\frac{1}{2}\left(\theta-\frac{1}{\theta}\right)$,

this becomes $-cQ=\left\{c-(1-b)^2\right\}\left\{\frac{1}{4}\left(\theta-\frac{1}{\theta}\right)^2+1\right\}$

$$=\left\{c-(1-b)^2\right\}\frac{1}{4}\left(\theta+\frac{1}{\theta}\right)^2,$$

that is, $\sqrt{Q}=\frac{1}{2}\sqrt{\left(\frac{c-(1-b)^2}{-c}\right)}\left(\theta+\frac{1}{\theta}\right)$;

and the corresponding value of $\sqrt{Q'}$ is

$$\sqrt{Q'}=\sqrt{\left\{\frac{1}{4}\frac{c-(1-b)^2}{-c}\left(\theta+\frac{1}{\theta}\right)^2-4q\right\}},$$

where q stands for its value

$$\frac{1}{c}\left\{1-b+\sqrt{c-(1-b)^2}\,\tfrac{1}{2}\left(\theta-\frac{1}{\theta}\right)\right\}.$$

We may write these in the form

$$1 : \tfrac{1}{2}\sqrt{Q} : \tfrac{1}{2}\sqrt{Q'} = M\theta : 1+\theta^2 : \sqrt{\Theta},$$

where M is a constant, and Θ is a quartic function of θ, such that $(1+\theta^2)^2-\Theta$ is a quadric function only of θ.

The equations $(x, y, z) = (1, u)^m (1, v)^n$

thus assume the form

$$(x, y, z) = (M\theta, 1+\theta^2+\sqrt{\Theta})^m (M\theta, 1+\theta^2-\sqrt{\Theta})^n;$$

and on the right-hand side the term of the highest order in θ is

$$(1+\theta^2+\sqrt{\Theta})^m (1+\theta^2-\sqrt{\Theta})^n,$$

viz., if $n =$ or $> m$, then this is

$$\{(1+\theta^2)^2-\Theta\}^m (1+\theta^2-\sqrt{\Theta})^{n-m},$$
$$= (1, \theta)^{2m} (1+\theta^2-\sqrt{\Theta})^{n-m},$$

which is of the order $2m+2(n-m), = 2n$ (which, in virtue of $n =$ or $> m$, is $=$ or $> m+n$). In particular, if $n=m$, then the highest order is $= 2n$; or the curve of the order $2n$, as represented by the equations

$$(x, y, z) = (1, u)^n (1, v)^n,$$

where (u, v) are connected by a quadriquadric equation, is also represented by the equations

$$(x, y, z) = (1, \theta)^n + (1, \theta)^{n-2}\sqrt{\Theta},$$

which is the required transformation of the original equations.

It is to be noticed that the foregoing form,

$$\begin{matrix} 1 & & +u^2 \\ & +2buv & \\ +v^2 & & +cu^2v^2 \end{matrix} = 0,$$

is the most special form to which the quadriquadric relation can be reduced by the linear transformations of u, v; in fact, by mere division, the equation is made to have the constant term 1; the number of the coefficients of transformation is then $3+3, = 6$; and to reduce the relation to the foregoing form, we have the six conditions, coeff. $u = 0$, coeff. $v = 0$, coeff. $vu^2 = 0$, coeff. $v^2u = 0$, coeff. $u^2 = 1$, coeff. $v^2 = 1$; but in this form, expressing v in terms of u, the radical is $\sqrt{\Omega}$, where

$$\Omega = (1+u^2)(1+cu^2)-b^2u^2,$$

which is not more general than if we had

$$\Omega = (1+u^2)(1+cu^2),$$

viz., there is a superfluous constant b. And we thus see how it is that the system of equations

$$(x, y, z) = (1, u)^n (1, v)^n,$$

(u, v) connected by a quadriquadric relation, contains, not $6n$, but $6n+1$ constants, one of these being superfluous.

Clebsch's transformation, above referred to, is as follows:

Starting with the equations

$$(x, y, z) = (1, \theta)^{n+k} + (1, \theta)^{n+k-2} \sqrt{\Theta},$$

if the function Θ is not originally of the standard form, we may, by a linear substitution, reduce it to this form, viz., we may write

$$\Theta = \theta (1-\theta)(1-k^2\theta);$$

and then writing $\theta = \mathrm{sn}^2 u$, $(\sin^2 \mathrm{am}\, u)$, we have

$$\begin{aligned} \sqrt{\Theta} &= \mathrm{sn}\, u\ \mathrm{cn}\, u\ \mathrm{dn}\, u \\ &= \mathrm{sn}\, u\ \mathrm{sn}'u; \end{aligned}$$

so that the formulæ become

$$(x, y, z) = (1, \mathrm{sn}^2 u)^{n+k} + (1, \mathrm{sn}^2 u)^{n+k-2}\ \mathrm{sn}\, u\ \mathrm{sn}'u.$$

Hermite's theorem, used in the demonstration, is that any such function of $\mathrm{sn}\, u$ is expressible in the form

$$C\, \frac{H(u-a_1)\ H(u-a_2) \ldots H(u-a_{2n+2k})}{\Theta^{2n+2k}(u)}$$

(H, Θ denoting here the two Jacobian functions), where

$$a_1 + a_2 \ldots + a_{2n+2k} = 0.$$

[Observe, in passing, that the equation

$$0 = (1, \mathrm{sn}^2 u)^{n+k} + (1, \mathrm{sn}^2 u)^{n+k-2}\ \mathrm{sn}\, u\ \mathrm{sn}'u$$

gives, when rationalised, an equation in $\mathrm{sn}^2 u$ of the order $2n+2k$; the roots of this equation are $\mathrm{sn}^2 a_1$, $\mathrm{sn}^2 a_2 \ldots \mathrm{sn}^2 a_{2n+2k}$. Considering the functions $(1, \mathrm{sn}^2 u)^{n+k}$ and $(1, \mathrm{sn}^2 u)^{n+k-2}$ as indeterminate, the coefficients can be found so that all but one of the roots of the equation in $\mathrm{sn}^2 u$ shall have any given values whatever, $\mathrm{sn}^2 a_1$, $\mathrm{sn}^2 a_2 \ldots \mathrm{sn}^2 a_{2n+2k-1}$; the theorem then shows that the remaining root is $\mathrm{sn}^2 a_{2n+2k}$, where

$$-a_{2n+2k} = a_1 + a_2 \ldots + a_{2n+2k-1},$$

which is, in fact, Abel's theorem.]

Now, supposing that the three functions of θ all vanish for $2k$ common values of θ, each of the functions of u will contain the same $2k$ H functions, say these are $H(u-a_{2n+1}) \ldots H(u-a_{2n+2k})$. Omitting these and also the denominator factor $\Theta^{2k}(u)$, we have the set of equations $(x, y, z) = C\, \dfrac{H(u-a_1)\ H(u-a_2) \ldots H(u-a_{2n})}{\Theta^{2n}(u)}$,

where, however, $\alpha_1+\alpha_2 \ldots +\alpha_{2n} \neq 0$,

[the values $\alpha_1, \alpha_2 \ldots \alpha_{2n}$ are of course different for the three coordinates x, a, z respectively]; viz., we have

$$\alpha_1+\alpha_2 \ldots +\alpha_{2n} = -(\alpha_{2n+1}+ \ldots +\alpha_{2n+2k})$$
$$= 2ns \text{ suppose.}$$

Writing, then,

$$u = u'+s, \quad \alpha_1 = \alpha_1'+s, \quad \alpha_2 = \alpha_2'+s, \ldots \alpha_{2n}' = \alpha_{2n}'+s,$$

and, consequently, $\alpha_1'+\alpha_2'+ \ldots +\alpha_{2n}' = 0$,

we have $$(x, y, z) = C\frac{H(u'-\alpha_1') \ldots H(u'-\alpha_{2n}')}{\Theta^{2n}(u'+s)};$$

or, changing the common denominator,

$$(x, y, z) = C\frac{H(u'-\alpha_1') \ldots H(u'-\alpha_{2n}')}{\Theta^{2n}(u')},$$

where $$\alpha_1'+\alpha_2' \ldots +\alpha_{2n}' = 0;$$

or, what is the same thing,

$$(x, y, z) = (1, \mathrm{sn}^2u')^n+(1, \mathrm{sn}^2u')^{n-2}\mathrm{sn}\,u'\,\mathrm{sn}'.u';$$

viz., writing $\mathrm{sn}\,u' = \theta'$, and $\Theta' = \theta'(1-\theta')(1-k^2\theta')$, this is

$$(x, y, z) = (1, \theta')^n+(1, \theta')^{n-2}\sqrt{\Theta'},$$

a normal representation of the curve of the order $2n$.

The relation between the parameters θ, θ' is given by $\theta' = \mathrm{sn}^2(u-s)$, $\mathrm{sn}^2u = \theta$, that is, we have

$$\sqrt{\theta'} = \frac{\sqrt{\theta}\,\mathrm{cn}s\,\mathrm{dn}s-\mathrm{sn}s\sqrt{(1-\theta)(1-k^2\theta)}}{1-k^2\mathrm{sn}^2s.\,\theta};$$

or, writing $\mathrm{sn}^2s = \sigma$, this is

$$\theta' = \frac{\sqrt{\theta}\sqrt{(1-\sigma)(1-k^2\sigma)}-\sqrt{\sigma}\sqrt{(1-\theta)(1-k^2\theta)}}{1-k^2\sigma\theta};$$

and, conversely, we have

$$\theta = \frac{\sqrt{\theta'}\sqrt{(1-\sigma)(1-k^2\sigma)}+\sqrt{\sigma}\sqrt{(1-\theta')(1-k^2\theta')}}{1-k^2\sigma\theta'}.$$

And the theorem, in fact, shows that, substituting this value of θ in the functions of θ which serve to express x, y, z, these become *proportional* to the functions of θ'; viz., they become equal to these functions, each multiplied by an irrational function $A'+B'\sqrt{\Theta'}$, (A', B' rational functions of θ').

Note on the Plückerian Characteristics of Epi- and Hypo-trochoids and allied Curves. By Samuel Roberts, M.A.

[*Read May 8th*, 1873.]

The following results are, it is true, obvious consequences of familiar theorems; yet, as they do not seem to have been explicitly mentioned,* and are essential to a satisfactory conception of the curves, I think it may be useful to place them on record.

Let a be the radius of the fixed circle, b that of the rolling circle; and suppose for epitrochoids $a = \frac{\mu}{\nu} b$, and for hypotrochoids $a = \frac{-\mu}{\nu} b$, where $\frac{\mu}{\nu}$ is in its lowest terms. If $a+b = mb$, we have for the co-ordinates of a point on the generated curve:

$$x = mb\cos\phi - d\cos m\phi$$
$$y = mb\sin\phi - d\sin m\phi,$$

d being measured from the centre of the rolling circle to the describing point, so that when $d=b$ the curve is a cycloid.

These equations may be reduced to the forms

$$x = \mathrm{A}\cos p\theta + \mathrm{B}\cos q\theta$$
$$y = \mathrm{A}\sin p\theta \pm \mathrm{B}\sin q\theta,$$

where p, q are positive whole numbers prime to one another. In fact, we get the following equations:

$$a = \frac{\mu}{\nu} b, \quad \frac{\mu}{\nu} + 1 = m, \quad p = \mu + \nu, \quad q = \nu, \quad \phi = \nu\theta,$$
$$x = mb\cos q\theta - d\cos p\theta,$$
$$y = mb\sin q\theta - d\sin p\theta.$$

$$a = -\frac{\mu}{\nu} b, \quad -\frac{\mu}{\nu} + 1 = m, \quad q = \nu, \text{ and } p = \pm(\nu - \mu) \text{ is positive,}$$

i. e., sign is upper or lower, as $\nu > \mu$ or $\nu < \mu$.

$$x = mb\cos q\theta - d\cos p\theta,$$
$$y = mb\sin q\theta \mp d\sin p\theta.$$

It is evident, on writing $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ for 2θ, and $\frac{\alpha}{\beta} - \frac{\beta}{\alpha}$ for $2i\theta$, that the curve is unicursal. We get, in fact, putting z for unity and assuming

$$p > q, \qquad \left.\begin{aligned} \rho x &= \frac{1}{2}\left\{ \mathrm{A}\,(\alpha^{2p} + \beta^{2p}) + \mathrm{B}\,(\alpha^{p+q}\beta^{p-q} + \alpha^{p-q}\beta^{p+q}) \right\} \\ \rho y &= \frac{1}{2i}\left\{ \mathrm{A}\,(\alpha^{2p} - \beta^{2p}) \pm \mathrm{B}\,(\alpha^{p+q}\beta^{p-q} - \alpha^{p-q}\beta^{p+q}) \right\} \\ \rho z &= \alpha^p\beta^p \end{aligned}\right\} \ldots\ldots (a).$$

* This was so when the present paper was read. The paragraph (p. 327 *ante*) of Prof. Wolstenholme's paper on Epicycloids and Hypocycloids is to some extent in opposition to the assertion. But I am allowed to state that the paragraph in question was not in the paper as read to the Society.

It appears, then, that the order of the curve is $2p$, and that the points of the curve at infinity lie contiguously at the circular points, which are therefore singular, tangentially or punctually, or in both ways.

For the tangent, we have

$$y\,(\mathrm{A}p\sin p\theta+q\mathrm{B}\sin q\theta)+x\,(\mathrm{A}p\cos p\theta\pm\mathrm{B}q\cos q\theta)$$
$$-\mathrm{A}^2p\mp\mathrm{B}^2q-\mathrm{AB}(p\pm q)\cos(p\mp q)\,\theta=0.$$

The class, therefore, is $2p$ for A, B of the same signs, and $2(p+q)$ for A, B of opposite signs.

Comparing these results with the values obtained by substituting μ, ν, we get the following

Trochoidal Curves.

$+\mu,\ +\nu$		Order $2(\mu+\nu)$	Class $2(\mu+\nu)$
$-\mu,\ +\nu,$	$\mu>2\nu$	,, $2(\mu-\nu)$	,, 2μ
	$\mu>\nu<2\nu$	,, 2ν	,, 2μ
	$\nu>\mu$	,, 2ν	,, 2ν.

Cycloidal Curves.

$+\mu,\ +\nu$		Order $2(\mu+\nu)$	Class $2\nu+\mu$
$-\mu,\ +\nu$	$\mu>2\nu$	,, $2(\mu-\nu)$	,, μ
	$\mu>\nu<2\nu$	,, 2ν	,, μ
	$\nu>\mu$	,, 2ν	,, $2\nu-\mu$.

In the case of cycloids there are μ cusps lying on the fixed circle, and these reduce the class by μ. The singularity at the circular points is the same for both classes of curves, since at infinity a finite length is evanescent, and the describing point may be considered as on the rolling circle. On account of the double mode of generating these curves, one and the same curve may come under different formulæ. The number of trochoidal curves of a given degree is twice the number of primes relative to and less than half the order. Half these are epitrochoids.

A characteristic difference exists between two classes of curves with reference to the circular points and the centre of the fixed circle. Consider the system derived from (a),

$$\rho\,(x+iy)=\mathrm{A}\alpha^{2p}+\mathrm{B}\alpha^{p+q}\beta^{p-q},$$
$$\rho\,(x-iy)=\mathrm{A}\beta^{2p}+\mathrm{B}\alpha^{p-q}\beta^{p+q},$$
$$\rho z=\alpha^p\beta^p.$$

The circular points are here multiple in the order p, and at each of them there is one sole tangent $x\pm iy=0$ meeting the curve in $p+q$ consecutive points. The centre of the fixed circle is the corresponding multiple focus.

The singularity at a circular point counts as $p-1$ cusps and $\frac{(p-1)(p-2)}{2}$ double points; and since the class is $2p$, there are no

other cusps. The above applies to the case $+\mu$, $+\nu$, and to the case $-\mu$, $+\nu$ when $\nu > \mu$. It includes therefore epicycloids and hypocycloids which can be generated as epicycloids.

Again, consider the system for $-B$ in the second equation,

$$\begin{aligned} \rho(x+iy) &= A\alpha^{2p} + B\alpha^{p-q}\beta^{p+q}, \\ \rho(x-iy) &= A\beta^{2p} + B\alpha^{p+q}\beta^{p-q}, \\ \rho z &= \alpha^p\beta^p. \end{aligned}$$

Here the circular points are multiple only in the degree $p-q$; and $z=0$ (line at infinity) is the sole tangent at both points, meeting the curve, however, in p consecutive points. The centre of the fixed circle loses its special focal character.

Each circular point counts as $p-q-1$ cusps and $\frac{(p-q-1)(p-q-2)}{2}$ double points. There are no other cusps, since the class is $2(p+q)$.

I do not dwell on the well-known exceptional cases; as, for instance, when $\frac{\mu}{\nu} = -2$, corresponding to $p=q=1$ in the above equations.

In the same manner, it is plain that curves described by the composition of two simple or complex harmonic motions along the axes of x and y are unicursal, and the line at infinity meets the curve in one or two groups of consecutive points. For we have

$$\rho z = \alpha^m\beta^{m-q} \text{ or } \alpha^m.$$

Thus for two simple harmonic motions we get

$$\begin{aligned} x &= A\sin r\theta + B\cos r\theta, \\ y &= A'\sin s\theta + B'\cos s\theta, \end{aligned}$$

where r and s are prime to one another. Therefore

$$\begin{aligned} \rho x &= \tfrac{1}{2}\left\{\frac{A}{i}(\alpha^{2r}-\beta^{2r}) + B(\alpha^{2r}+\beta^{2r})\right\}, \\ \rho y &= \tfrac{1}{2}\left\{\frac{A'}{i}(\alpha^{2s}-\beta^{2s}) + B'(\alpha^{2s}+\beta^{2s})\right\}\alpha^{r-s}\beta^{r-s}, \\ \rho z &= \alpha^r\beta^r, \end{aligned}$$

r being taken $> s$.

The order of the curve is therefore generally $2r$, and the points at infinity lie at $y=0$, $z=0$. The class is generally $2(r+s)$. There are well-known cases of degeneracy. These degenerate forms are, however, described as curves of higher order; thus, for $r=2$, $s=1$ we have a portion of a parabola described twice. The nature of the singularity at infinity is thus preserved.

It is not worth while to develope the results for curves described by a combination of uniform circular movements, the describing point being a point on the last circle whose centre moves in the next pre-

ceding one, and so on. The general order and class for any particular case can be determined without difficulty. Most of the curves used in ornamental turning are of the foregoing kinds, and meet the line at infinity only at or adjacently to the circular points, except for very special adjustments.

June 12*th*, 1873.

Dr. HIRST, F.R.S., President, in the Chair.

Mr. Robert Rawson, Head-Master of Portsmouth Dockyard School, was elected a Member, and Mr. G. S. Carr was proposed for election.

The following papers were read :—

"Some general Theorems relating to Vibrations," by the Hon. J. W. Strutt, F.R.S.

"Invariant Conditions of Multiple Concurrence of Three Conics," by Mr. J. J. Walker, M.A.

"On a new form of Biquaternion, being the ratio of two systems of forces," by Prof. W. K. Clifford.

A further Note "On Geodesic Lines," by Prof. Cayley, F.R.S.

"The Locus of the Point of Concourse of Tangents to an Epicycloid, inclined to each other at a constant angle," by Prof. Wolstenholme. (This last paper, in the Author's absence, was taken as read.)

Conversation ensued upon the subject of Prof. Clifford's paper, in which the President, Prof. Cayley, and Mr. Roberts took part.

The following presents were received :—

"Bulletin des Sciences Mathématiques et Astronomiques," Juin, 1873.

"Jahrbuch über die Fortschritte der Mathematik, zweiter Band, Jahrgang 1869 & 1870": from Drs. Ohrtmann and Müller.

"Crelle," 76 Band, 1es Heft, 1873.

"Untersuchungen über Elasticität unter Berücksichtigung der Wärme," von C. W. Borchardt. Berlin, 1873.

"Ueber das Ellipsoid von kleinsten Volumen bei gegebenem Flächeninhalt einer Anzahl von Centralschnitten," von C. W. Borchardt. Berlin, 1872.

"Ueber die Transformation der Electricitätsgleichungen in allgemeine orthogonale Coordinaten," von C. W. Borchardt: (from Crelle). The last three from the Author.

"Monatsbericht," Jan. 1873.

"Sur une classe remarquable de courbes et de surfaces algébriques et sur la théorie des imaginaires," par G. Darboux. Paris, 1873. From the Author.

The following twelve memoirs from "Det Kgl. Norske Universitet à Christiania":—

"Om ligningen af 5te grad," af Dr. A. S. Guldberg.

"Om ligningen af 3die grad," by the same.

"Om forringelsen af modularligningernes grad," af L. Sylow.

"Om den algebraiske ligning af nte grad, hvis Rodder representeres, ved formelen $x = R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}}$," af Dr. A. S. Guldberg.

"Om Vœdskers Udvidelse ved konstant Tryk," by the same.

"Bidrag til Legemernes Molekylartheori," af Cato M. Guldberg.

"Sur le mouvement simultané de corps sphériques variables dans un fluide indéfini et incompressible: premier mémoire," par C. A. Bjernkes, Christiania.

"Ny Interpolationsmethode," af J. J. Åstrand.

"Over en classe geometriske Transformationer," af Sophus Lie.

"Ueber eine Classe geometrischer Transformationen," (Fortsetzung): by the same.

"Om den Gruppe af Substitutioner, der tilhrer ligningen for division af Perioderne ved de Elliptiske Funktioner," af L. Sylow.

"Bidrag til Theorien for de ubestemte chemiske Forbindelser," af Cato M. Guldberg.

Some General Theorems relating to Vibrations. By the Hon. J. W. Strutt, M.A. (Lord Rayleigh).

[*Read June 12th*, 1873.]

This paper contains a short account of some general theorems, with which I have lately become acquainted during the preparation of a work on Acoustics. As they seem to possess considerable interest, I take the present opportunity of bringing them before the Society.

Section I.

The natural periods of a conservative system, vibrating freely about a configuration of stable equilibrium, fulfil the stationary condition.

Let the system be referred, in the usual manner, to independent coordinates $\psi_1, \psi_2, \psi_3, \ldots$ whose origin is taken to correspond with the configuration of equilibrium. Then, the square of the motion being neglected, the kinetic and potential energies are expressible in the form

$$T = \tfrac{1}{2}[11]\dot{\psi}_1^2 + \tfrac{1}{2}[22]\dot{\psi}_2^2 + \ldots\ldots + [12]\dot{\psi}_1\dot{\psi}_2 + \ldots\ldots \quad \ldots(1),$$

$$V = \tfrac{1}{2}\{11\}\psi_1^2 + \tfrac{1}{2}\{22\}\psi_2^2 + \ldots\ldots + \{12\}\psi_1\psi_2 + \ldots\ldots \quad \ldots(2),$$

where $[11]\ldots$, $\{11\}\ldots$ are constants, subject to the condition of making T and V always positive. For the present purpose, it is convenient, though not necessary, to transform the coordinates in the manner explained in Thomson and Tait's "Natural Philosophy," § 337, so as to reduce T and V to a sum of squares;

$$T = \tfrac{1}{2}[1]\dot{\phi}_1^2 + \tfrac{1}{2}[2]\dot{\phi}_2^2 + \ldots \quad \ldots\ldots(3),$$

$$V = \tfrac{1}{2}\{1\}\phi_1^2 + \tfrac{1}{2}\{2\}\phi_2^2 + \ldots \quad \ldots\ldots(4),$$

where the coefficients are necessarily positive. The natural vibrations of the system are those represented by the separate variation of the coordinates $\phi_1, \phi_2, \ldots$; and the corresponding differential equations obtained by Lagrange's method are of the form

$$[s]\ddot{\phi}_s + \{s\}\phi_s = 0 \quad \ldots\ldots(5),$$

showing that the period of the natural vibration ϕ_s is given by

$$\tau_s = 2\pi[s]^{\frac{1}{2}} \div \{s\}^{\frac{1}{2}} \quad \ldots\ldots(6).$$

Let us now suppose that the system is no longer allowed to choose its type of vibration, but that an arbitrary type is imposed upon it by a suitable constraint, leaving only one degree of freedom. Thus, let

$$\phi_1 = A_1\theta, \quad \phi_2 = A_2\theta \quad \ldots\ldots(7),$$

where $A_1, A_2, \ldots$ are given real coefficients. The expressions for T and V become

$$T = \left\{\tfrac{1}{2}[1]A_1^2 + \tfrac{1}{2}[2]A_2^2 + \ldots\ldots\right\}\dot{\theta}^2 \quad \ldots\ldots(8),$$

$$V = \left\{\tfrac{1}{2}\{1\}A_1^2 + \tfrac{1}{2}\{2\}A_2^2 + \ldots\ldots\right\}\theta^2 \quad \ldots\ldots(9),$$

whence, if θ varies as $\cos pt$,

$$p^2 = \frac{\{1\}A_1^2 + \{2\}A_2^2 + \ldots\ldots}{[1]A_1^2 + [2]A_2^2 + \ldots\ldots} \quad \ldots\ldots(10).$$

This gives the period of the vibration of constrained type; and it is evident that the period is stationary, when all but one of the coefficients $A_1, A_2, \ldots$ vanish, that is to say, when the type coincides with one of those natural to the system.

By means of this theorem we may prove that an increase in the mass of any part of a vibrating system is attended by a prolongation of all the natural periods, or at any rate that no period can be diminished. Suppose the increment of mass to be infinitesimal. After the alteration, the types of vibration will in general be changed; but, by a suitable constraint, the system may be made to retain any one of the former types. If this be done, it is certain that any vibration which involves a motion of the part whose mass is increased, has its period prolonged. Only as a particular case (as, for example, when a load is placed at the node of a vibrating string) can the period remain unchanged. The theorem now allows us to assert that the removal of

the constraint, and the consequent change of type, can only affect the period by a quantity of the second order; and that therefore, in the limit, the free period cannot be less than before the change. By integration we infer that a finite increase of mass must prolong the period of every vibration which involves a motion of the part affected, and that in no case can any period be diminished; but in order to see the correspondence of the two sets of periods, it may be necessary to suppose the alteration made by steps. The converse of this and corresponding theorems relating to an alteration in the potential energy of a system will now be obvious.

A very useful application of the principle may be made to the approximate calculation of the natural periods of a system whose constitution, though complicated, is but slightly different from one of a much simpler nature. The main difficulty of the general problem consists in the determination of the free types, which may involve the solution of a difficult differential equation. We now see that an approximate knowledge of the type may be sufficient for practical purposes, and that, in the class of cases referred to, the adoption of the type natural to the approximate simpler system in the calculation of T and V will entail an error of the second order only in the final result.

To illustrate this question, we may take a case not without interest of its own—namely, the transverse motion of a stretched string of nearly, but not quite, uniform longitudinal density. If the uniformity were exact, the type of the sth component vibration would be

$$y = \phi_s \sin \frac{s\pi x}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

where l is the length, x the distance of any particle from one end, and y the transverse displacement. In accordance with the plan proposed, we are to calculate the period for the variable string on the supposition that (11) is also applicable to it. We find

$$\begin{aligned} \mathrm{T} &= \tfrac{1}{2}\dot{\phi}_s^2 \int_0^l \rho \sin^2 \frac{s\pi x}{l}\, dx \\ &= \frac{\rho_0 l}{4}\dot{\phi}_s^2 \left\{ 1 - 2\int_0^l \frac{\Delta\rho}{\rho_0} \sin^2 \frac{s\pi x}{l}\, dx \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12), \end{aligned}$$

if $\rho = \rho_0 + \Delta\rho$, $\Delta\rho$ being small.

For the potential energy we have (T_1 being the tension) the usual expression

$$\mathrm{V} = \tfrac{1}{2}\mathrm{T}_1 \int_0^l \left(\frac{dy}{dx}\right)^2 dx = \frac{\mathrm{T}_1 l}{4} \cdot \frac{s^2\pi^2}{l^2}\phi_s^2 \quad \ldots\ldots\ldots\ldots\ldots (13).$$

Hence, if the solution be

$$\phi_s = \mathrm{A} \cos\left(\frac{2\pi t}{\tau_s} - \epsilon\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

the period τ_s is given by

$$\tau_s^2 = \frac{4\rho_0 l^2}{s^2 \mathrm{T}_1}\left\{1 - 2\int_0 \frac{\Delta\rho}{\rho_0}\sin^2\frac{s\pi x}{l}\,dx\right\} \quad \ldots\ldots\ldots\ldots (15).$$

As might be expected, the effect of an alteration of density vanishes at the nodes, and is a maximum midway between them.

A similar method applies to a great variety of problems, and gives the means of calculating the correction due to the necessary deviation of any actual system, on which experiments can be made, from the ideal simplicity assumed in theory.

Another point of importance with reference to this application has yet to be noticed. It appears from (10) that the period of the vibration corresponding to any hypothetical type is included between the greatest and least of the periods natural to the system. In the case of systems like strings and plates, which are treated as capable of continuous deformation, there is no least natural period; but we may still assert that the period calculated from any hypothetical type cannot exceed that belonging to the gravest normal type. When, therefore, the object is to estimate the longest natural period of a system by calculations founded on an assumed type, we have, *à priori*, the assurance that the result will come out too small. For example, the value for τ_1, given in (15), is certainly less than the truth, while the error is of the second order in $\Delta\rho$.

In the choice of a hypothetical type, judgment must be used, the object being to approach the truth as nearly as can be done without too great a sacrifice of simplicity. The type for a string heavily weighted at any point might suitably be taken from the extreme case of an infinite load, that is to say, the two parts of the string might be supposed to be straight. Even with a uniform unloaded string, the result of the above hypothesis is not so very far from the truth. Taking $y = mx\cos pt$ from $x=0$ to $x = \frac{l}{2}$, we find, for the whole string,

$$\mathrm{T} = \frac{\rho p^2 m^2 l^3}{24}\sin^2 pt, \quad \mathrm{V} = m^2\frac{\mathrm{T}_1 l}{2}\cos^2 pt;$$

whence

$$p^2 = \frac{12\mathrm{T}_1}{\rho l^2} \quad \ldots\ldots\ldots\ldots (16).$$

The correct result for a uniform string is

$$p^2 = \frac{\pi^2\mathrm{T}_1}{\rho l^2} \quad \ldots\ldots\ldots\ldots (17);$$

so that the period calculated from the assumed type is too small in the ratio of $\pi : \sqrt{12}$ or ·907 : 1.

A much closer approximation would be obtained by the assumption

of a parabolic form

$$y = m\left(1 - \frac{4x^2}{l^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

Proceeding in the same way as before, we should find a period too short in the ratio $\pi : \sqrt{10}$, or ·9936 : 1. In order that the natural type should be parabolic, the density of the string would have to vary as $(l^2 - 4x^2)^{-1}$, being thus a minimum in the middle, and becoming infinite at either end.

The gravest tone of a square plate is obtained when the type of vibration is such that the nodal lines form a cross passing through the centre of the plate, and parallel to the edges. The next type in order of importance gives the diagonals for the nodal lines. Chladni found experimentally that the interval between the two tones was about a fifth. It so happens that the second kind of vibration can be completely treated theoretically, being referable to the simpler case of the vibration of bars; but the first has not hitherto been successfully attacked. I find that if we assume for the type of vibration

$$z = xy \cos pt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

the nodal lines being taken for axes of x and y, the boundary conditions are satisfied, and the calculated period comes out greater than that corresponding to the diagonal position of the nodal lines in the ratio of 1·37 : 1. Since this ratio is certainly too small, Chladni's result is about what might have been expected from theoretical considerations.

Before leaving the subject of natural vibrations, I wish to direct the attention of mathematicians to a point which does not appear to have been sufficiently considered: I refer to the expansion of arbitrary functions in series of others of specified types. The best known example of such expansions is that generally called after Fourier, in which an arbitrary periodic function is analysed into a series of harmonics, whose periods are sub-multiples of that of the given function. It is well known that the difficulty of the question is confined to the proof of the *possibility* of the expansion; if this be assumed, the determination of the coefficients is easy enough. What I wish now to draw attention to is, that in this, and an immense variety of similar cases, the possibility of the expansion may be inferred from physical considerations.

To fix our ideas, let us consider the small vibrations of a uniform string stretched between two fixed points. We know, from the general theory, that the whole motion, whatever it may be, can be analysed into a series of harmonic functions of the time, representing component vibrations, each of which can exist by itself. If we can discover these normal types, we shall be in a position to represent the most general possible vibration by combining them, each with arbitrary amplitude

and phase. The nature of the normal types is given by the solution of the differential equation

$$\frac{d^2y}{dx^2} + \kappa^2 y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

whence it appears that they are expressed by

$$y = \sin\frac{\pi x}{l}, \quad y = \sin\frac{2\pi x}{l}, \quad y = \sin\frac{3\pi x}{l}, \text{ \&c.}$$

We infer that the most general position which the string can assume is capable of representation by a series of the form

$$A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + \ldots\ldots,$$

which is a particular case of Fourier's theorem. There would be no difficulty in proving it in its most general form.

So far the string has been supposed uniform. But we have only to introduce a variable density, or even a single load at one point of the string, in order completely to alter the expansion whose possibility may be inferred from dynamical theory. It is evident that corresponding to any system, whether string, bar, membrane, plate, or what not, there is an appropriate expansion for an arbitrary function of one or more variables. Thus the expansion in La Place's series may be proved by considering the motion of a thin layer of gas between two concentric spherical surfaces, the expansion in Bessel's functions from the vibrations of a circular membrane, or of the air contained within a rigid cylinder, &c. When the difficulty of a direct analytical proof of even these simple cases is considered, the advantage of the physical point of view will be admitted.

The method of definite integration (or summation, if the system have only finite freedom), by which the constants are determined to suit arbitrary initial circumstances, is well known, and has been applied to a great variety of problems, dealing not only with vibrations, but with other physical questions such as the conduction of heat; but I have never seen the reason of its success distinctly stated. It may be said to depend on the characteristic property of the normal coordinates, namely, their power of expressing the energy of the system as a sum of squares only. In the case of a string, for example, we have

$$y = \phi_1 \sin\frac{\pi x}{l} + \phi_2 \sin\frac{2\pi x}{l} + \ldots\ldots,$$

where ϕ_1, ϕ_2 ..... are the normal coordinates. The expression for the energy is

$$\frac{\rho}{2}\int_0^l \dot{y}^2 dx, \quad \text{or} \quad \frac{\rho}{2}\int_0^l dx \left\{\dot{\phi}_1 \sin\frac{\pi x}{l} + \dot{\phi}_2 \sin\frac{2\pi x}{l} + \ldots\ldots\right\}^2.$$

If by the solution of the differential equation, or otherwise, we have

assured ourselves as to the nature of the normal types, we may assume without further proof, that the products of the coordinates will disappear from the expression for the energy,—in the above instance, that

$$\sin\frac{r\pi x}{l}\sin\frac{s\pi x}{l}\,dx$$

will vanish, if r and s be different.

Section II.

The Dissipation Function.

The original equation of motion of a system in rectangular coordinates, as obtained at once by an application of the Principle of Virtual Velocities, is

$$\Sigma m\,(\ddot{x}\delta x+\ddot{y}\delta y+\ddot{z}\delta z)=\Sigma\,(\mathrm{X}\delta x+\mathrm{Y}\delta y+\mathrm{Z}\delta z)\ \ldots\ldots\ldots\ (21).$$

When transformed to independent coordinates, and restricted so as to give the equations of vibratory motion in their simplest form, this becomes

$$\frac{d}{dt}\left(\frac{d\mathrm{T}}{d\dot{\psi}_1}\right)=\Psi_1\ \&c.\ \ldots\ldots\ldots\ (22),$$

where $\Psi_1\psi_1+\Psi_2\delta\psi_2+\ldots$ is the transformation of $\Sigma\,(\mathrm{X}\delta x+\mathrm{Y}\delta y+\mathrm{Z}\delta z)$, denoting the work done on the system by the applied forces during the hypothetical displacement.

If we separate from Ψ the forces which depend only on the position of the system, we obtain

$$\frac{d}{dt}\left(\frac{d\mathrm{T}}{d\dot{\psi}_1}\right)+\frac{d\mathrm{V}}{d\psi_1}=\Psi_1,\ \&c.\ \ldots\ldots\ldots\ (23).$$

The principal object of the present section is to show that another group of forces may be advantageously treated in a similar manner.

The forces referred to are those which vary in direct proportion to the component velocities of the parts of the system. It is well known that friction, and other sources of dissipation, may be usefully represented as following this law approximately; and even when the true law is different, the principal features of the case will be brought out. The effect of such forces will be to introduce into the original equation terms of the form

$$\Sigma\,(\kappa_x\dot{x}\delta x+\kappa_y\dot{y}\,\delta y+\kappa_z\dot{z}\,\delta z)\ \ldots\ldots\ldots\ (24),$$

where κ_x, κ_y, κ_z are the coefficients of friction, parallel to the axes, for the particle xyz. The transformation to the independent coordinates is effected in a similar manner to that of

$$\Sigma m\,(\dot{x}\delta x+\dot{y}\delta y+\dot{z}\,dz),$$

and gives

$$\frac{d\mathrm{F}}{d\dot{\psi}_1}\delta\psi_1+\frac{d\mathrm{F}}{d\dot{\psi}_2}\delta\psi_2+\ldots\ldots,$$

where $$F = \tfrac{1}{2}\Sigma(\kappa_x\dot{x}^2+\kappa_y\dot{y}^2+\kappa_z\dot{z}^2)$$
$$= \tfrac{1}{2}(11)\dot{\psi}_1^2 + \tfrac{1}{2}(22)\dot{\psi}_2^2 + \ldots\ldots + (12)\dot{\psi}_1\dot{\psi}_2 + \ldots\ldots \quad \ldots(25).$$

F, it will be observed, is, like T and V, a necessarily positive quadratic function of the coordinates, and represents the rate at which energy is dissipated.

The above investigation refers to forces proportional to the absolute velocities; but it is equally important to include such as depend on the *relative* velocities of the parts of a system, and this fortunately can be done without any increase of complication. For example, if a force acts on the particle x_1 proportional to $\dot{x}_1-\dot{x}_2$, there must be at the same moment, by the law of action and reaction, an equal and opposite force acting on x_2. The additional terms in the fundamental equation will be of the form $$\kappa(\dot{x}_1-\dot{x}_2)\,\delta x_1+\kappa(\dot{x}_2-\dot{x}_1)\,\delta x_2,$$

which may be written

$$\kappa(\dot{x}_1-\dot{x}_2)\,\delta(x_1-x_2) = \delta\psi_1\frac{d}{d\dot{\psi}_1}\left\{\tfrac{1}{2}(\dot{x}_1-\dot{x}_2)^2\right\}+\ldots\ldots;$$

and so on for any number of pairs of mutually influencing particles. The only effect is the addition of new terms to F, which still appears in the form (25).*

The existence of the function F does not seem to have been recognised hitherto, and indeed is expressly denied in the excellent "Acoustics" of the late Prof. Donkin (p. 101). We shall see that its existence implies certain relations between the coefficients in the generalized equations of motion, which carry with them important consequences.

Lagrange's equation, after the separation from Ψ of the forces proportional to the displacements *and velocities*, whether absolute or relative, becomes $$\frac{d}{dt}\left(\frac{d\mathrm{T}}{d\dot{\psi}}\right)+\frac{d\mathrm{F}}{d\dot{\psi}}+\frac{d\mathrm{V}}{d\psi} = \Psi \quad \ldots\ldots\ldots\ldots\ldots\ldots (26),$$

where
$$\left.\begin{aligned} \mathrm{T} &= \tfrac{1}{2}[11]\dot{\psi}_1^2+\ldots\ldots+[12]\dot{\psi}_1\dot{\psi}_2+\ldots\ldots \\ \mathrm{F} &= \tfrac{1}{2}(11)\dot{\psi}_1^2+\ldots\ldots+(12)\dot{\psi}_1\dot{\psi}_2+\ldots\ldots \\ \mathrm{V} &= \tfrac{1}{2}\{11\}\psi_1^2+\ldots\ldots+\{12\}\psi_1\psi_2+\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots(27).$$

On substitution, we obtain a system of equations, which may be written:

$$\left.\begin{aligned} \overline{11}\,\psi_1+\overline{12}\,\psi_2+\overline{13}\,\psi_3+\ldots\ldots &= \Psi_1 \\ \overline{21}\,\psi_1+\overline{22}\,\psi_2+\overline{23}\,\psi_3+\ldots\ldots &= \Psi_2 \\ \overline{31}\,\psi_1+\overline{32}\,\psi_2+\overline{33}\,\psi_3+\ldots\ldots &= \Psi_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\} \ldots\ldots\ldots\ldots (28),$$

* The differences referred to in the text may of course pass into differential coefficients in the case of a continuous body.

where a coefficient such as $\overline{rs}$ is an abbreviation for the quadratic operator

$$[rs]\frac{d^2}{d} + (rs)\frac{d}{dt} + \{rs\}.$$

It is to be carefully noticed that since $[rs] = [sr]$, $(rs) = (sr)$, $\{rs\} = \{sr\}$, it follows that $\overline{rs} = \overline{sr}$.

The small vibrations of a system free from dissipative influences can always be analysed into a series of normal components, each of which is similar in character to that of a system possessing but one degree of freedom. It is, in general, otherwise with the vibrations of a dissipative system. These may, indeed, be analysed into components of the quasi-harmonic type (Thomson and Tait, § 343); but these last are different in character from the vibration of a simple dissipative system. For instance, the system, supposed to be animated by one component, does not pass simultaneously through the configuration of equilibrium. The reason of the difference will appear at once. When there is no friction, a suitable transformation of coordinates will always reduce T and V to a sum of squares, and the equations of motion become

$$[1]\ddot{\phi}_1 + \{1\}\phi_1 = \Phi_1, \ \&c.,$$

the same as for a simple system. The presence of friction will not interfere with the reduction of T and V; but the transformation proper for them will not in general suit also the requirements of F. The equation can then only be reduced to the form

$$[1]\ddot{\phi}_1 + (11)\dot{\phi}_1 + (12)\dot{\phi}_2 + \ldots\ldots + \{1\}\phi_1 = \Phi_1 \quad \ldots\ldots\ldots (29),$$

and not to the simple form expressing the vibration of a system of one degree of freedom,

$$[1]\ddot{\phi}_1 + (1)\dot{\phi}_1 + \{1\}\phi_1 = \Phi_1 \quad \ldots\ldots\ldots\ldots\ldots (30).$$

We may, however, choose which two of the three functions we shall reduce, and the selection would vary according to circumstances.

Cases, however, arise in which, owing to the special character of the system, the same transformation of coordinates will reduce all three functions to a sum of squares, and then the motion possesses an exceptional simplicity. Under this head the most important are probably when F is of the same form as T or V. In the problem of the string, if we assume a direct retarding force proportional to the velocity, we have F proportional to T; if the dissipation is due to viscosity, we might have F proportional to V. The same exceptional reduction is possible when F is a linear function of T and V, or when T is itself of the same form as V. In any of these cases the equations of motion for each component are of the same form as for a dissipative system with one degree of freedom, and the elementary types are characterised by the fact that the whole system passes simultaneously through the con-

figuration of equilibrium. It appears that the law of friction usually assumed for a string is of an exceptional character, and leads to results of, in some respects, delusive simplicity.

Section III.

The present section is devoted to the proof and illustration of a very important law of a reciprocal character, connecting the forces and motions of any two types. Particular cases of it have been noticed by previous writers; but the general theorem is, I believe, new, and indeed could not be proved without the results of the preceding section.

The following partial statement will convey an idea of its nature:—

Let a periodic force Ψ_s, equal to $A_s \cos pt$, act on a system either conservative, or subject to dissipation represented by the function F, giving the forced vibration $\psi_r = \kappa A_s \cos(pt-\epsilon)$, where κ is the coefficient of amplitude, and ϵ the retardation of phase. The theorem asserts that if the system be acted on by the force $\Psi_r = A_r \cos pt$, the corresponding forced motion of type s will be

$$\psi_s = \kappa A_r \cos(pt-\epsilon).$$

The solution of the general equations (28) may be expressed in the form

$$\left.\begin{aligned} \nabla\psi_1 &= \frac{d\nabla}{d.\overline{11}}\Psi_1 + \frac{d\nabla}{d.\overline{12}}\Psi_2 + \ldots\ldots \\ \nabla\psi_2 &= \frac{d\nabla}{d.\overline{21}}\Psi_1 + \frac{d\nabla}{d.\overline{22}}\Psi_2 + \ldots\ldots \\ \ldots\quad &\ldots\quad \ldots\quad \ldots\quad \ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots (31),$$

where ∇ denotes the determinant

$$\begin{vmatrix} \overline{11}, & \overline{12}, & \overline{13}, & \ldots\ldots \\ \overline{21}, & \overline{22}, & \overline{23}, & \ldots\ldots \\ \overline{31}, & \overline{32}, & 33, & \ldots\ldots \\ \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (32),$$

and the partial differentiations are made without recognition of the relations $\overline{sr} = \overline{rs}$, &c. By the nature of determinants it follows that, since $\overline{sr} = \overline{rs}$,

$$\frac{d\nabla}{d.\overline{rs}} = \frac{d\nabla}{d.\overline{sr}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (33).$$

Thus the component displacement ψ_r due to a force Ψ_s is given by

$$\nabla\psi_r = \frac{d\nabla}{d.\overline{rs}}\Psi_s \ldots\ldots\ldots\ldots\ldots\ldots (34).$$

If, now, we inquire what the effect of a force Ψ_r will be in producing

the displacement of type s, we find

$$\nabla\psi_s = \frac{d\nabla}{d \cdot sr}\,\Psi_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35),$$

so that in virtue of (33) the relation of ψ_s to Ψ_r, in the second case, is the same as the relation of ψ_r to Ψ_s in the first.

Distinguishing the second case by a dash affixed to the corresponding quantities, let us take

$$\Psi_s = A_s \epsilon^{ipt}, \quad \Psi'_r = A'_r \epsilon^{ipt},$$

where the coefficients A_s, A'_r may, without loss of generality, be supposed to be real. The solution may be expressed in the form

$$\left.\begin{aligned} \psi_r &= A_s \frac{d \log \nabla (ip)}{d \cdot rs}\, \epsilon^{ipt} \\ \psi'_s &= A'_r \frac{d \log \nabla (ip)}{d \cdot sr}\, \epsilon^{ipt} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (36),$$

where $\frac{d}{dt}$ is replaced by ip in ∇ and its differentials. Hence by (33) we see that

$$A'_r \psi_r = A_s \psi'_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37),$$

which is the symbolical expression of the reciprocal theorem with respect both to amplitude and phase. If $\Psi'_r = \Psi_s$, then will $\psi'_s = \psi_r$; but it must be remembered that the forces and displacements of different types are not necessarily comparable. The following statement will, however, hold good in all cases:—The force Ψ'_r does as much work on the motion due to Ψ_s, as Ψ_s does on the motion due to Ψ'_r.

There is an important class of cases to which our principle, general as is the proof just given, would not at first sight appear to apply. Among these may be noticed systems in which the cause of the dissipation, or of part of it, is the conduction and radiation of heat. The dissipation cannot always be represented by a function F, which shall be the same in form under all circumstances. I am not at present in a position to discuss this question completely; but there is one consideration which may here be referred to as sufficient to bring a large additional field within the sweep of the demonstration. Since the investigation is concerned only with harmonic motions of period (p), it will be sufficient for the establishment of the theorem if the dissipation function exist *for all vibrations of the given period.*

A few examples may promote the comprehension of a theorem which, on account of its extreme generality, may appear vague.

Let A and B be two points of a uniform or variable stretched string. If a periodic transverse force act at A, the same vibration will be produced at B as would have ensued at A had the force acted at B.

In a space occupied by air, let A and B be two sources of disturbance. The vibration excited at A will have at B the same relative amplitude and phase as if the places were exchanged. Helmholtz (Crelle, Band LVII.) has proved this result in the case of a uniform fluid without friction, in which may be immersed any number of rigid fixed solids; but we are now in a position to assert that the reciprocity will not be interfered with, whatever number of strings, membranes, forks, &c. may be present, even though they are subject to damping.

The theorem includes the optical law, that if one point can be seen from a second, the second can also be seen from the first, whatever reflections or refractions the light may have to undergo on its passage.

A last example may be taken from electricity. Let there be two linear conducting circuits A and B, in whose neighbourhood there may be any number of others (either closed or terminating in condensers), or solid conducting masses. The theorem asserts that an electromotive force acting in A gives the same variable current in B as would be produced in A if the electromotive force were transferred to B.

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*Addition to the Memoir on Geodesic Lines, in particular those of a Quadric Surface.** *By* Prof. Cayley.

[*Read June 12th*, 1873.]

38. In the Memoir above referred to, speaking of the geodesic lines on the skew hyperboloid, I say (No. 35),—"The geodesic of initial direction M1 touches at M the oval curve of curvature M1, and lies wholly above this curve; *it makes an infinity of convolutions round the upper part of the hyperboloid*, cutting all the oval curves of curvature for which p has a positive value greater than p_1 (if p_1 is the value of p corresponding to the oval curve through M), and ascending to infinity." The statement as to the infinity of convolutions is incorrect; I was led to it by the assumption that the geodesic could not touch any hyperbolic curve of curvature. The fact is, that it touches at infinity (has for asymptotes) in general two hyperbolic curves of curvature; viz., the geodesic descending from infinity in the direction of a hyperbolic curve of curvature, so as to touch the oval curve through M, again ascends to infinity in the direction of a hyperbolic curve of curvature (the same as the first-mentioned one, or a different curve),

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* See pp. 191—211. The articles are numbered consecutively with those of the original Memoir.

making in its whole course say k convolutions, where m is a positive finite number; viz., if $k<1$, there is no complete convolution, and when $k=1$ or any integer number, then the two hyperbolic curves are one and the same curve; k is infinite only in the special case afterwards referred to in the same No. 35, where the oval curve of curvature is the ellipse which is a principal section of the hyperboloid, and does not even attain to the value 1 except for an oval curve exceedingly close to this ellipse. The error was, on consideration, obvious enough, though I was, in fact, led to perceive it by the numerical calculations about to be referred to, which gave me geodesics not making a complete convolution.

39. I have effected, for a particular skew hyperboloid and oval curve of curvature thereof, the numerical calculations for laying down the geodesic lines which touch this curve of curvature. Taking in general the equation of the hyperboloid to be

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1,$$

and the θ-curve of curvature to be the intersection by the confocal surface

$$\frac{x^2}{a-\theta}+\frac{y^2}{b-\theta}+\frac{z^2}{c-\theta}=1,$$

then the selected values for the hyperboloid, and oval curve of curvature touched by the geodesics, are

$$\begin{aligned} a &= \quad 900,\\ b &= \quad 400,\\ c &= -c' = -1600,\\ \theta &= -\theta' = -1650; \end{aligned}$$

so that a, b, c', θ' are the positive values 900, 400, 1600, and 1650 respectively. I recall that $p=c'$ to $p=\infty$ gives the oval curves of curvature, viz., $p=c'$, the elliptic principal section; $p=\theta'$, the given oval curve: and that we are in the sequel concerned only with the oval curves above this, for which p extends from θ' to ∞. Moreover, $q=-b$ to $-a$ gives the hyperbolic curves of curvature, viz., $q=-b$ the xz-principal section; and $q=-a$ the yz-principal section of the hyperboloid. We have, in fact, to deal with the integrals

$$\Pi(p)=\int_{\theta'} dp\sqrt{\frac{p}{(p+a)(p+b)(p-c')(p-\theta')}},$$

and

$$\Psi(q)=\int_{-b} dq\sqrt{\frac{q}{(q+a)(q+b)(q-c')(q-\theta')}};$$

or if $p=\theta'+u$, u extending from 0 towards ∞, then

$$\Pi(p)=\int_{0}^{u} du\sqrt{\frac{u+\theta'}{(u+a+\theta')(u+b+\theta')(u+\theta'-c')u}},$$

and so if $q = -b-v$, v extending from 0 towards $(a-b)$, which is its limit,

$$\Psi(q) = \int_0^v dv \sqrt{\frac{v+b}{(a-b-v)\,v\,(v+b+c')\,(v+b+\theta')}};$$

the relation for any particular geodesic of the series being

$$\pm\Pi(p) \pm \Psi(q) = C.$$

40. To avoid discontinuity as to sign, it is convenient to take the integral $\Psi(q)$ in a particular manner. The hyperboloid is by the xz- and yz-principal planes divided into four quadrants; or since we attend only to the upper half of the hyperboloid, say this upper half is thus divided into four quadrants, x to y, y to x', x' to y', and y' to x; or call them the first, second, third, and fourth quadrants. But we may consider the quadrants as forming an infinite succession, first, second, third, fourth, fifth, and so on; or we may take them in the reverse order, -1, -2, -3, &c. For a hyperbolic curve $q = -b-v$ in the first quadrant the integral is to be taken $v=0$ to $v=v$; for a curve in the second quadrant $v=0$ to $a-b$, and thence positively $a-b$ to v; for a curve in the third quadrant $v=0$ to $a-b$, thence $a-b$ to 0, and thence 0 to v; and so on: and so for a point in the quadrant -1, the integral is from 0 to v, taken negatively, &c.; that is, as the hyperbolic curve travels from the xz-position in the positive direction, the integral $\Psi(q)$ continually increases from zero; and if the curve travels from the xz-position in the negative direction, then the integral $\Psi(q)$ continually decreases from zero; that is, it increases negatively. It is to be remarked that the integral $v=0$ to $a-b$ is finite, say it is $= K'$; and of course it is only thus far that the integral requires to be calculated, the subsequent values differing from the preceding ones only by multiples of this complete integral.

The integral $\Pi(p)$ requires no explanation; it is taken from $u=0$, giving a certain oval curve, up to any positive value of u, giving the oval curves above this one; and, in particular, taking the integral to $u = \infty$ (or, what is the same thing, to $p = \infty$) its value is finite, $= K$, suppose.

41. Consider the geodesic which touches the given oval curve at a point for which $\Psi(q)$ has a given value Q; at this point $p=\theta'$, or $\Pi(p)=0$; so that, taking for the equation of the geodesic $\Psi(q) \pm \Pi(p) = C$, we have $Q=C$, and consequently

$$\Psi(q) = Q \mp \Pi(p).$$

Taking the positive sign, then as p increases from θ', $\Psi(q)$ increases, or the describing point of the geodesic moves upwards from the point of contact in the direction of positive rotation; and taking the negative sign, then $\Psi(q)$ decreases, or the describing point of the geodesic moves

upwards from the point of contact in the direction of negative rotation; and, in particular, p becoming infinite, then the first mentioned branch touches at infinity the hyperbolic curve, for which q is such that $\Psi(q) = Q+K$, and the second branch that for which q is such that $\Psi(q) = Q-K$.

42. The graphical process is as follows: we describe, on the hyperboloid, a series of hyperbolic curves of curvature, numbering them according to the values of $\Psi(q)$; viz., considering the hyperbolic branches which form the xz, yz, $x'z$ and $y'z$ sections respectively, these are 0, K′, 2K′, 3K′, and on going round a second time they would be 4K′, 5K′, 6K′, 7K′, and so on respectively. We similarly describe, say on the upper half of the hyperboloid, the oval curves of curvature, numbering them according to the values of $\Pi(p)$, viz., beginning with that for which $p=\theta'$, which is 0, we go successively up to the oval curve at infinity, which is K.

In the example, and drawing belonging thereto,* where, for convenience, the values of the integrals have been multiplied by 100,000, we have, as will appear, K=12490, K′=34726: the two sets of curves are drawn at intervals of 2000; viz., we have the hyperbolic curves 0, 2000, 4000, ... 34000, 34726; and the oval curves 0, 2000, 4000, 6000, 8000, 10000; the distance of the successive oval curves increases very rapidly, since the curve at infinity would be K, =12490, and the curve K=10000 is the last which comes into the limits of the figure.

43. The two sets of curves of curvature being thus drawn at equal intervals of $\Pi(p)$ and $\Psi(q)$ respectively, dividing the hyperboloid into quadrilateral spaces (which of course should theoretically be indefinitely small), the diagonals of these quadrilateral spaces are the elements of the geodesic lines; and by a series of such elements we have a particular geodesic line. The general character comes out in the drawing very distinctly; viz., the geodesic is a hyperbola-like curve descending from infinity to touch the oval curve, and again ascending to infinity; by reason of the small value of K in comparison of K′, there is nothing like a complete convolution, but the whole curve is included within a quadrant of the hyperboloid.

44. I remark that the calculations were performed roughly. I made no attempt to estimate or allow for errors arising from the intervals being too great; and there are very probably accidental errors of calculation. But starting with the value 10411 of $\Pi(p)$ for $p=10000$, I found, with some care, superior and inferior limits of the remainder of the integral, $p=10000$ to $p=\infty$; and the process is, I think, an interesting one. Consider in general the integral

* This drawing was exhibited at the meeting of the Society.

$$\mathrm{I} = \int_{0}^{\infty} dp \sqrt{\frac{p}{(p+a)(p+b)(p-c')(p-\theta')}}$$
$$= \mathrm{I}_1 + \int_{p_1}^{\infty} dp \sqrt{\frac{p}{(p+a)(p+b)(p-c')(p-\theta')}},$$

where I_1 is the integral calculated up to a somewhat large value $p = p_1$.

Writing

$$\alpha = \tfrac{1}{2}(a+b),$$
$$\beta = \tfrac{1}{2}(c'+\theta'),$$
$$\gamma = \tfrac{1}{2}(a+b) - m,$$
$$\delta = \tfrac{1}{2}(c'+\theta') + n,$$

where m and n are as yet undetermined, we have

$$(p+a)(p+b) = (p+\alpha)^2 - \tfrac{1}{4}(a-b)^2, \quad > (p+\alpha)^2,$$
$$(p-c')(p-\theta') = (p-\beta)^2 - \tfrac{1}{4}(\theta'-c')^2, \quad > (p-\beta)^2;$$

and the integral is thus

$$> \mathrm{I}_1 + \int^{\infty} dp \frac{\sqrt{p}}{(p+\alpha)(p-\beta)}.$$

But we may determine m and n, so that for $p =$ or $> p_1$,

$$(p+a)(p+b) < (p+\gamma)^2,$$
$$(p-c')(p-\theta') < (p-\delta)^2;$$

and the integral is then

$$< \mathrm{I}_1 + \int_{p_1}^{\infty} dp \frac{\sqrt{p}}{(p+\gamma)(p-\delta)}.$$

The determination thus depends on the last-mentioned integrals, the values of which are at once obtainable by writing therein $\sqrt{p} = x$; viz., we have

$$\int dp \frac{\sqrt{p}}{(p+\alpha)(p-\beta)} = 2\int \frac{x^2 dx}{(x^2+\alpha)(x^2-\beta)}$$
$$= \mathrm{C} + \frac{2}{\alpha+\beta}\left\{\tfrac{1}{2}\beta\, h.l\frac{x-\sqrt{\beta}}{x+\sqrt{\beta}}\right\} + \frac{2\sqrt{\alpha}}{\alpha+\beta}\tan^{-1}\frac{x}{\sqrt{\alpha}};$$

and hence, substituting in the formula, for $h.lx$ its value $\frac{\log x}{\log e}$, the superior limit is

$$\mathrm{I}_1 + \left\{\frac{\sqrt{\delta}}{\gamma+\delta}\frac{1}{\log e}\log\frac{\sqrt{p_1}+\sqrt{\delta}}{\sqrt{p_1}-\sqrt{\delta}} + \frac{2\sqrt{\gamma}}{\gamma+\delta}\cot^{-1}\frac{\sqrt{p_1}}{\sqrt{\gamma_1}}\right\},$$

and the inferior limit is

$$\mathrm{I}_1 + \left\{\frac{\sqrt{\beta}}{\alpha+\beta}\frac{1}{\log e}\log\frac{\sqrt{p_1}+\sqrt{\beta}}{\sqrt{p_1}-\sqrt{\beta}} + \frac{2\sqrt{\alpha}}{\alpha+\beta}\cot^{-1}\frac{\sqrt{p_1}}{\sqrt{\alpha_1}}\right\}.$$

45. The numerical values are $p_1 = 10{,}000$; $a, b, c', \theta' = 900, 400$ 1600, 1650; and thence determining by trial values of m and n,

$$\alpha = 650, \qquad \gamma = 650 - 4 = 646,$$
$$\beta = 1625, \qquad \delta = 1625 + 160 = 1785.$$

I obtained for the logarithmic and circular terms of the two limits respectively

	Superior	Inferior
Logarithmic	·015668	·015144
Circular	·005202	·005593
	·020870	·020737

The value of I_1 was $10411 \div 100{,}000 = \cdot 104110$, and the two limits thus are ·124980 and ·124850; or restoring the factor 100,000, they are 12498 and 12485; the mean of these, say 12490, was taken for the value $\Pi(p)$, $p = \infty$; that is, $K = 12490$.

46. As regards the calculation of the integrals $\Pi(p)$ and $\Psi(q)$, introducing the numerical values, and multiplying by the before-mentioned factor 100,000, we have $(q = -400 - v)$,

$$\Psi(q) = 100{,}000 \int_0^v dv \sqrt{\frac{v+400}{(500-v)\,v\,(v+2000)\,(v+2050)}},$$

which for any small value of v is

$$= 100{,}000 \sqrt{\frac{400}{500.2000.2050}} \left(\int \frac{dv}{\sqrt{v}}, = 2\sqrt{v} \right);$$

viz., this is $= 883{\cdot}45\ (\log = 2{\cdot}9461830)\ \sqrt{v}$,

which was used for the values $v = 1, 2, \ldots 10$, that is, to $q = -410$; after which the calculation was continued by quadratures giving to v the values 10, 20, 30, ... up to $v = 490$, or $q = -890$. For the remainder of the integral, writing $500 - v = w$ (that is, $q = -900 + w$), we have

$$\Psi(q) - \Psi(-890) = 100{,}000 \int dw \sqrt{\frac{900-w}{w\,(500-w)\,(2500-w)\,(2550-w)}}$$

$$= 100{,}000 \sqrt{\frac{900}{500.2500.2550}} \left\{ \int \frac{dw}{\sqrt{w}}, = 2\,(\sqrt{10} - \sqrt{w}) \right\}$$

$$= 1062{\cdot}7\ (\log = 3{\cdot}0264261)\ (\sqrt{10} - \sqrt{w}),$$

which was used for the values $w = 9, 8, 7, \ldots 1, 0$, to complete the calculation up to $q = -900$.

47. We have in like manner, $p = 1650 + u$,

$$\Pi(p) = 100{,}000 \int_0^u du \sqrt{\frac{u+1650}{(u+2550)\,(u+2050)\,(u+50)\,u}},$$

which for small values of u is

$$= 100{,}000 \sqrt{\frac{1650}{2550.2050.50}} \left(\int \frac{du}{\sqrt{u}}, = 2\sqrt{u} \right);$$

viz., this is $502{\cdot}5\ (\log = 2{\cdot}7011399)\ \sqrt{u}$,

used for $u = 1, 2, \ldots 10$, that is to $p = 1660$. The calculation was

afterwards continued by quadratures, by giving to u a succession of values, at intervals at first of 10, and afterwards of 20, 50, 100, 200, and 500, up to $p = 10{,}000$, giving for the integral the value 10411; and thence, as appearing above, the value for $p = \infty$ was found to be $= 12490$.

48. After the calculation of the values of $\Pi(p)$ and $\Psi(q)$, it was easy by interpolation to revert these tables, so as to obtain a table which, for Π or Ψ as argument, gives the values of p and q. The arguments are taken at intervals of 500; up to 10,000 as regards p, since the original table was only calculated thus far; and up to 34726 as regards q. I had thus calculated the annexed Table III., when it occurred to me that there was a convenience in taking the arguments to be submultiples of the complete integral 34726; say we divide this into 90 parts, or, as it were, graduate the quadrant of the hyperboloid by means of hyperbolic curves of curvature adapted for the geodesics in question. Taking every fifth part, or in fact dividing the quadrant into 18 parts, we have the Table IV.

49. It will be remembered that the foregoing results apply only to the geodesics which touch the oval curve of curvature $p = +1650$; for the geodesics touching any other oval curve of curvature, the values of the integrals, and the mutual distances of the curves of curvature used for tracing the geodesics, would be completely altered. But it is possible to derive some general conclusions as to the geodesics that touch a given oval curve of curvature.

Observe that the integral K′ ($= 34726$ in the case considered) measures the quadrant of the hyperboloid; viz., $\Psi(q) = 0$, $\Psi(q) = \mathrm{K}'$ determine two hyperbolic curves of curvature (principal sections), the mutual distance whereof is a quadrant. Each geodesic touches the given oval curve of curvature, and it touches at infinity the two hyperbolic curves $\Psi(q) = \mathrm{Q} \pm \mathrm{K}$ ($\mathrm{K} = 12490$ in the case considered); viz., the distance of these in regard to the circuit of four quadrants, or say the amplitude of the geodesic, is measured by the ratio $\frac{\mathrm{K}}{2\mathrm{K}'}$.

50. Now it is easy to see that as the oval curve of curvature approaches the principal elliptic section, that is, as θ' approaches c' (or writing $\theta' = c' + m$, as m diminishes towards zero), the integral K′ alters its value only slowly, increasing towards a certain constant limit; but, contrarywise, K increases without limit, its value for any small value of m being of the form $\mathrm{A} - \mathrm{B} \log m$, $= \infty$ in the limit; wherefore, as m diminishes, the value of $\frac{\mathrm{K}}{2\mathrm{K}'}$, or amplitude of the geodesic, continually increases. If this is $= 1$, the geodesic touching at infinity a certain hyperbolic curve of curvature, in descending to touch the oval

curve, makes round the hyperboloid a half-convolution, and then again ascends through another half-convolution to touch at infinity the same hyperbolic curve of curvature; viz., it makes in all one entire convolution, or say in descending it makes a half-convolution. But if $K \div 2K' = 2$, then the curve makes in descending a complete convolution; and so, if $K \div 2K' = 2s$, then the geodesic makes in descending s convolutions; and, as already mentioned, ultimately when $m=0$ the geodesic makes an infinity of convolutions; that is, it never actually reaches the elliptic principal section, but has this line for an asymptote.

51. To sustain the foregoing statements, I write $\theta' (= c' + m) = 1600 + m$, and I consider the integral

$$K'_m = 100{,}000 \int_0^\infty du \sqrt{\frac{u + 1600 + m}{(u + 2500 + m)(u + 2000 + m)(u + m)\, u}},$$

say for a moment this is $K'_m = \int_0^\infty U_m du.$

Supposing m to be small, we divide the integral into two parts, say from 0 to a [where a, = for example 50 or 100, is large in comparison with m, but small in comparison with the numbers (c', &c.), 1600, &c.], and from a to ∞. In the second part, the expression under the integral sign and the value of the integral varies slowly with m, and we may, as an approximation, write $m=0$. We have thus

$$K_m = \int_0^a U_m du + \int_a^\infty U_0 du,$$

and the first part hereof is

$$= 100{,}000 \sqrt{\left(\frac{1600}{2500 \,.\, 2000}\right)} \int_0^a \frac{du}{\sqrt{u\,(u+m)}};$$

viz., the integral is here

$$h \,.\, l \left\{ u + \tfrac{1}{2}m + \sqrt{u\,(u+m)} \right\}$$

$$= h \,.\, l \frac{a + \frac{1}{2}m + \sqrt{a\,(a+m)}}{\frac{1}{2}m} = h \,.\, l \frac{4a}{m} \text{ approximately;}$$

or say this is

$$= \frac{1}{\log e} \log \frac{4a}{m}.$$

The first term is thus

$$= 100{,}000 \sqrt{\frac{1600}{2500 \,.\, 2000}} \frac{1}{\log e} \log \frac{4a}{m},$$

which is

$$= 4119 \log \frac{4a}{m},$$

or we have

$$K'_m = 4119 \log \frac{4a}{m} + \int_a^\infty U_0 du.$$

We ought to have the same value of the integral, whatever, within

proper limits, the assumed value of a may be. Taking, for instance, $a=50$ and $a=100$, we ought to have

$$K'_m = 4119 \log \frac{400}{m} + \int_{100}^{\infty} U_0\, du$$

$$= 4119 \log \frac{200}{m} + \int_{50}^{\infty} U_0\, du\,;$$

that is,

$$4119 \log 2 = \int_{50}^{100} U_0\, du.$$

In verification, I calculated the second side by quadratures; viz., for the values 50, 60, 70, 80, 90, 100, the values of U_0 are 35·532, 29·570, 25·311, 22·373, 19·632, 17·645; whence, adding the half sum of the extreme terms to the sum of the mean terms, and multiplying by 10, the value of the integral is $=1234{\cdot}74$ The value of the left-hand side is $=1239{\cdot}94$, which is a sufficient agreement.

52. Returning to the formula for K'_m, this may be written

$$K'_m = \left(4119 \log 400 + \int_{100}^{\infty} U_0\, du\right) - 4119 \log m.$$

I did not calculate the value of the integral in this formula, but determined the term in ( ) in such wise that the formula should be correct for the foregoing value $m=50$; viz., the term thus is

$$= 12490 + 4119 \log 50 = 12490 + 6998, = 19488, \text{ or say } 19500.$$

We thus have

$$K_m = 19500 - 4119 \log m\,;$$

and we may roughly assume that, for any small value of m, K'_m has the same value as for $m=50$; viz., we may write

$$K'_m = 34726, \text{ or say } = 35000.$$

We thus see how to give to m such a value that the quantity $\frac{K}{2K'}$, which is the number of convolutions of the geodesic, may have any given value; and, in particular, we see how exceedingly small m must be for any moderately large number of convolutions; for instance, $m = \frac{1}{100{,}000{,}000}$ or $\log m = -8$, $K = 19500 + 32952$, = say 52500, or the number is $= \frac{525}{700}$, about five-sevenths of a convolution.

Correction.—Instead of speaking, as above, of a geodesic as touching at infinity a hyperbolic curve of curvature, the accurate expression is that the geodesic at infinity is parallel to a certain hyperbolic curve of curvature. The geodesic has, in fact, for asymptote the right line on the surface parallel at infinity to such curve of curvature.—Added Dec. 1873.—A. C.

TABLE I.

p	$\Pi(p)$.
1650	000
1	502
2	711
3	870
4	1005
5	1124
6	1231
7	1329
8	1421
9	1507
1660	1589
70	2188
80	2605
90	2933
1700	3205
10	3438
20	3642
30	3824
40	3989
50	4130
60	4287
70	4414
80	4532
90	4642
1800	4745
20	4945
40	5106
60	5261
80	5403
1900	5533
20	5654
40	5766
60	5871
80	5970
2000	6063
50	6275
100	6462
150	6629
200	6779
250	6916
300	7041
350	7156
400	7262
450	7362

TABLE II.

q	$\Psi(q)$.
—400	000
1	883
2	1249
3	1530
4	1767
5	1975
6	2164
7	2337
8	2499
9	2650
410	2794
20	4016
30	4952
40	5752
50	6467
60	7124
70	7739
80	8321
90	8877
500	9413
10	9931
20	10436
30	10923
40	11406
50	11830
60	12347
70	12809
80	13266
90	13720
600	14171
10	14619
20	15067
30	15513
40	15960
50	16408
60	16857
70	17308
80	17762
90	18220
700	18682
10	19149
20	19623
30	20101
40	20588

TABLE I., *continued.*

p	$\Pi(p)$.
2500	7454
600	7625
700	7777
800	7913
900	8037
3000	8151
200	8352
400	8526
600	8679
800	8815
4000	8936
500	9216
5000	9423
500	9593
6000	9737
500	9861
7000	9970
500	10066
8000	10152
500	10229
9000	10299
500	10363
10000	10411
⋮	
∞	12490

TABLE II., *continued.*

q	$\Psi(q)$.
−750	21086
60	21595
70	22117
80	22653
90	23206
800	23778
10	24374
20	24998
30	25655
40	26355
50	27105
60	27928
70	28861
80	29936
890	31365
1	31538
2	31720
3	31914
4	32123
5	32350
6	32600
7	32886
8	33224
9	33663
−900	34726

TABLE III.

Π or Ψ	p	D	q	D
0	1650·0		−400 0	
		1		·3
500	1651·0		400·3	
		3		1·0
1000	1654·0		401·3	
		5		1·6
1500	1659·0		402·9	
		7·8		2·2
2000	1666·8		405·1	
		10·7		2·9
2500	1677·5		408 0	
		14·9		3·7
3000	1692·4		411·7	
		20·6		4·0
3500	1713·0		415·7	
		27·8		4·3
4000	1740·8		420·0	
		36·6		5·2
4500	1777·4		425·2	
		49·4		5·4
5000	1826·8		430·6	
		68·1		6·2
5500	1894·9		436·8	
		91·5		6·6
6000	1986·4		443·4	
		124·6		7·1

TABLE III., *continued.*

Π or Ψ	p	D	q	D
6500	2111	124·6	−450·5	7·1
7000	2283	172	458·1	7·6
7500	2527	244	466·1	8·0
8000	2870	353	474·5	8·4
8500	3285	415	483·2	8·7
9000	4114	829	492·3	9·1
9500	5226	1112	501·7	9·4
10000	7156	1930	511·3	9·6
500	⋮		521·1	9·8
11000	⋮		531·5	10·1
500	⋮		542·0	10·5
12000	⋮		552·5	10·8
500	∞		563·3	10·8
13000			574·2	10·9
500			585·1	10·9
14000			596·2	11·1
500			607·3	11·1
15000			618·5	11·2
500			629·7	11·2
16000			641·0	11·3
500			652·1	11·1
17000			663·2	11·1
500			674·2	11·0
18000			685·2	11·0
500			696·1	10·9
19000			706·8	10·7
500			717·4	10·6
20000			728·0	10·6
500			738·4	10·4
21000			748·3	9·9
500			758·1	9·8
22000			767·7	9·6
500			777·4	9·4
23000			786·4	9·0
500			795·1	8·7
24000			803·7	8·6
500			812·0	8·3
25000			820·0	8·0
500			827·6	7·6
26000			834·9	7·3
500			841·9	7·0
27000			848·6	6·7
500			854·8	6·2
28000			860·8	6·0
500			866·1	5·3
				5·2

TABLE III. (*continued.*)

Π or Ψ	*p*	D	*q*	D
				5·2
29000			−871·3	4·7
500			876·0	4·6
30000			880·6	3·4
500			884·0	3·5
31000			887·5	3·3
500			890·8	3·6
32000			893·4	2·2
500			895·6	1·7
33000			897·5	1·1
500			898·6	0·7
34000			899·3	0·5
500			899·8	0·2
34726			−900	

TABLE IV.

	Π or Ψ	*p*	Diff.	*q*	Diff.
0	0	1650		−400	
			15·7		4·8
1	1929	1665·7		404·8	
			66·4		13·9
2	3858	1732·1		418·7	
			212·1		21·8
3	5788	1944·2		440·5	
			713·5		29·1
4	7717	2657·7		469·6	
			3040·0		34·9
5	9646	5697·7		504·5	
					39·1
6	11575			543·6	
					41·6
7	13504			585·2	
					42·8
8	15434			628·0	
					43·2
9	17363			671·2	
					41·8
10	19292			713·0	
					39·7
11	21221			752·7	
					36·3
12	23150			789·0	
					32·2
13	25079			821·2	
					27·4
14	27008			848·6	
					22·1
15	28938			870·7	
					15·8
16	30868			886·5	
					10·2
17	32797			896·7	
					3·3
18	34726			−900	

Preliminary Sketch of Biquaternions. *By* Prof. Clifford, M.A.

[*Read June 12th*, 1873.]

I.

The *vectors* of Hamilton are quantities having magnitude and direction, but no particular position; the vector AB being regarded as identical with the vector CD when AB is equal and parallel to CD and in the same sense. The translation of a rigid body is an example of such a quantity; for since all particles of the body move through equal distances along parallel straight lines in the same sense, the motion is entirely specified by a straight line of the given length and direction drawn through any point whatever. A couple, again, may be adequately represented by a vector; since the axis of a couple is any line of length proportional to its moment drawn perpendicular from a given face of its plane.

For many purposes, however, it is necessary to consider quantities which have not only magnitude and direction, but *position* also. The rotational velocity of a rigid body is about a certain definite axis, and equal rotations about two parallel axes are not equivalent to one another. A force acting upon a solid has a definite line of action, and equal forces acting along parallel lines differ by a certain couple. The difference between the two kinds of quantities is clearly seen when we consider the geometric calculus which is used for the study of each. In studying the motions of a particle or the composition of couples, the only construction required is that of the "force-polygon," and the theory involved is that of the addition of vectors; but in the static or kinematic of solids we require in addition the construction of the "link-polygon," and there is involved the theory of the involution of lines in space, or of the linear complex.

The name *vector* may be conveniently associated with a velocity of *translation*, as the simplest type of the quantity denoted by it. In analogy with this, I propose to use the name *rotor* (short for *rotator*) to mean a quantity having magnitude, direction, and position, of which the simplest type is a velocity of *rotation* about a certain axis. A rotor will be geometrically represented by a length proportional to its magnitude measured upon its axis in a certain sense. The rotor AB will be identical with CD if they are in the same straight line, of the same length, and in the same sense; *i. e.*, a vector may move anywise parallel to itself, but a rotor *only* in its own line.

The *addition* of rotors will proceed by the rules which govern the composition of forces and rotations. Here, however, we come upon a very important break in the analogy between rotors and vectors. While

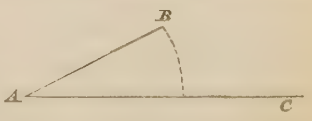
the sum of any number of vectors is always a vector, it will only happen in special cases that the sum of a number of rotors is a rotor. In fact, the composition of two forces whose lines of action do not intersect, or of two rotation-velocities whose axes do not intersect, gives rise to a system of forces on the one hand, and the most general velocity of a rigid body on the other. These still more complex quantities have been studied, and the theory of their addition or composition completely worked out, by Dr. Ball.

A system of forces may be reduced in one way to a single force P, and a couple G whose plane is perpendicular to the line of action of the force, or *central axis.* Dr. Ball speaks of the system of forces as a *wrench* about a certain *screw;* the axis of the screw being the central axis, and the pitch being the ratio $\frac{G}{P}$ of the couple to the force. Similarly the velocity of a rigid body may be represented in one way only as a rotation-velocity ω about a certain axis combined with a translation-velocity v along that axis. Dr. Ball speaks of this velocity as a *twist*-velocity about a certain screw; the axis of the screw being the axis of rotation, and its pitch the ratio $\frac{v}{\omega}$ of the translation to the rotation. A *screw* is here a geometrical form resulting from the combination of an *axis* or straight line given in position with a *pitch* which is a linear magnitude. A *wrench* is the association with this geometrical form of a magnitude whose dimensions are those of a force; a *twist*-velocity the association of a magnitude whose dimensions are those of an angular velocity. The extreme convenience of this nomenclature is well exemplified in the remarkable memoir above referred to.

Just as a vector (translation-velocity or couple) is magnitude associated with direction, and as a rotor (rotation-velocity or force) is magnitude associated with an axis; so this new quantity, which is the sum of two or more rotors (twist-velocity or wrench) is magnitude associated with a screw. Following up the analogy thus indicated, I propose to call this quantity a *motor;* the simplest type of it being the general motion of a rigid body. And we shall say that in general the sum of rotors is a motor, but that in particular cases it may degenerate into a rotor or a vector.

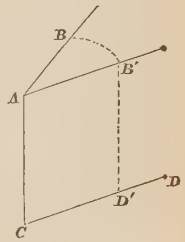
II.

A *quaternion* is the ratio of two vectors, or the operation necessary to make one into the other. Let the vectors be AB and AC, as they may both be made to start from any arbitrary point A. Then AB is made into AC by turning it round an axis through A perpendicular to the plane BAC until its direction coincides with that of AC, and then



magnifying or diminishing it until it is of the same length as AC. The ratio of two vectors then is the combination of an ordinary numerical ratio with a *rotation;* or, as Hamilton expresses it, a quaternion is the product of a tensor and a versor. Since the point A is perfectly arbitrary, this rotation is not about a definite axis; but is completely specified when its angular magnitude and the direction of its axis are given.

This quaternion* $\frac{AC}{AB} = q$, then, is an operation which, being performed on AB, converts it into AC, so that $q \,.\, AB = AC$. The axis of the quaternion is perpendicular to the plane BAC; and it is clear that the quaternion operating upon any other vector AD in this plane will convert it into a fourth vector AE in the same plane, the angle DAE being equal to BAC and the lengths of the four lines proportionals. But a quaternion can *only* operate upon a vector which is perpendicular to its axis. If AF be any vector not in the plane BAC, the expression $q \,.\, AF$ is absolutely unmeaning. A meaning is indeed subsequently given to an analogous expression *in which the signification of* AF *is different.* But it is very important to remark that so long as AF means a *vector* not perpendicular to the axis of q, the expression $q \,.\, AF$ has no meaning at all.



Let us now consider what is the operation necessary to convert one *rotor* into another. There is one straight line which meets at right angles the axes of any two rotors, and part of which constitutes the shortest distance between them. Let AC be the shortest distance between the rotors AB and CD. Then AB may be converted into CD by a process consisting of three steps. First, turn AB about the axis AC into the position AB′, parallel to CD. Then slide it along this axis into the position CD′. Lastly, magnify or diminish it in the ratio of CD′ to CD. The first two operations may be regarded as together forming a twist about a screw whose axis is AC and whose pitch is

$$\frac{AC}{\text{circ. meas. of } BAB'}.$$

The ratio of two rotors, then, is the combination of an ordinary numerical ratio with a *twist.* This twist is associated with a perfectly

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* Professor Cayley, by a very convenient notation, distinguishes $\frac{|AC}{AB|}$ and $\frac{AC|}{|AB}$; viz., $AB\frac{|AC}{AB|} = 1$, but $\frac{AC|}{|AB}AB = 1$. It should, I think, be a convention that $\frac{X}{Y}$ is *always* to mean $\frac{X|}{|Y}$, viz., the operation which converts Y into X, *or* which, coming after the operation Y, is equivalent to the operation X.

definite screw, and is only specified when its angular magnitude and the screw (involving direction, position, and pitch) are given. We may say also that just as the rotation (versor) involved in a quaternion is the ratio of two directions, so the twist involved in the ratio of two rotors is really the ratio of their axes.

Here again a remark must be made about the range of this operation. Using the expression *tensor-twist* to mean the ratio of two rotors (which is in fact a twist multiplied by a tensor), we may say that a tensor-twist can operate upon any rotor which meets its axis at right angles. Let t denote the operation which converts AB into CD, so that $t = \frac{\mathrm{CD}}{\mathrm{AB}}$, and $t \,.\, \mathrm{AB} = \mathrm{CD}$; then if EF be any other rotor which meets AC at right angles, the expression $t \,.\, \mathrm{EF}$ will have a definite meaning, viz., it will mean a rotor obtained by sliding EF along a distance equal to AC, turning it about AC as axis through an angle equal to BAB′, and altering its length in the ratio AB : CD. But if EF be a rotor not meeting AC, or meeting it at any other than a right angle, the expression $t \,.\, \mathrm{EF}$ will have no meaning whatever.

We have now defined the ratio of two rotors, and shown that like a quaternion it has a restricted range of operation. The question naturally arises, what now is the operation which converts one *motor* into another? We can answer this question very easily in the case in which the two motors have the same pitch; for in this case their ratio is a tensor-twist whose tensor is the ratio of their magnitudes and whose twist is the ratio of their axes. We are led to this by considering each motor as the sum of two rotors which do not intersect. Let α and β be two such rotors, t a tensor-twist whose axis meets them both at right angles; then $t\alpha$ is a rotor, say γ, and $t\beta$ is another rotor, say δ. If therefore we assume the distributive law, we have

$$t(m\alpha + n\beta) = m\gamma + n\delta,$$

or

$$t = \frac{m\gamma + n\delta}{m\alpha + n\beta}.$$

It is a mere translation of known theorems to say that the axis of t meets at right angles the axes of the motors $m\alpha + n\beta$ and $m\gamma + n\delta$, and that one of these axes is converted into the other by the same twist that makes α into γ or β into δ.

The solution of this problem in the general case, in which the pitches are different, is not so easy. In the first place, we must remember that every motor consists of a rotor part and a vector part, and that its pitch is determined by the ratio of these two parts. By combining a suitable vector with a motor, therefore, we may make the pitch of it anything we like, without altering the rotor part. Now let it be required to find the operation which will convert a motor A into a motor

B. Let B′ be a motor having the same rotor part as B, and the same pitch as A; and let $B = B' + \beta$, where β is a vector parallel to the axis of B. Then the ratio $\frac{B}{A} = \frac{B'}{A} + \frac{\beta}{A}$; but $\frac{B'}{A}$ is a tensor-twist, say t, and we may write

$$\frac{B}{A} = t + \frac{\beta}{A},$$

where it now only remains to find an operation which will convert a motor A into a vector β.

In order to do this, we must introduce a symbol whose nature and operation will at first sight appear completely arbitrary, but will be justified in the sequel. *The symbol ω, applied to any motor, changes it into a vector parallel to its axis and proportional to the rotor part of it.* That is to say, it changes rotation about any axis into translation parallel to that axis, and a force into a couple in a plane perpendicular to its line of action. But if the rotation is accompanied by translation or the force by a couple, the symbol takes no account whatever of these accompaniments; and if made to operate directly on a vector, reduces it to zero. It follows from this that if it be made to operate twice upon a motor, it reduces it to zero; or $\omega^2 A = 0$ always. The portion of any expression which involves ω must therefore be treated as an infinitesimal of the first order; all higher orders being uniformly neglected.

Since then $\omega A = a$, a vector, and the ratio $\frac{\beta}{a}$ is a quaternion q so that $qa = \beta$, we may write successively

$$\beta = qa = q\omega A,$$

$$\frac{\beta}{A} = q\omega,$$

and then

$$\frac{B}{A} = t + q\omega,$$

or *the ratio of two motors may be expressed as the sum of two parts, one of which is a tensor-twist, and the other is ω multiplied by a quaternion.*

The same ratio may be expressed in another form. Let an arbitrary point O be assumed as the origin; then every motor may be expressed in one way as the sum of a rotor passing through O and a vector. Now the theory of rotors passing through a fixed point is exactly the same as that of vectors in general, and the ratio of any two of them is a tensor-twist whose pitch is zero, or what is the same thing, a quaternion whose axis is constrained to pass through the fixed point. If we use cursive Greek letters (as α, β) in general to represent rotors through the origin, we may distinguish vectors from them by prefixing the symbol ω; thus $\omega\alpha$ denotes a vector parallel and proportional to the rotor α. The ratio $\frac{\beta}{\alpha}$ will then be a quaternion q, which is also the

ratio $\frac{\omega\beta}{\omega a}$.* The general expression for a motor is then $a+\omega\beta$. Let it now be required to find the ratio of two motors $a+\omega\beta$, $\gamma+\omega\delta$; or the value of the expression $\dfrac{\gamma+\omega\delta}{a+\omega\beta}$.

First, let $\frac{\gamma}{a}=q$; then $q(a+\omega\beta)=\gamma+q\omega\beta=\gamma+\omega q\beta$.

The symbol $q\beta$ has at present no geometrical meaning; for in general the rotors a, β, γ will not be coplanar, and cannot therefore be operated on by the same quaternion q. If however (as in the Calculus of Quaternions) we consider all these quantities as expressed in terms of three rectangular unit rotors through the origin, $\frac{\delta-q\beta}{a}$ will be a perfectly definite quaternion r. The equation

$$ra=\delta-q\beta$$

is, like the equation $q(a+\omega\beta)=\gamma+\omega q\beta$,

at present purely literal and devoid of meaning. Yet if (remembering the properties of the symbol ω) we add ω times the first equation to the second and assume the distributive law, we obtain

$$(q+\omega r)(a+\omega\beta)=\gamma+\omega\delta.$$

In this way the ratio $\frac{\gamma+\omega\delta}{a+\omega\beta}$ is expressed in the form $q+\omega r$, which expression may conveniently be called a *biquaternion*.† The final equation, however, is not susceptible of interpretation in the same sense as the equation $qa=\gamma$. The expression $q+\omega r$ does not denote the sum of geometrical operations which can be applied to the motor $a+\omega\beta$ as a whole; and the ratio of two motors is only expressed by a symbol as the sum of two parts, each of which separately has a definite meaning in certain other cases, but not in the case in point. In following sections this difficulty will be partly overcome by showing that the system here sketched is the limit of another in which it does not occur.

The preceding remarks may however explain, and be illustrated by, the following table :—

Geometrical Form	Quantity	Example	Ratio
Sense on st. line	Vector on st. line	Addition or Subtraction	Signed Ratio
Direction in plane	Vector in plane	Complex quantity	Complex Ratio
Direction in space	Vector in space	Translation, Couple	Quaternion
Axis	Rotor	Rotation-Velocity, Force	Twist
Screw	Motor	Twist-Velocity, System of Forces	Biquaternion

* It follows from this that $\omega q=q\omega$, or ω is commutative with quaternions.

† Hamilton's *biquaternion* is a quaternion with complex coefficients; but it is convenient (as Prof. Peirce remarks) to suppose from the beginning that all scalars may be complex. As the word is thus no longer wanted in its old meaning, I have made bold to use it in a new one.

III.

That geometry of three-dimensional space which assumes the Euclidian postulates has been called by Dr. Klein the *parabolic* geometry of space, to distinguish it from two other varieties, which assume uniform positive and negative curvature respectively, and which he calls the *elliptic* and *hyperbolic* geometry of space. The investigations which follow involve the postulates of elliptic geometry. As, however, the postulate of uniform positive curvature is not sufficient to define this, it may be worth while to devote a short space to an explanation of its nature.

Space of three dimensions is that the points of which may be associated with systems of values of three variables x, y, z. It is not in general possible, however, so to make this association that to every system of values there shall correspond in general one point, and to every point in general one system of values. When this is the case, the space is called *unicursal*. An *algebraic* space is one in which the position of a point may be uniquely defined by a set of values of periodic algebraic integrals, without exceptions which form a part of the space. Thus, unicursal spaces are a particular case of algebraic. Attending now to unicursal spaces only, we must observe that there are in general exceptions to the unique correspondence of points and value-systems; namely, there are certain points to each of which correspond an infinite number of values of the coordinates satisfying a certain equation or equations; and there are certain value-systems to which correspond, not points, but loci in the space. The assignment of these point-equations and loci-values and of their relations with one another serves to determine the *projective-connection* of the space; and when once these are known, the whole of its projective geometry may be worked out. The point-equations and loci-values may or may not involve imaginary values of the variables or their coefficients; but in all cases they must be taken into account. The points which correspond to real systems of values are called real points; those which correspond to imaginary systems, imaginary points: the study of these latter, which does not strictly belong to that of three-dimensional space, is undertaken only for the sake of the former.

Loci which correspond to linear equations between the coordinates may at present be called *planes*, and their intersections *lines;* this is a purely projective definition, and these loci are not necessarily *flat* planes and *straight* lines in the metrical sense. Points, lines, and planes are included in the name *elements*.

The *metric* geometry of space* is the theory of the projective relations of certain fixed geometrical forms with all other geometrical forms, or of the invariant relations of certain fixed algebraic forms with all other

* This theory of metric geometry is due to Prof. Cayley: Sixth Memoir on Quantics, Phil. Trans., 1859.

algebraic forms. The word *power* will be explained as much as is wanted in the sequel; meanwhile it may be said that these fixed forms (called all together *the absolute*) are given when we know the points, the lines, and the planes of the absolute, or say the elements of the absolute; and that the power of an element of the absolute in regard to any arbitrary element is infinite. In other words, we *require* in general equations of the absolute in point-, line-, and plane-coordinates respectively.

A unicursal space the points of which may be represented uniquely by value-systems of the coordinates x, y, z, without the exception of any point-equations or loci-values, is called a *linear* space. This is merely a projective definition, and leaves the absolute, therefore the whole of metric geometry, undetermined.

There is a particular determination of the absolute in a linear space which is of the utmost importance. It is that in which the points of the absolute are those of a certain quadric surface, while the lines and planes of the absolute are those which touch this surface; or in which the three equations of the absolute are of the second degree. There are three cases* to be considered, as being the only ones of which observed space can form a part:—

(1) *Elliptic* geometry; all the elements of the absolute are imaginary.

(2) *Hyperbolic* geometry; the absolute contains no real straight lines, and surrounds us. In this case, real points situate on the other side of the surface are called *ideal.*

(3) *Parabolic* geometry; the surface degenerates into an imaginary conic in a real plane. The points of the absolute are points in the (real) plane of this conic; the lines and planes are the imaginary lines and planes which meet and touch the conic respectively.

The *first* of these suppositions will be made in what follows. It may be well here to set down in what it consists.

(1) The space to be considered is such that there is one point of it for every set of values of the coordinates x, y, z, and one set of values for every point, without any exception whatever.

(2) There is a certain quadric surface, called the absolute, all whose points and tangent planes are imaginary. If the line joining two points a, b meet the absolute in i, j, the quantity

$$\frac{ab \,.\, ij}{\sqrt{(ai \,.\, aj \,.\, bi \,.\, bj)}} \equiv \overline{ab},$$

(which is a function of anharmonic ratios, and therefore an invariant,) is called the *power* of the points a, b in regard to one another, or of

* On this division see Dr. Klein, "Ueber die so-genannte Nicht-Euklidische Geometrie," Math. Annalen, Bd. 4. The second case is the geometry of Lobatschewsky and Bolyai.

either in regard to the other. The *distance* of these two points is an angle θ such that $\sin\theta = \overline{ab}$.

Similarly, if through the line of intersection of the planes A, B there be drawn the tangent planes I, J to the absolute, the power of the planes A, B in regard to one another is the quantity

$$\frac{\mathrm{AB}\,.\,\mathrm{IJ}}{\sqrt{(\mathrm{AI}\,.\,\mathrm{AJ}\,.\,\mathrm{BI}\,.\,\mathrm{BJ})}} = \overline{\mathrm{AB}},$$

and the angle between them is an angle ϕ such that

$$\sin\phi = \overline{\mathrm{AB}}.$$

(3) If two points are conjugate in regard to the absolute, they are distant a *quadrant* from one another; if two lines or planes are conjugate in regard to the absolute, they are at right angles. Thus all the points at a quadrant distance from a given point are situate on its polar plane in regard to the absolute, and every plane through it cuts this polar plane at right angles. Every line has a polar line in regard to the absolute, such that every point on the polar line is distant a quadrant from every point on the line; and every line which is at right angles to either meets the other. Through an arbitrary point can in general be drawn *one* line perpendicular to a given plane; namely, the line joining the point to the pole of the plane. If, however, the point *is* the pole of the plane, every line through it is perpendicular to the plane. Similarly, from a point not on the polar of a given line can be drawn one and only one perpendicular to the line; namely, the line through the point which meets the given line and its polar.

(4) *In general, two lines can be drawn so that each meets two given lines at right angles, and these are polars of one another.* One line may therefore be converted into another by rotation about two polar axes. These axes are determined as the lines which meet the two given lines and their polars. If we travel continuously along one of these lines and draw perpendiculars on the other, one of these axes determines the shortest distance between the lines, and the other the longest. If then these two are equal, the lines are equidistant along their whole length. Thus *there is a case of exception in which two lines and their polars belong to the same set of generators of a hyperboloid; the lines are then equidistant along their whole length, and meet the same two generators of one system of the absolute.* I shall use the word *parallel* to denote two lines so situated; and they shall be called *right* parallel or *left* parallel according as one is converted into the other by a right-handed or left-handed twist. Through an arbitrary point can be drawn one right parallel and one left parallel to a given line; the angle between them is twice the distance of the point from the line. There are many points of analogy between the *parallels* here defined and those of parabolic geometry. Thus, if a line meet two parallel lines, it makes equal

angles with them; and a series of parallel lines meeting a given line constitute a ruled surface of zero curvature. The geometry of this surface is the same as that of a finite parallelogram whose opposite sides are regarded as identical.

(5) A twist-velocity of a rigid body must be regarded as having *two* axes. For a motion of translation along any axis is the same thing as a rotation about the polar axis, and *vice versâ*. Hence a twist-velocity is compounded of rotation-velocities about two polar axes; say these are θ, ϕ. Then the motion may be regarded either as a twist-velocity about a screw whose pitch is $\frac{\phi}{\theta}$ and whose axis is the first axis, or about a screw whose pitch is $\frac{\theta}{\phi}$ and whose axis is the polar axis. In general, then, a motor has two axes, and is expressible in one way only as the sum of two polar rotors. There is, however, one case of exception in which the axes of a motor are indeterminate; that, namely, in which the magnitudes of the two polar rotors are equal.* If a rigid body receive at the same time a rotation about an axis and an equal translation along it, all the points of the body will describe parallel straight lines; and the motion of the body is at the same time a rotation about any one of these lines combined with an equal translation along it. Such a motion may be adequately represented by a line of given length drawn through any point whatever parallel to a given line. A motor of pitch unity, or which is its own polar, may therefore be regarded as having the nature of a *vector*, and shall in future be denoted by that name. For we may define a vector as a motor whose axes are indeterminate; and the case we are now considering is the only case of such indetermination which occurs in elliptic geometry. Vectors will be called *right* or *left* according as the twist of them is right- or left-handed.

Prop.: *Every motor is the sum of a right and a left vector.* For let A be a motor, and A′ the polar motor; then we have $A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$. Now $A+A'$ and $A-A'$ are both motors of pitch unity, but one right-handed and the other left-handed.

IV.

A fixed point being chosen as origin, let three lines perpendicular to one another be drawn through it, and let three unit-rotors having these lines as axes be denoted by the symbols i, j, k. Then every rotor through the origin will be denoted by an expression of the form $ix+jy+kz$, where x, y, z are scalar quantities, or the ratios of magnitudes. The symbols i, j, k shall have also another meaning; viz., each

* This motion is described in another connection by Drs. Klein and Lie. Math. Annalen, Bd. 4; it is a transformation of the absolute into itself in which two generators remain unaltered.

shall signify the rotation through a right angle about its axis of any rotor which meets that axis at right angles. When they are performed on rotors passing through the origin, these operations satisfy the equations $i^2 = j^2 = k^2 = ijk = -1$, by the ordinary rules of quaternions; and it is easy to see that the same equations hold good when the operations are performed on rotors not passing through the origin. The compound symbol $ix+jy+kz$ is also to have an analogous secondary meaning; viz., a rectangular rotation about the axis of the rotor which it previously denoted, combined with a tensor $\sqrt{(x^2+y^2+z^2)}$. It can operate only on a rotor which meets its axis at right angles. This being so, the ratio of any two rotors through the origin is a *quaternion* of the form $q \equiv w+ix+jy+kz \equiv w+\rho$, say. The axis ρ of this quaternion is perpendicular to the plane of the two rotors. If α be a rotor through the origin and q a quaternion, the product $q\alpha$ can be formed according to the Hamiltonian rules of multiplication, and is in general a quaternion r. In this general case the equation $q\alpha = r$ can only be interpreted by giving to α its *secondary* meaning; and the translation of this statement into words is as follows:—If a rotor be capable of being successively operated upon by the rectangular versor α and the quaternion q, the final result will be the same as if it had been originally operated upon by the quaternion r. If, however, the axes of q and α are at right angles, the scalar part of r will be wanting, and we may write the equation $q\alpha = \rho$. This equation is now susceptible of a *primary* interpretation; viz., the quaternion q operating on the rotor α produces the rotor ρ; although the *secondary* interpretation does not cease to be true.

With such conventions, the two sides of the equation

$$(q+r)\,s = qs+rs$$

(in which q, r, s are quaternions) have always the same meaning when both are interpretable; which is what is meant by saying that the distributive law holds good for these symbols.

The ratio of two rotors which do not meet is a twist which in general has perfectly definite axes. But when the rotors are polars of one another, the axes of the twist are indeterminate; for any line meeting both meets them at right angles, and will serve for an axis. It is therefore always possible to find a twist which shall simultaneously convert two given rotors into their polars; and any two rectangular twists with pitch 1 or -1 have a pair of common rotors on which they can operate, and which they convert into one another. Hence we may consider that

All rectangular twists of pitch 1 *are equivalent to one another; and all rectangular twists of pitch* -1 *are equivalent to one another.*

The rectangular twist of pitch 1 shall be denoted by the symbol ω; the expression $\omega\alpha$ will denote the rotor polar to α and equal to it in magnitude, obtained from it by a left-handed twist. During the

operation of this twist, every point of the rotor describes a straight line; if therefore the twist be continued through two right angles, the rotor will be replaced in its original position, *not* reversed; we have therefore $\omega^2 = 1$.

Every motor can be expressed as the sum of two rotors, one passing through the origin and the other being polar to a rotor through the origin. The general expression for a motor is therefore

$$\alpha + \omega\beta.$$

This will represent a *rotor* if the two rotor constituents intersect, or if each is perpendicular to the polar of the other; or if $S\alpha\beta = 0$.

Let now $$\xi = \frac{1+\omega}{2}, \quad \eta = \frac{1-\omega}{2};$$

then
$$\xi^2 = \frac{1+2\omega+\omega^2}{4} = \frac{2+2\omega}{4} = \xi,$$
$$\eta^2 = \frac{1-2\omega+\omega^2}{4} = \frac{2-2\omega}{4} = \eta,$$
$$\xi\eta = \frac{1-\omega^2}{4} = 0.$$

Any motor $\alpha+\omega\beta$ can also be expressed in the form $\xi\gamma+\eta\delta$. It is clear that $\xi\gamma$ is the right vector part of this motor, and that $\eta\delta$ is the left vector part. If we multiply $\xi\gamma+\eta\delta$ by ξ, the result is merely $\xi\gamma$; so the effect of multiplying a motor by ξ is merely to pick out the right vector part of it. The symbols ξ, η are thus in a certain sense *selective* symbols, and are analogous to the S and V of quaternions.

Ratio of two motors.—We can find immediately now the operation which converts a motor $\xi\gamma+\eta\delta$ into a motor $\xi\alpha+\eta\beta$. For if we perform the operation
$$\left(\xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta}\right)(\xi\gamma+\eta\delta),$$
remembering the laws of multiplication of ξ, η, we obtain the result $\xi\alpha+\eta\beta$. If then $\frac{\alpha}{\gamma} = q$, $\frac{\beta}{\delta} = r$, we may write

$$\frac{\xi\alpha+\eta\beta}{\xi\gamma+\eta\delta} = \xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta} = \xi q + \eta r,$$

and the latter may be written in the form

$$\frac{q+r}{2} + \omega\cdot\frac{q-r}{2} = s + \omega t,$$

showing that *the ratio of two motors is a biquaternion.*

The motor $\xi\alpha+\eta\beta$ will be a *rotor* if

$$S(\alpha+\beta)(\alpha-\beta) = 0,$$

or if
$$T\alpha = T\beta;$$

and it is easy to see from this that the biquaternion $\xi q+\eta r$ will be a *twist*, or the ratio of two rotors, if $Tq = Tr$.

V.

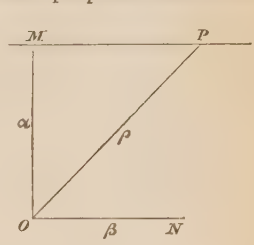
1. *Position-Rotor of a Point.*—The coordinates of a point in regard to a quadrantal tetrahedron 1234 being x_1, x_2, x_3, x_4, the equation to the absolute is $\Sigma x^2 = 0$. The rotor from the origin (the point 4) to the point x is represented by $i_1\frac{x_1}{x_4} + i_2\frac{x_2}{x_4} + i_3\frac{x_3}{x_4}$, or $\Sigma i_k\frac{x_k}{x_4}$ $(k = 1, 2, 3)$, where i_1, i_2, i_3 are rotors along the edges of the tetrahedron from the origin to the middle points of the edges. The tensor of this rotor is the tangent of the angular distance from the origin to the point it represents. For if

$$\rho = i_1\frac{x_1}{x_4} + i_2\frac{x_2}{x_4} + i_3\frac{x_3}{x_4},$$

$$[\mathrm{T}\rho]^2 = \frac{x_1^2 + x_2^2 + x_3^2}{x_4^2} = \tan^2\widehat{ox}, \text{ where } o \text{ is the origin.}$$

The angular distance from the origin to a point has an infinite number of values, which differ by multiples of π. If therefore a rotor be considered to have this angular distance as its length, the rotor of a point can only be defined by such an equation as $\breve{\rho} = \breve{\alpha} \pmod{\breve{\pi}_a}$. To obviate this indetermination, there is required a one-valued unicursal function having the period π; the tangent of the angular distance is hereby completely singled out.

2. *Equation of a Straight Line.*—Let OM be the perpendicular from the origin O upon the straight line MP; and let ON be a line perpendicular to OM in the plane MOP. Then from the triangle MOP we have $\frac{\tan \mathrm{OM}}{\tan \mathrm{OP}} = \cos \mathrm{MOP}$;



or if $\mathrm{OM} = \alpha$, $\mathrm{OP} = \rho$, $\mathrm{ON} = \beta$,

$$\mathrm{T}\alpha = \mathrm{T}\rho \cos \mathrm{MOP};$$

so that α is the component of ρ in the direction OM, and we have $\rho = \alpha + \beta x$, where x is some scalar.

By varying x, then, we get all the points in the line MP. But if α_1 is *any* particular value of ρ, the equation may just as well be written

$$\rho = \alpha_1 + \beta x,$$

where now α_1 is not necessarily perpendicular to β.

This form may be reduced to the preceding as follows:

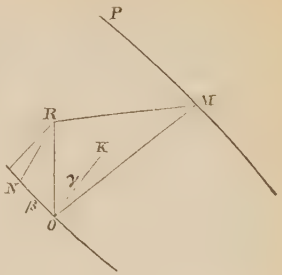
To find the perpendicular from O, put $\delta\mathrm{T}\rho = 0$; this gives

$$\mathrm{S}\alpha_1\beta + \beta^2 x = 0,$$

and the equation becomes $\rho = \alpha_1 - \beta\,\mathrm{S}\frac{\alpha_1}{\beta} - \beta x$,

where $\alpha_1 - \beta\,\mathrm{S}\frac{\alpha_1}{\beta} = \alpha$ of the former equation.

3. *Rotor along Straight Line whose Equation is given.*



Let OR be the rotor through the origin which has right parallelism with MP. Then $\angle \mathrm{NOR} = \mathrm{OM}$. Let OK be perpendicular to ON and OM, and of such length that $\frac{\tan \mathrm{OK}}{\tan \mathrm{ON}} = \tan \mathrm{NOR}$. Then, if $\gamma = \mathrm{OK}$, $\mathrm{OR} = \beta + \gamma$. Now $\frac{\mathrm{T}\gamma}{\mathrm{T}\beta} = \mathrm{T}\alpha$, and $\mathrm{U}\gamma = \mathrm{U}\alpha\beta$, since γ is perpendicular to α and β. Hence $\gamma = \alpha\beta$; and if R be a rotor along MP, m a scalar,

$$\text{right vector of } \mathrm{R} = \xi\mathrm{R} = m\xi(\beta+\gamma) = m\xi(\beta+\alpha\beta),$$

so $$\text{left vector of } \mathrm{R} = \eta\mathrm{R} = m\eta(\beta-\gamma) = m\eta(\beta-\alpha\beta);$$

therefore $$\mathrm{R} = m(\beta+\omega\alpha\beta).$$

Now if R have the same length as β, we have

$$\beta^2 = \mathrm{R}^2 = m^2(\beta^2+\overline{\alpha\beta}^2) = m^2\beta^2(1-\alpha^2);$$

therefore $$\mathrm{R} = \frac{\beta+\omega\alpha\beta}{\sqrt{(1-\alpha^2)}}.$$

Conversely, equation to axis of rotor $\gamma+\omega\delta$ is

$$\rho = \frac{\delta}{\gamma} + \gamma x.$$

This finds the rotor in the case in which $\rho = \alpha+\beta x$, where $\mathrm{S}\alpha\beta = 0$. But in the general case we have only to write the equation in the form

$$\rho = \alpha - \beta\,\mathrm{S}\frac{\alpha}{\beta} + \beta x,$$

whence
$$\mathrm{R} = \frac{\beta+\omega\left(\alpha-\beta\,\mathrm{S}\frac{\alpha}{\beta}\right)\beta}{\sqrt{\left(1-\alpha^2-\beta^2\mathrm{S}^2\frac{\alpha}{\beta}+2\mathrm{S}\alpha\beta\,\mathrm{S}\frac{\alpha}{\beta}\right)}}$$
$$= \frac{\beta+\omega\mathrm{V}\alpha\beta}{\sqrt{\left(1+\mathrm{S}\alpha\beta\,\mathrm{S}\frac{\alpha}{\beta}-\alpha^2\right)}}.$$

4. *Rotor ab joining Points whose Position-Rotors are α, β.*

The equation of this rotor is

$$\rho = \alpha + (\beta-\alpha)x,$$

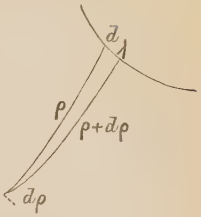
whence $$m\mathrm{R} = \beta-\alpha+\omega\mathrm{V}\alpha\beta.$$

Now if a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4 are the coordinates of the points, we have $[\mathrm{TR}]^2 = \tan^2 ab = \frac{\Sigma(a_hb_k-a_kb_h)^2}{(\Sigma a_hb_h)^2} = -\frac{(\alpha-\beta)^2+(\mathrm{V}\alpha\beta)^2}{(1-\mathrm{S}\alpha\beta)^2}$,

therefore $$R = \frac{\beta - \alpha + \omega V\alpha\beta}{1 - S\alpha\beta}.$$

Cor.—If ρ be the rotor of a variable point on a curve, $d\lambda$ a rotor along the tangent of length equal to the arc of the curve between ρ and $\rho + d\rho$, we have

$$d\lambda = \frac{d\rho + \omega V\rho d\rho}{1 - \rho^2}.$$



5. *Rotor parallel to β through Point whose Position-Rotor is α.*

The general equation to a line through the point α is $\rho = \alpha + \lambda x$, where λ is any rotor through the origin. A rotor along this line is $\lambda + \omega V\alpha\lambda$; if this is right parallel to β, we have

$$\xi(\lambda + V\alpha\lambda) = \xi\beta, \qquad [\xi\omega = \xi]$$

or $$\lambda + V\alpha\lambda = \beta.$$

Operating by $S\alpha$, we have, since $S \cdot \alpha V\alpha\lambda = 0$,

$$S\alpha\lambda = S\alpha\beta,$$

whence, by addition, $$\lambda + \alpha\lambda = \beta + S\alpha\beta,$$

and $$\lambda = (1+\alpha)^{-1}(\beta + S\alpha\beta) = \beta - (1+\alpha)^{-1} V\alpha\beta.$$

The rotor required is $\lambda + \omega V\alpha\lambda$, or $\lambda + \omega(\beta - \lambda)$. This becomes, then,

$$\beta - (1+\alpha)^{-1} V\alpha\beta + \omega(1+\alpha)^{-1} V\alpha\beta = \beta - 2\eta(1+\alpha)^{-1} V\alpha\beta.$$

Instead of operating by $S\alpha$ on the equation

$$\lambda + V\alpha\lambda = \beta,$$

we might have operated with $V\alpha$, and got

$$V\alpha\lambda + \alpha V\alpha\lambda = V\alpha\beta, \quad \text{since} \quad V \cdot \alpha V\alpha\lambda = \alpha V\alpha\lambda,$$

therefore $$V\alpha\lambda = (1+\alpha)^{-1} V\alpha\beta,$$

and $$\lambda = \beta - V\alpha\lambda = \beta - (1+\alpha)^{-1} V\alpha\beta.$$

Similarly, we have for the rotor *left* parallel to β,

$$\lambda = \beta + (1-\alpha)^{-1} V\alpha\beta,$$

and the rotor is

$$\lambda + \omega(\lambda - \beta) = \beta + (1-\alpha)^{-1} V\alpha\beta + \omega(1-\alpha)^{-1} V\alpha\beta$$
$$= \beta + 2\xi(1-\alpha)^{-1} V\alpha\beta.$$

On a Correspondence of Points in relation to two Tetrahedra.

By Prof. Cayley, LL.D., F.R.S.

[*Read June 12th*, 1873.]

The following question has been considered by R. Sturm in an interesting paper, "Das Problem der Projectivität und seine Anwendung auf die Flächen zweiten Grades," Clebsch and Neumann, t. I. (1870), pp. 533—574:—Given *in plano* two groups of the same number (5, 6, or 7) of points, to find points P, P′ homographically related to these two groups respectively; viz., the lines from P to the points of the first group and those from P′ to the points of the second group are to be homographic pencils. In the present paper I require only a particular form of these results; viz., in each group two of the points are the circular points at infinity; or, disregarding these, we have two groups of 3, 4, or 5 points such that the points of the first group at P, and those of the second group at P′, subtend equal angles. I give for this particular case an independent analytical investigation; but I will first state the results included in the more general ones obtained by Sturm.

If the points A, B, C at P and the points A′, B′, C′ at P′ subtend equal angles, then to any given position of the one point corresponds a single position of the other point; viz., the two points have a (1, 1) correspondence; the nature of this being, that to any line in the one figure corresponds in the other figure a quintic curve, having 6 dps.; viz., the three points, the two circular points at infinity I, J, and one other fixed point of that figure [say for the first figure this fixed point is (ABC)].

If the points A, B, C, D at P and the points A′, B′, C′, D′ at P′ subtend equal angles, then the locus of each point is a cubic curve; viz., the locus of P passes through A, B, C, D, I, J and the four fixed points (ABC), (ABD), (ACD), (BCD); and the like for the locus of P′.

Finally (although this is a theorem which I do not require), if the points A, B, C, D, E at P and the points A′, B′, C′, D′, E at P′ subtend equal angles, then there are three positions of each point.

The problem I propose to consider is: Given the tetrahedra ABCD and A′B′C′D′, it is required in the planes ABC and A′B′C′ respectively to find the points P, P′ such that A, B, C, D at P, and A′, B′, C′, D′ at P′, subtend equal angles. I was led to this by the more general problem, which I do not at present discuss: Given the two tetrahedra, it is required to find the loci of the points P, P′ such that A, B, C, D at P, and A′, B′, C′, D′ at P′, subtend equal angles.

Here, drawing from D, D′ the perpendiculars DK, D′K′ on the planes

ABC and A′B′C′ respectively, we have A, B, C, K at P, and A′, B′, C′, K′ at P′, subtending equal angles, and such that the distances PK and P′K′ are proportional to the heights of the tetrahedra (viz., the triangles PDK and P′D′K′ are obviously similar). The required points P, P′ are each the intersection of two loci, viz.:

1. P is such that A, B, C, K at P, and A′, B′, C′, K′ at P′, subtend equal angles; locus is a cubic through A, B, C, K, I, J, (ABC), (ABK), (ACK), (BCK).
2. P is such that A, B, K at P, and A′, B′, K′ at P′, subtend equal angles, and that PK and P′K′ are in a given ratio; locus is a certain octic curve Ω.

And the required positions of P are obtained as the intersections of the two loci.

I proceed to the analytical investigation.

Preliminary Formulæ.

1. Consider a triangle ABC, and let the position of a point P be determined by means of its coordinates x, y, z, which are equal to the perpendicular distances of P from the sides, each divided by the perpendicular distance of the opposite vertex (as usual, x, y, z are positive for a point within the triangle); or what is the same thing, x, y, z = PBC, PCA, PAB, divided each by ABC, whence identically $x+y+z=1$.

Suppose for a moment the rectangular coordinates of A, B, C are (α_1, β_1), (α_2, β_2), (α_3, β_3) respectively; and that those of P are X, Y. Also let the sides BC, CA, AB be $= a, b, c$ respectively.

We have

$$\begin{aligned} X &= \alpha_1 x + \alpha_2 y + \alpha_3 z, \\ Y &= \beta_1 x + \beta_2 y + \beta_3 z, \\ 1 &= \quad x + \quad y + \quad z. \end{aligned}$$

And if we consider a second point P′, the coordinates of which are x', y', z' and X′, Y′, we have the like relations between these quantities. Calling δ the distance of the two points P, P′, we may write

$$\begin{aligned} \delta^2 = \quad & \{\alpha_1(x-x') + \alpha_2(y-y') + \alpha_3(z-z')\}^2 \\ & + \{\beta_1(x-x') + \beta_2(y-y') + \beta_3(z-z')\}^2 \\ & - \{(\alpha_1^2+\beta_1^2)(x-x') + (\alpha_2^2+\beta_2^2)(y-y') + (\alpha_3^2+\beta_3^2)(z-z')\} \\ & \qquad \times \{(x-x') + (y-y') + (z-z')\}, \end{aligned}$$

the last term being in fact $= 0$; viz., this is

$$\begin{aligned} \delta^2 = & -\{(\alpha_2-\alpha_3)^2 + (\beta_2-\beta_3)^2\}(y-y')(z-z') \\ & -\{(\alpha_3-\alpha_1)^2 + (\beta_3-\beta_1)^2\}(z-z')(x-x') \\ & -\{(\alpha_1-\alpha_2)^2 + (\beta_1-\beta_2)^2\}(x-x')(y-y'); \end{aligned}$$

or what is the same thing, the expression for the distance δ^2 of the two points P, P′ is

$$\delta^2 = -a^2(y-y')(z-z') - b^2(z-z')(x-x') - c^2(x-x')(y-y'),$$

which expression may be modified by means of the identical equations

$$1 = x+y+z, \quad 1 = x'+y'+z';$$

viz., writing $yz'-y'z,\ zx'-z'x,\ xy'-x'y = \xi, \eta, \zeta,$

we have
$$\begin{aligned} x-x' &= x(x'+y'+z') - x'(x+y+z) = \zeta-\eta, \\ y-y' &\qquad\qquad\qquad\qquad = \xi-\zeta, \\ z-z' &\qquad\qquad\qquad\qquad = \eta-\xi; \end{aligned}$$

and consequently
$$\begin{aligned} \delta^2 = \ & a^2(-\xi^2-\eta\zeta+\zeta\xi+\xi\eta) \\ & + b^2(-\eta^2+\eta\zeta-\zeta\xi+\xi\eta) \\ & + c^2(-\zeta^2+\eta\zeta+\zeta\xi-\xi\eta). \end{aligned}$$

2. Treating x', y', z' as constants and x, y, z as current coordinates, the formula for δ^2 is of course the equation of a circle, centre x', y', z' and radius δ. It thus also appears that the general equation of a circle is

$$-a^2yz-b^2zx-c^2xy+(Lx+My+Nz)(x+y+z) = 0;$$

viz., writing $-a^2yz-b^2zx-c^2xy = U$, and $x+y+z = \Omega$, this is

$$U+(Lx+My+Nz)\Omega = 0,$$

where $U=0$ is the circle circumscribed about the triangle ABC, and $\Omega=0$ is the line infinity. Of course the general equation of a circle passing through the points (B, C) is $U+Lx\Omega = 0$, and similarly those of circles through (C, A) and through (A, B) are $U+My\Omega = 0$ and $U+Nz\Omega = 0$ respectively. But we require the interpretation of the coefficients L, M, N which enter into these equations.

3. Considering the triangle ABC, if through B, C we have a circle, this is by the side BC divided into two segments, and I consider that lying on the same side with A as the positive segment, and define the angle of the circle to be the angle in this positive segment. It is clear that if we have *within* the triangle a point P, and, through this point and (B, C), (C, A), (A, B) respectively, three circles, then if α, β, γ be the angles of these circles, we have $\alpha+\beta+\gamma = 2\pi$; and conversely, if the circles through (B, C), (C, A), (A, B) are such that their angles α, β, γ satisfy the relation $\alpha+\beta+\gamma = 2\pi$, then the three circles meet in a point. But it is further to be noticed, that if, producing the sides of the triangle so as to divide the plane into seven spaces, the triangle, three trilaterals, and three bilaterals, we take the point P within one of the bilaterals, we still have $\alpha+\beta+\gamma = 2\pi$; but taking it within one of the trilaterals, we have $\alpha+\beta+\gamma = \pi$. And the converse theorem is, that if the three circles (B, C), (C, A), (A, B) are such that $\alpha+\beta+\gamma = \pi$ or 2π, then the circles meet in a point; viz., if the sum

is 2π, then this point lies in the triangle or one of the bilaterals; but if the sum is $=\pi$, then this point lies in a trilateral.

4. I seek for the equation of a circle through the points B, C, and containing the angle L. The equation in rectangular coordinates is easily seen to be

$$(X-\alpha_2)(X-\alpha_3)+(Y-\beta_2)(Y-\beta_3) - \cot L\{(\beta_2-\beta_3)X-(\alpha_2-\alpha_3)Y+\alpha_2\beta_3-\alpha_3\beta_2\} = 0.$$

In fact, this is the equation of a circle through (B, C); and taking fo a moment the origin at B, and axis of X to coincide with BC, or writing $\alpha_2, \beta_2 = 0, 0$; $\alpha_3, \beta_3 = a, 0$, the equation is

$$X(X-a)+Y^2-aY\cot L = 0,$$

viz., the equation of the tangent at B is $-aX-aY\cot L = 0$, that is, $Y = -X\tan L$, or the angle in the positive segment is $= L$.

If for a moment λ, μ, ν are the inclinations of the sides of the triangle ABC to the axis of X, then A, B, C being the angles, we may write

$$\begin{aligned} \mu-\nu &= \pi-A, \\ \nu-\lambda &= \pi-B, \\ \lambda-\mu &= -\pi-C, \end{aligned}$$

and

$$\begin{aligned} X-\alpha_2 &= (\alpha_1x+\alpha_2y+\alpha_3z)-\alpha_2(x+y+z) = c\cos\nu\,.\,x-a\cos\lambda\,.\,z, \\ X-\alpha_3 &= (\alpha_1x+\alpha_2y+\alpha_3z)-\alpha_3(x+y+z) = -b\cos\mu\,.\,x+a\cos\lambda\,.\,y, \\ Y-\beta_2 &= \beta_1x+\beta_2y+\beta_3z-\beta_2(x+y+z) = c\sin\nu\,.\,x-a\sin\lambda\,.\,z, \\ Y-\beta_3 &= \beta_1x+\beta_2y+\beta_3z-\beta_3(x+y+z) = -b\sin\nu\,.\,x+a\sin\lambda\,.\,z; \end{aligned}$$

whence

$$(X-\alpha_2)(X-\alpha_3)+(Y-\beta_2)(Y-\beta_3)$$
$$= -a^2yz-b^2zx-c^2xy+bc\cos A\,.\,x^2+(b^2-ab\cos C)zx+(c^2-ac\cos B)xy;$$

viz., this is $= -a^2yz-b^2zx-c^2xy+bc\cos A\,.\,x(x+y+z)$.

Moreover, if $\Delta =$ twice the area of the triangle, then

$$\begin{aligned} (\beta_2-\beta_3)X-(\alpha_2-\alpha_3)Y+\alpha_2\beta_3-\alpha_3\beta_2 &= \Delta x(x+y+z) \\ &= bc\sin A\,.\,x(x+y+z); \end{aligned}$$

so that the equation becomes

$$-a^2yz-b^2zx-c^2xy+bc\sin A(\cot A-\cot L)\,x(x+y+z) = 0,$$

or what is the same thing,

$$-a^2yz-b^2zx-c^2xy+\Delta(\cot A-\cot L)\,x(x+y+z) = 0,$$

or if we please, $-a^2yz-b^2zx-c^2xy+\Delta(\cot A-\cot L)x = 0$.

Writing as before,

$$\begin{aligned} -a^2yz-b^2zx-c^2xy &= U, \\ x+y+z &= \Omega, \end{aligned}$$

the equation is

$$U+\Delta(\cot A-\cot L)\Omega x = 0;$$

or forming the like equations of two other similar circles, we have the

circles (B, C), (C, A), (A, B) containing the angles L, M, N respectively; and the equations are

$$\begin{aligned} &U+\Delta(\cot A-\cot L)\,\Omega x=0,\\ &U+\Delta(\cot B-\cot M)\,\Omega y=0,\\ &U+\Delta(\cot C-\cot N)\,\Omega z=0. \end{aligned}$$

Correspondence, A, B, C at P, and A′, B′, C′ at P′, subtending equal angles.

4. Consider now the two figures A′, B′, C′, subtending at P′ the same angles L, M, N which A, B, C subtend at P; then we have

$$\begin{aligned} &\frac{U}{\Omega\Delta x}+\cot A-\cot L=0, \quad \frac{U'}{\Omega'\Delta' x'}+\cot A'-\cot L=0,\\ &\frac{U}{\Omega\Delta y}+\cot B-\cot M=0, \quad \frac{U'}{\Omega'\Delta' y'}+\cot B'-\cot M=0,\\ &\frac{U}{\Delta\Omega z}+\cot C-\cot N=0, \quad \frac{U'}{\Omega'\Delta' z'}+\cot C'-\cot N=0; \end{aligned}$$

and thence

$$\begin{aligned} &\frac{U}{\Omega\Delta x}+\cot A=\frac{U'}{\Omega'\Delta' x'}+\cot A',\\ &\frac{U}{\Omega\Delta y}+\cot B=\frac{U'}{\Omega'\Delta' y'}+\cot B',\\ &\frac{U}{\Omega\Delta z}+\cot C=\frac{U'}{\Omega'\Delta' z'}+\cot C'; \end{aligned}$$

and consequently

$$\begin{aligned} \frac{1}{x'}:\frac{1}{y'}:\frac{1}{z'} &= \frac{U}{\Omega\Delta x}+\cot A-\cot A'\\ &:\frac{U}{\Omega\Delta y}+\cot B-\cot B'\\ &:\frac{U}{\Omega\Delta z}+\cot C-\cot C', \end{aligned}$$

or what is the same thing,

$$\begin{aligned} x':y':z' &= \frac{x}{U+\Delta(\cot A-\cot A')\,\Omega x}\\ &:\frac{y}{U+\Delta(\cot B-\cot B')\,\Omega y}\\ &:\frac{z}{U+\Delta(\cot C-\cot C')\,\Omega z}, \end{aligned}$$

where observe that the equations

$$\begin{aligned} &U+\Delta(\cot A-\cot A')\,\Omega x=0,\\ &U+\Delta(\cot B-\cot B')\,\Omega y=0,\\ &U+\Delta(\cot C-\cot C')\,\Omega z=0, \end{aligned}$$

represent circles (B, C), (C, A), (A, B) containing the angles A′, B′, C′;

and since $A'+B'+C'=\pi$, these meet in a point O. We may for convenience write

$$x' : y' : z' = \frac{BC}{BCO} : \frac{CA}{CAO} : \frac{AB}{ABO},$$

where $BC=0$ denotes $(x=0)$ the line BC; $BCO=0$ the circle through BCO. And of course, in like manner,

$$x : y : z = \frac{B'C'}{B'C'O'} : \frac{C'A'}{C'A'O'} : \frac{A'B'}{A'B'O'};$$

so that the points P, P′ have a rational, or (1, 1), correspondence.

Writing

$$x' : y' : z' = BC \,.\, CAO \,.\, ABO : CA \,.\, ABO \,.\, BCO : AB \,.\, BCO \,.\, CAO$$
$$= X : Y : Z$$

suppose, X, Y, Z are quintic functions of x, y, z, and the curve in the first figure corresponding to the line $\alpha x' + \beta y + \gamma z' = 0$ of the second figure is

$$\alpha X + \beta Y + \gamma Z = 0;$$

viz., this is a quintic curve having dps. at each of the points A, B, C, O, I, J. In fact, if for BCO we write BCOIJ, and so for the other two circles respectively, we have in an algorithm which will be at once understood $X = BC \,.\, CAOIJ \,.\, ABOIJ, = (ABCOIJ)^2$, and similarly $Y = Z, = (ABCOIJ)^2$, or the curve is $(ABCOIJ)^2 = 0$.

A, B, C, D at P and A′, B′, C′, D′ at P′ subtending equal angles.

5. Consider now *in plano* the points A, B, C, D which at P, and the points A′, B′, C′, D′ which at P′, subtend equal angles. Let a, b, c, f, g, h denote the perpendicular distances of P from the lines BC, CA, AB, AD, BD, CD respectively; and the like as to a', b', c', f', g', h'. Observe that, neglecting constant factors, a, b, c are what were before represented by x, y, z; we may consider the coordinates of P in regard to the triangles ABC, BCD, CAD, ABD to be (a, b, c), (a, h, g), (b, f, h), (c, g, f) respectively. We have in regard to ABC the point O as before, and in regard to BCD, CAD, ABD the points O_1, O_2, O_3 respectively. Then A, B, C at P and A′, B′, C′ at P′ subtending equal angles, we may write

$$a' : b' : c' = \frac{a}{BCO} : \frac{b}{CAO} : \frac{c}{ABO};$$

viz., $BCO=0$ is here the circle through B, C, O, and the like for CAO and ABO, the expressions being multiplied into the proper constant factors to take account of the constant factors whereby a, b, c and a', b', c' differ from x, y, z and x', y', z' respectively.

We have in like manner

$$a' : h' : g' = \frac{a}{\mathrm{BCO}_1} : \frac{h}{\mathrm{CDO}_1} : \frac{g}{\mathrm{BDO}_1},$$

$$b' : f' : h' = \frac{b}{\mathrm{CAO}_2} : \frac{f}{\mathrm{ADO}_2} : \frac{h}{\mathrm{CDO}_2},$$

$$c' : g' : f' = \frac{c}{\mathrm{ABO}_3} : \frac{g}{\mathrm{BDO}_3} : \frac{f}{\mathrm{ADO}_3}.$$

From the ratios of (f', g', h'), (b', c', f'), (c', a', g'), (a', b', h') respectively we deduce

$$\mathrm{CDO}_1 . \mathrm{ADO}_2 . \mathrm{BDO}_3 - \mathrm{BDO}_1 . \mathrm{CDO}_2 . \mathrm{ADO}_3 = 0,$$
$$\mathrm{CAO} . \mathrm{ABO}_3 . \mathrm{ADO}_2 - \mathrm{ABO} . \mathrm{ADO}_3 . \mathrm{CAO}_2 = 0,$$
$$\mathrm{ABO} . \mathrm{BCO}_1 . \mathrm{BDO}_3 - \mathrm{BCO} . \mathrm{BDO}_1 . \mathrm{ABO}_3 = 0,$$
$$\mathrm{BCO} . \mathrm{CAO}_2 . \mathrm{CDO}_1 - \mathrm{CAO} . \mathrm{CDO}_2 . \mathrm{BCO}_1 = 0,$$

each of which equations represents a sextic curve; and admitting that it can be shown that these pass through O, O_1, O_2, O_3 respectively, the forms are

$$\mathrm{D^3A\,B\,C\,O}\,O_1O_2O_3\mathrm{I^3J^3} = 0,$$
$$\mathrm{D\,A^3B\,C\,O}\,O_1O_2O_3\mathrm{I^3J^3} = 0,$$
$$\mathrm{D\,A\,B^3C\,O}\,O_1O_2O_3\mathrm{I^3J^3} = 0,$$
$$\mathrm{D\,A\,B\,C^3O}\,O_1O_2O_3\mathrm{I^3J^3} = 0.$$

6. Now the locus of P is evidently a curve, and this can only happen by reason that the four left-hand functions contain a common factor, and the form of them suggests that this common factor is $\mathrm{ABCDO}O_1O_2O_3\mathrm{IJ}$, the four extraneous factors being $\mathrm{D^2I^2J^2}$, $\mathrm{A^2I^2J^2}$, $\mathrm{B^2I^2J^2}$, $\mathrm{C^2I^2J^2}$; viz., $\mathrm{ABCDO}O_1O_2O_3\mathrm{IJ} = 0$ is a cubic curve passing through the ten points; and $\mathrm{D^2I^2J^2} = 0$ a cubic curve through each of the points D, I, J twice; viz., it is the triad of lines IJ, DI, DJ; and the like as to the other extraneous factors $\mathrm{A^2I^2J^2}$, $\mathrm{B^2I^2J^2}$, and $\mathrm{C^2I^2J^2}$. I have not worked out the analysis to verify this *à posteriori;* but, the conclusion agreeing with Sturm, I accept it without further investigation, viz., the result is that A, B, C, D at P and A′, B′, C′, D′ at P′ subtending equal angles, the locus of P is a cubic curve $\mathrm{ABCDO}O_1O_2O_3O_4\mathrm{IJ} = 0$ through the ten points thus represented; and of course the locus of P′ is in like manner a cubic curve $\mathrm{A'B'C'D'O'}O_1'O_2'O_3'O_4'\mathrm{IJ} = 0$ through the ten points thus represented.

A, B, C, D, E at P and A′, B′, C′, D′, E′ at P′ subtending equal angles.

7. We may go a step further, and consider A, B, C, D, E at P and A′, B′, C′, D′, E′ at P′ subtending equal angles. Attending only to the points A, B, C, D and A′, B′, C′, D′, the locus of P is a cubic curve

$$\mathrm{ABCDO}O_1O_2O_3O_4\mathrm{IJ} = 0;$$

and similarly attending to the points A, B, C, E and A′, B′, C′, E′, the locus of P is a cubic curve

$$ABCEOQ_1Q_2Q_3IJ = 0.$$

(Observe that O, as depending only on A, B, C, is the same point as before; but that Q_1, Q_2, Q_3, as depending on E instead of D, are not the same as O_1, O_2, O_3.) The two cubic curves have in common the points A, B, C, I, J, O, and they consequently intersect in three other points; that is, there are three positions of the point P, and of course three corresponding positions of P′.

A, B, C at P and A′, B′, C′ at P′ subtending equal angles, and AP, A′P′ in a given ratio.

Consider, as before, A, B, C at P and A′, B′, C′ at P′ subtending equal angles, and the points P, P′ being moreover such that the distances AP, A′P′ are in a given ratio. And I write for shortness

$$x' : y' : z' = \frac{x}{L} : \frac{y}{M} : \frac{z}{N},$$

where L, M, N denote

$$U + \Delta(\cot A - \cot A')\,\Omega x, \quad U + \Delta(\cot B - \cot B')\,\Omega y,$$
$$U + \Delta(\cot C - \cot C')\,\Omega z,$$

respectively. We have

$$\begin{aligned}(AP)^2 &= -a^2yz - b^2z(x-1) - c^2(x-1)y \\ &= -a^2yz - b^2zx - c^2xy + (b^2z + c^2y)(x+y+z) \\ &= \quad c^2y^2 + b^2z^2 + (b^2 + c^2 - a^2)yz;\end{aligned}$$

or what is the same thing,

$$= c^2y^2 + b^2z^2 + 2bc\cos A \,.\, yz;$$

and similarly

$$(A'P')^2 = c'^2y'^2 + b'^2z'^2 + 2b'c'\cos A' \,.\, y'z'.$$

The required relation therefore is

$$\frac{c'^2y'^2 + b'^2z'^2 + 2b'c'\cos A' \,.\, y'z'}{(x'+y'+z')^2} = \theta^2 \,.\, \frac{c^2y^2 + b^2z^2 + 2bc\cos A \,.\, yz}{(x+y+z)^2};$$

viz., substituting for x', y', z' their values, this is

$$\begin{aligned}L^2(x+y+z)^2\,(b'^2z^2M^2 + c'^2y^2N^2 + 2b'c'\,yzMN\cos A) \\ = \theta^2(b^2z^2 + c^2y^2 + 2bcyz\cos A)\,(xMN + yNL + zLM)^2,\end{aligned}$$

which is an equation of the 12th order. I say that the points A, B, C, O, I, J are each quadruple. In fact, according to the foregoing algorithm, we may write

$$x+y+z = IJ, \quad zM = AB \,.\, CAOIJ, \quad \&c.,$$
$$xL = yM = zN = A^2BCOIJ,$$
$$y = z = A, \quad xMN = BC \,.\, CAOIJ \,.\, ABOIJ, \quad \&c.,$$
$$xMN = yNL = zLM = (ABCOIJ)^2;$$

and the equation is

$$(BCOIJ)^2\,(I\dot{J})^2\,(A^2BCOIJ)^2 = \theta^2.\,A^2(ABCOIJ)^4,$$

that is $$(IJ)^2\,(ABCOIJ)^4 = \theta^2.\,A^2(ABCOIJ)^4;$$

so that the points are each quadruple.

The two Tetrahedra; A, B, C, D at P in ABC and A, B′, C′, D′ at P′ in A′B′C′ subtending equal angles.

I consider now the before-mentioned problem of the two tetrahedra; viz., on the two bases ABC and A′B′C′ respectively, letting fall the perpendiculars DK and D′K′, then first A, B, C, K at P and A′, B′, C′, K′ at P′ subtend equal angles; the locus of P is a cubic curve $ABCKOO_1O_2O_3IJ = 0$ through these ten points. ($O = ABC$ is derived from the points A, B, C; and in like manner $O_1 = BCK$, $O_2 = CAK$, $O_3 = ABK$.)

Next, B, C, K at P and B′, C′, K′ at P′ subtend equal angles, and moreover the distances KP and K′P′ are in a given ratio; the locus of P is a 12-thic curve $(BCKO_1IJ)^4 = 0$,

having each of these six points as a quadruple point. Hence among the 36 intersections of the two curves we have the points B, C, K, O_1, I, J each 4 times, and there remain $36-24$, $=12$ intersections.

The conclusion is that A, B, C, D at a point P of ABC, and A′, B′, C′, D′ at a point P′ of A′B′C′, subtending equal angles, there are 12 positions of P, and of course 12 corresponding positions of P′.

Invariant Conditions for Three Conics having Common Points.

By J. J. Walker, M.A.

[*Read June 12th*, 1873.]

1. The notation adopted in the discussion of the invariant conditions of the concurrence of three conics,

$$u = ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0,$$
$$v = a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy = 0,$$
$$w = a''x^2+b''y^2+c''z^2+2f''yz+2g''zx+2h''xy = 0,$$

is as follows. There are twelve independent invariants of the system, viz., the three invariants of the single conics,

$$\Delta = abc+2fgh-af^2-bg^2-ch^2,$$
$$\Delta' = a'b'c'+ \ldots\ldots\ldots\ldots -c'h'^2,$$
$$\Delta'' = a''b''c''+ \ldots\ldots\ldots\ldots -c''h''^2;$$

the three pairs of invariants of the conics taken two and two,

$$\begin{aligned}\Theta &= a''(b'c'-f'^2)+b''(c'a'-g'^2)+c''(a'b'-g'^2)\\ &\quad +2f''(g'h'-a'f')+2g''(h'f'-b'g')+2h''(f'g'-c'h')\\ \Theta' &= a'(b''c''-f''^2)+\ \ldots\ldots\\ &\quad +2f'(g''h''-a''f'')+\ \ldots\ldots\\ \Theta_1 &= a(b''c''-f''^2)+\ \ldots\ldots\\ &\quad +2f(g''h''-a''f'')+\ \ldots\ldots\\ \Theta_1' &= a''(bc-f^2)+\ \ldots\ldots\\ &\quad +2f''(gh-af)+\ \ldots\ldots\\ \Theta_2 &= a'(bc-f^2)+\ \ldots\ldots\\ &\quad +2f'(gh-af)+\ \ldots\ldots\\ \Theta_2' &= a(b'c'-f'^2)+\ \ldots\ldots\\ &\quad +2f(g'h'-a'f')+\ \ldots\ldots\end{aligned}$$

lastly, the three invariants into which the coefficients of all three enter symmetrically, which may be considered as fundamental:

1°. Of the first order in the coefficients of each,

$$\begin{aligned}\Phi &= a(b'c''+b''c'-2f'f'')+b(c'a''+c''a'-2g'g'')+c(a'b''+a''b'-2h'h'')\\ &\quad +2f(g'h''+g''h'-a'f''-a''f')+2g(h'f''+h''f'-b'g''-b''g')\\ &\quad +2h(f'g''+f''g'-c'h''-c''h')\\ &= a'(bc''+b''c-2f''f)+\ \ldots\ldots\\ &= a''(bc'+b'c-2ff')+\ \ldots\ldots\end{aligned}$$

2°. Of the second order in the coefficients of each,

$$\begin{aligned}X^* &= (ab'c'')^2+4\{(ab'f'')(ac'f'')+4(bc'g'')(b'a'g'')+4(ca'h'')(cb'h'')\}\\ &\quad +8\{(af'g'')(bf'g'')+(af'h'')(cf'h'')+(cg'h'')(bg'h'')-(ag'h'')(bc'f'')\\ &\quad -(cf'g'')(ab'h'')-(bh'f'')(ca'g'')\}+4\{(ab'c')-2(f'g'h')\}\{(fg'h'),\end{aligned}$$

where $(ab'c'')$, $(ab'f'')$ ... $(ag'h'')$ ... stand for the determinants

$$\begin{vmatrix} a & b & c\\ a' & b' & c'\\ a'' & b'' & c''\end{vmatrix},\qquad \begin{vmatrix} a & b & f\\ a' & b' & f'\\ a'' & b'' & f''\end{vmatrix}\ \ldots\ldots\ \begin{vmatrix} a & g & h\\ a' & g' & h'\\ a'' & g'' & h''\end{vmatrix}\ \ldots\ldots$$

3°. Of the fourth order in the coefficients of each conic; viz., the quartic invariant 'S' (Salmon's "Higher Plane Curves," 2nd ed. p. 184) of the Jacobian of the system, multiplied by -81,

$$\begin{aligned}\Psi &= -a_1b_2c_3m+b_2c_3a_2a_3+c_3a_1b_3b_1+a_1b_2c_1c_2+(a_1b_3c_2+b_2c_1a_3+c_3a_2b_1)\,m\\ &\quad -\{a_1(b_3^2c_1+b_1c_2^2)+b_2(c_1^2a_2+c_2a_3^2)+c_3(a_2^2b_3+a_3b_1^2)\}+m^4\\ &\quad -2(b_1c_1+c_2a_3+a_3b_3)\,m^2+3(a_2b_3c_1+a_3b_1c_2)\,m+b_1^2c_1^2+c_2^2a_2^2+a_3^2b_3^2\\ &\quad -(b_1c_1c_2a_2+c_2a_2a_3b_3+a_3b_3b_1c_1),\end{aligned}$$

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* Dr. Salmon's "Conics," p. 348, 5th ed., who, after mentioning its discovery by Mr. Sylvester, points out that the invariant may be obtained by operating on the Jacobian with the contravariant which is the condition that any transversal should be cut in involution by the three conics.

where $a_1 = 3(ag'h'')$, $b_2 = 3(bh'f'')$, $c_3 = 3(cf'g'')$,

$$a_2 = -(ab'g'') - (ah'f''), \quad b_1 = -(ab'f'') - (bg'h''),$$
$$c_1 = -(ca'f'') - (cg'h''), \quad a_3 = -(ca'h'') - (af'g''),$$
$$b_3 = -(bc'h'') - (bf'g''), \quad c_2 = -(bc'g'') - (ch'f''),$$
$$2m = -(ab'c'') - 2(fg'h'').$$

This invariant is Dr. Salmon's 'M' ("Conics," p. 349).

2. I may remark—and the remark is new, as far as I am aware—that the sextic invariant of the Jacobian of three conics is not an independent invariant of the system, but calling the 'T' ("Higher Plane Curves," p. 185) of the Jacobian, multiplied by -729, Ω,

$$16\Omega - 48\Psi X + X^3 \equiv 0.$$

To prove this identity, suppose the triangle of reference to be the common self-conjugate triangle of v and w; then $f' = 0$, $g' = 0$, $h' = 0$, $f'' = 0$, $g'' = 0$, $h'' = 0$ simultaneously. The coefficients of three times the Jacobian reduce to

$$a_1 = 0, \quad b_2 = 0, \quad c_3 = 0, \quad 2m = -(ab'c''),$$
$$a_2 = (a'b'' - a''b')\,g, \quad b_3 = (b'c'' - b''c')\,h, \quad c_1 = (c'a'' - c''a')\,f,$$
$$a_3 = (c'a' - c''a')\,h, \quad b_1 = (a'b'' - a''b')\,f, \quad c_2 = (b'c'' - b''c')\,g,$$

[whence $a_2b_3c_1 - a_3b_1c_2 \equiv 0$], while in this case

$$X = 4m^2 - 4(b_1c_1 + c_2a_2 + a_3b_3),$$
$$\Psi = m^4 - 2(b_1c_1 + c_2a_2 + a_3b_3)\,m^2 + 3(a_2b_3c_1 + a_3b_1c_2)\,m + b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2 - (c_2a_2a_3b_3 + a_3b_3b_1c_1 + b_1c_1c_2a_2),$$
$$\Omega = 8m^6 - 24(b_1c_1 + \ldots)\,m^4 + 36(a_2b_3c_1 + a_3b_1c_2)\,m^3 + (12\,c_2a_2a_3b_3 + \ldots)\,m^2 + 24(b_1^2c_1^2 + \ldots)\,m^2 - 36(a_2b_3c_1 + a_3b_1c_2)(b_1c_1 + \ldots) - 8(b_1^3c_1^3 + \ldots) + 27(a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2) + 6\,b_1c_1c_2a_2a_3b_3 + 12\{b_1^2c_1^2(c_2a_2 + a_3b_3) + c_2^2a_2^2(a_3b_3 + b_1c_1) + a_3^2b_3^2(b_1c_1 + c_2a_2)\}.$$

Hence $\Omega - \frac{1}{8}X^3 = 36\{(a_2b_3c_1 + a_3b_1c_2)\,m^3 - (c_2a_2a_3b_3 + \ldots)\,m^2 - (a_2b_3c_1 + a_3b_1c_2)(b_1c_1 + \ldots)\,m + b_1^2c_1^2(c_2a_2 + a_3b_3) + \ldots\} + 27(a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2) + 54\,b_1c_1c_2a_2a_3b_3.$

Also $\Psi - \frac{1}{16}X^2 = 3\{(a_2b_3c_1 + a_3b_1c_2)\,m - (c_2a_2a_3b_3 + \ldots)\}$,

whence

$$3X(\Psi - \tfrac{1}{16}X^2) = 36\{(a_2b_3c_1 + a_3b_1c_2)\,m^3 - (c_2a_2a_3b_3 + \ldots)m^2 - (a_2b_3c_1 + a_3b_1c_2)(b_1c_1 + \ldots)\,m + b_1^2c_1^2(c_2a_2 + a_3b_3) + 3b_1c_1c_2a_2a_3b_3\} + 27(a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2)$$
$$= \Omega - \tfrac{1}{8}X^3 - 27(a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2) + 54\,b_1c_1c_2a_2a_3b_3$$
$$= \Omega - \tfrac{1}{8}X^3 - 27(a_2b_3c_1 - a_3b_1c_2)^2.$$

It has been shown above that $a_2b_3c_1 - a_3b_1c_2$ vanishes in this case, consequently, clearing of fractions,

$$3X(16\Psi - X_c^2) = 16\Omega - 2X^3,$$

or
$$16\Omega - 48\Psi X + X^3 = 0.$$

3. Dr. Salmon has discussed the condition for the three conics having one point in common, or a single concurrence, and shown it to be ("Conics," p. 349) $X^2 = 64\Psi$.

In fact, taking the common point as one of the corners of the triangle of reference, in this case

$$a = 0, \quad a' = 0, \quad a'' = 0$$

simultaneously, whence X reduces to

$$X = 8\{(bg'h'')(cg'h'') - (fg'h'')^2\}.$$

But in this case, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, and consequently Ψ reduces to

$$\Psi = m^4 - 2b_1c_1m^2 + b_1^2c_1^2 = (m^2 - b_1c_1)^2.$$

Now, however, b_1, c_1, m reduce to

$$b_1 = -(bg'h''), \quad c_1 = -(cg'h''), \quad m = -(fg'h''),$$

consequently,
$$X = 8(b_1c_1 - m^2),$$

and
$$X^2 = 64(b_1c_1 - m^2)^2 = 64\Psi.$$

The form of the Jacobian, wanting the terms x^3, x^2y, x^2z, shows that the point common to the three conics is a double point on it. Conversely, it may be shown, that if the Jacobian have a double point, the three conics must pass through this point, or else the relation $X^2 = 16\Psi$ must hold.

For, generally, the discriminant of the Jacobian is proportional to

$$\Omega^2 - 64\Psi^3.$$

Equating this with zero, and eliminating Ω between the resulting equation and the identity $16\Omega = (48\Psi - X^2)X$,
as the condition for a double point,

$$16384\Psi^3 - (48\Psi - X^2)^2X^2 = 0,$$

which may be transformed into

$$(16\Psi - X^2)^2(64\Psi - X^2) = 0.$$

Consequently the Jacobian will have a double point if either

$$64\Psi = X^2,$$

which is the condition for the three conics having a common point of intersection, or if
$$16\Psi = X^2.$$

4. Suppose, now, the three conics to have two points in common,

and let these be taken as two corners of the triangle of reference; then u, v, w simultaneously reduce to

$$u \equiv ax^2+2fyz+2gzx+2hxy = 0,$$
$$v \equiv a'x^2+2f'yz+2g'zx+2h'xy = 0,$$
$$w \equiv a''x^2+2f''yz+2g''zx+2h''xy = 0,$$

whence $\Delta = 2fgh-af^2$, $\Delta' = 2f'g'h'-a'f'^2$, $\Delta'' = 2f''g''h''-a''f''^2$,

$$\Theta = -a''f'^2-2f''a'f'+2(f''g'h'+g''h'f'+h''f'g'),$$
$$\Theta' = -a'f''^2-2f'a''f''+2(f'g''h''+g'h''f''+h'f''g''),$$
$$\Theta_1 = -af''^2-2fa''f''+2(fg''h''+gh''f''+hf''g''),$$
$$\Theta_1' = -a''f^2-2f''af+2(f''gh+g''hf+h''fg),$$
$$\Theta_2 = -a'f^2-2f'af+2(f'gh-g'hf+h'fg),$$
$$\Theta_2' = -af'^2-2fa'f'+2(fg'h'+gh'f'+hf'g'),$$
$$\Phi = -2(af'f''+a'f''f+a''ff')+2f(g'h''+g''h')$$
$$+2g(h'f''+h''f')+2h(f'g''+f''g').$$

From these values there follow, identically,

$$\Delta'f''^3-\Theta f''^2f'+\Theta'f''f'^2-\Delta''f'^3 = 0,$$
$$\Delta''f'^3-\Theta_1f'^2f''+\Theta_1'ff''^2-\Delta f''^3 = 0,$$
$$\Delta f''^3-\Theta_2f''^2f+\Theta_2'f'f^2-\Delta'f^3 = 0,$$
$$(\Theta f''-\Theta'f')f'^2+(\Theta_1f-\Theta_1'f'')f'^2+(\Theta_2f'-\Theta_2'f)f''^2 = 0,$$

which last may be derived from the first three by multiplying them by f'^3, f''^3, f'''^3 respectively, adding and dividing by $-ff'f''$.

The resultant of the first three of the above four equations, it has already been pointed out (Salmon's "Conics," 5th ed., p. 345), is the condition that u, v, w should have double contact with the same conic; and accordingly the "point-pair" common to the three conics in this case form the vertices of a degenerate conic which is touched by every line drawn through them. This resultant, however, has never yet, as far as I am aware, been exhibited. I obtain it as follows:

Multiply the first of the above system of four equations by f'^2, $f'f''$, $f''f$, ff' successively; the second by $f''^2, f'f'', f''f, ff'$ successively; the third by $f'''^2, f'f''', f''f, ff''$ successively, and the fourth by f'^2, f''^2, f'''^2, $f'f''', f''f, ff'$ successively. Thus will be formed a system of 18 equations, from which the 18 quantities

$$f'^4f'', f'''^4f', f''^4f, f^4f''', f'^4f''', f'^4f, f'^3f''^2, f''^3f'^2, f'''^3f^2, f^3f''^2, f^3f'^2, f'^3f^2,$$
$$f^3f'f'', f'^3f''f, f''^3ff', f'^2f''^2f, f''^2f^2f', f^2f'^2f'',$$

may be eliminated linearly; so that the resultant may be exhibited in the form of a determinant of the 18th degree:

...	...	...	...	...	...	Δ	...	...	$-\Delta'$	...	...	...	...	...	$-\Theta_2$	Θ_2'	...
Δ	...	...	...	...	...	...	...	...	...	...	...	$-\Delta'$	$-\Theta_2$	...	...	...	Θ_2'
...	...	...	$-\Delta'$	...	...	...	...	...	...	...	...	Θ_2'	Δ	...	...	...	$-\Theta_2$
...	...	...	...	$-\Delta'$	Δ	...	...	...	...	Θ_2'	$-\Theta_2$	...	...	...	...	...	...
...	...	...	...	...	...	...	$-\Delta$	...	...	Δ''	...	...	...	...	Θ_1'	...	$-\Theta_1$
...	$-\Delta'$	...	...	...	...	...	...	...	...	...	...	Δ''	...	Θ_1'	...	$-\Theta_1$	...
...	...	$-\Delta$	Δ''	...	...	...	...	Θ_1'	$-\Theta_1$	...	...	...	...	...	...	...	...
...	...	...	...	Δ''	...	...	...	...	...	...	...	$-\Theta_1$	...	$-\Delta$	...	Θ_1'	...
...	...	...	...	...	...	...	...	Δ'	...	...	$-\Delta''$	...	...	...	...	$-\Theta$	Θ'
$-\Delta''$	Δ'	...	...	...	...	Θ'	$-\Theta$	...	...	...	...	...	...	...	...	...	...
...	...	Δ'	...	...	...	...	...	...	...	...	...	...	$-\Delta''$	$-\Theta$	Θ'	...	...
...	...	...	...	...	$-\Delta''$	...	...	...	...	...	...	...	Θ'	Δ'	$-\Theta$	...	...
...	...	...	$-\Theta$	Θ'	...	...	...	...	Θ_2'	$-\Theta_1$	...	...	...	...	...	$-\Theta_2$	Θ_1'
Θ_1'	...	...	...	...	$-\Theta_1$	$-\Theta_2$	...	...	...	...	Θ'	...	...	...	Θ_1'	...	$-\Theta$
...	$-\Theta_2$	Θ_2'	...	...	...	...	Θ_1'	$-\Theta$	...	...	...	...	...	...	Θ_1	Θ'	...
...	...	...	...	...	...	Θ_1'	$-\Theta_2$	...	...	...	...	...	$-\Theta_1$	Θ_2'	...	$-\Theta$	Θ'
...	...	...	...	...	...	...	...	Θ_2'	$-\Theta$	...	...	Θ'	...	$-\Theta_2$	Θ_1'	...	$-\Theta_1$
...	...	...	...	...	...	...	...	...	...	Θ'	$-\Theta_1$	$-\Theta$	Θ_1'	...	$-\Theta_2$	Θ_2'	...

The resultant of the three equations

$$\Delta' f'''^3 - \Theta f''^2 f' + \Theta' f'' f'^2 - \Delta'' f'^3 = 0,$$
$$\Delta'' f^3 - \Theta_1 f^2 f'' + \Theta_1' f f''^2 - \Delta f'''^2 = 0,$$
$$\Delta f'^3 - \Theta_2 f'^2 f + \Theta_2' f' f^2 - \Delta' f^3 = 0,$$

may be reduced to that of a pair of cubics thus: since

$$\frac{f'}{f''} = \frac{f'}{f} \div \frac{f''}{f},$$

the equation $\Delta'' x^3 - \Theta' x^2 + \Theta x - \Delta' = 0$

must have a common root with the equation (of the ninth degree) whose roots are the quotients of those of

$$\Delta x^3 - \Theta_2 x^2 + \Theta_2' x - \Delta' = 0$$

divided by those of $\Delta x^3 - \Theta_1' x^2 + \Theta_1 x - \Delta'' = 0.$

This equation of the ninth degree is

$$\begin{aligned}
&\Delta^2 \Delta''^3 x^9 - \Delta\Delta''^2 \Theta_1 \Theta_2 x^8 + \Delta'' (\Delta \Theta_1'^2 \Theta_2' + \Delta'' \Theta_2^2 \Theta_1' - 2\Delta\Delta'' \Theta_1' \Theta_2') x^7 \\
&- \{\Delta''^2 (\Theta_2^3 - 3\Delta\Theta_2\Theta_2' + 3\Delta^2\Delta') + \Theta_1 (\Delta'' \Theta_2 \Theta_1' \Theta_2' + \Delta\Delta' \Theta_1'^2 - 3\Delta\Delta'\Delta'' \Theta_1')\} x^6 \\
&+ \{\Delta'' \Theta_1 (\Theta_2^2 \Theta_2' - 2\Delta\Theta_2'^2 - \Delta\Delta' \Theta_2) + \Delta' \Theta_2 \Theta_1' (\Theta_1'^2 - 2\Delta'' \Theta_1') + \Delta'' \Theta_1'^2 \Theta_2'^2\} x^5 \\
&- \{\Delta' \Theta_2' (\Theta_1'^2 \Theta_1 - 2\Delta\Theta_1'^2 - \Delta\Delta'' \Theta_1') + \Delta'' \Theta_1' \Theta_2 (\Theta_2'^2 - 2\Delta' \Theta_2) + \Delta' \Theta_2'^2 \Theta_1'^2\} x^4 \\
&+ \{\Delta'^2 (\Theta_1'^2 - 3\Delta\Theta_1'\Theta_1 + 3\Delta^2\Delta'') + \Theta_2' (\Delta' \Theta_1' \Theta_2 \Theta_1 + \Delta\Delta'' \Theta_2'^2 - 3\Delta\Delta'\Delta'' \Theta_2\} x^3 \\
&- \Delta' (\Delta \Theta_2'^2 \Theta_1 + \Delta' \Theta_1'^2 \Theta_2 - 2\Delta\Delta' \Theta_2 \Theta_1) x^2 + \Delta\Delta'^2 \Theta_2' \Theta_1' x - \Delta^2 \Delta'^3 = 0.
\end{aligned}$$

If this is subtracted from $\Delta'' x^3 - \Theta' x^2 + \Theta x - \Delta' = 0$, multiplied by

$$\Delta^2\Delta''^2x^6+\Delta\Delta''(\Delta\Theta'-\Theta_1\Theta_2)x^5$$
$$+\{\Delta^2(-\Delta''\Theta+\Theta'^2)+\Delta\Theta_2'(-\Delta''\Theta_1'+\Theta_1^2)+\Delta''\Theta_1'(-\Delta\Theta_2'+\Theta_2^2)-\Delta\Theta_1\Theta_2\Theta'\}x^4$$
$$+\Delta^2(2\Delta'\Delta''-\Theta\Theta')x^3$$
$$+\{\Delta^2(\Theta^2-\Delta'\Theta')+\Delta\Theta_1(\Theta_2'^2-\Delta'\Theta_2)+\Delta'\Theta_2(\Theta_1'^2-\Delta\Theta_1)-\Delta\Theta_2'\Theta_1'\Theta\}x^2$$
$$+\Delta\Delta'(\Delta\Theta-\Theta_2'\Theta_1')x+\Delta^2\Delta'^2,$$

the result is the cubic

$$\{3\Delta\Delta''(\Delta\Theta\Theta'-\Delta'\Theta_1\Theta_1'-\Delta''\Theta_2\Theta_2')+(\Delta\Theta_1\Theta'-\Delta''\Theta_2\Theta_1')(\Theta_2\Theta'-\Theta_1\Theta_2')$$
$$+\Delta''^2\Theta_2^3-\Delta^2\Theta'^3+\Delta\Delta''(2\Theta'\Theta_1'\Theta_2'-\Theta\Theta_1\Theta_2)+\Delta\Delta'\Theta_1'^3\}x^3$$
$$-\{\Delta''(3\Delta\Theta_2'-\Delta\Theta_2^2)(\Theta\Theta_1'-\Theta_1\Theta_2')-3\Delta'\Delta''\Theta_2\Theta_1'^2+\Delta''\Theta_1'^2\Theta_2'^2$$
$$+\Delta\Theta\Theta_1(\Theta_2\Theta'-\Theta_1\Theta_2')+\Delta'\Theta_1^2\Theta_2\Theta_1'\}x^2$$
$$+\{\Delta'(3\Delta\Theta_1-\Theta_1'^2)(\Theta'\Theta_2-\Theta_1\Theta_2')-3\Delta'\Delta''\Theta_2^2\Theta_1'+\Delta'\Theta_1^2\Theta_2^2$$
$$+\Delta\Theta'\Theta_2'(\Theta\Theta_1'-\Theta_1\Theta_2')+\Delta''\Theta_2'^2\Theta_1'\Theta_2\}x$$
$$-\{3\Delta\Delta'(\Delta\Theta\Theta'-\Delta''\Theta_2\Theta_2'-\Delta'\Theta_1\Theta_1'+(\Delta\Theta_2'\Theta-\Delta'\Theta_2\Theta_1')(\Theta\Theta_1'-\Theta_1\Theta_2')$$
$$+\Delta'^2\Theta_1'^3-\Delta^2\Theta'^3+\Delta\Delta'(2\Theta\Theta_1\Theta_2-\Theta'\Theta_1'\Theta_2')+\Delta\Delta''\Theta_2'^3\}=0.$$

The resultant of this cubic and

$$\Delta''x^3-\Theta'x^2+\Theta x-\Delta'=0$$

may be calculated by the known formula, and will evidently be of the 18th degree in $\Delta, \Delta', \Delta''\Theta \ldots \Theta_2'$.

5. The condition just discussed does not involve Φ; but when the coefficients of y^2, z^2 are zero in the equations of the three conics simultaneously, any one of the following system of six equations may be verified to hold identically, and the other five to subsist in virtue of symmetry, viz.:

$$\Phi f'^2f''+3\Delta'f''^2f-2\Theta ff'f''-\Theta_2'f''^2f'+\Theta'f'^2f-\Theta_1f'^3=0,$$
$$\Phi f''^2f'+3\Delta''f'^2f-2\Theta'ff'f''-\Theta_1f'^2f''+\Theta f''^2f-\Theta_2'f''^3=0,$$
$$\Phi f''^2f+3\Delta''f^2f'-2\Theta_1ff'f''-\Theta'f^2f''+\Theta_1'f''^2f'-\Theta_2f''^3=0,$$
$$\Phi f^2f''+3\Delta f''^2f'-2\Theta_1'ff'f''-\Theta_2f''^2f+\Theta_1f^2f'-\Theta'f^3=0,$$
$$\Phi f^2f'+3\Delta f'^2f''-2\Theta_2ff'f''-\Theta_1'f'^2f+\Theta_2'f^2f''-\Theta_4f^3=0,$$
$$\Phi f'^2f+3\Delta'f^2f''-2\Theta_2'ff'f''-\Theta f^2f'+\Theta_2f'^2f''-\Theta_2f''^3=0.$$

Among these six and the four equations first deduced the ten quantities $f^3, f'^3, f''^3, f'^2f'', \ldots\ldots fff''$ may be eliminated linearly, and the resultant is the following determinant of the tenth order, in which the highest power of Φ is Φ^4:

$$\begin{vmatrix}
-\Delta' & \Delta & 0 & 0 & 0 & 0 & 0 & \Theta_2' & -\Theta_2 & 0 \\
\Delta'' & 0 & -\Delta & 0 & 0 & \Theta_1' & -\Theta_1 & 0 & 0 & 0 \\
0 & -\Delta'' & \Delta' & \Theta' & -\Theta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Theta_1' & -\Theta_2 & \Theta_2' & -\Theta & \Theta' & -\Theta_1 & 0 \\
0 & -\Theta_1 & 0 & \Phi & -\Theta_2' & 3\Delta' & 0 & 0 & \Theta' & \Theta \\
0 & 0 & -\Theta_2' & -\Theta_1 & \Phi & \Theta & 0 & 0 & 3\Delta'' & \Theta' \\
0 & 0 & -\Theta_2 & 0 & \Theta_1' & \Phi & -\Theta' & 3\Delta'' & 0 & \Theta_1 \\
-\Theta' & 0 & 0 & 0 & 3\Delta & -\Theta_2 & \Phi & \Theta_1 & 0 & \Theta_1' \\
-\Theta & 0 & 0 & 3\Delta & 0 & 0 & \Theta_2' & \Phi & -\Theta_1' & \Theta_2 \\
0 & -\Theta_1' & 0 & \Theta_2 & 0 & 0 & 3\Delta' & -\Theta & \Phi & \Theta_2'
\end{vmatrix} = 0.$$

6. It has been pointed out by Dr. Salmon that when three conics have two points in common, their Jacobian breaks up into a conic and the right line joining those points. The cubic

$$\Delta x^3+\Delta' y^3+\Delta'' z^3+\Theta y^2 z+\Theta' yz^2+\Theta_1 z^2 x+\Theta_1' zx^2+\Theta_2 x^2 y+\Theta_2' xy^2+\Phi xyz$$

likewise breaks up in this case into a right line and conic. For the invariants Δ, Δ', ... Φ being the coefficients of λ^3, μ^3, ... $\lambda\mu\nu$ in the discriminant of $\lambda u+\mu v+\nu w$ in general, this discriminant, when the coefficients of y^2, z^2 can be made to vanish simultaneously in u, v, w, is

$$(\lambda f+\mu f'+\nu f'')\{2(\lambda g+\mu g'+\nu g'')(\lambda h+\mu h'+\nu h'') - (\lambda a+\mu a'+\nu a'')(\lambda f+\mu f'+\nu f'')\};$$

consequently, referred to a triangle having as two of its corners the two points common to the three conics,

$$\Delta x^3+\Delta' y^3+\Delta'' z^3+\Theta y^2 z+\ldots\ldots+\Phi xyz,$$

is identical with

$$(fx+f'y+f''z)\{2(gx+g'y+g''z)(hx+h'y+h''z) - (ax+a'y+a''z)(fx+f'y+f''z)\}.$$

7. If three conics have three points in common, their equations may simultaneously be reduced to

$$u \equiv 2f\,yz+2g\,zx+2h\,xy = 0,$$
$$v \equiv 2f'yz+2g'zx+2h'xy = 0,$$
$$w \equiv 2f''yz+2g''zx+2h''xy = 0,$$

so that the invariants reduce to

$$\Delta = 2fgh, \quad \Delta' = 2f'g'h', \quad \Delta'' = 2f''g''h'';$$
$$\Theta = 2(f''g'h'+g''h'f'+h''f'g'), \quad \Theta' = 2(f'g''h''+g'h''f''+h'f''g''),$$
$$\Theta_1 = 2(fg''h''+gh''f''+hf''g''), \quad \Theta_1' = 2(f''gh+g''hf+h''fg),$$
$$\Theta_2 = 2(f'gh+g'hf+h'fg), \quad \Theta_2' = 2(fg'h'+gh'f'+hf'g');$$
$$\Phi = 2\{f(g'h''+g''h')+g(h'f''+h''f')+h(f'g''+f''g')\};$$

whence
$$\Theta_1'\Theta_2 - \Delta\,\Phi = 4\,(ff''g'^2h'^2 + g'g''h'^2f'^2 + h'h''f'^2g'^2),$$
$$\Theta\,\Theta_2' - \Delta'\Phi = 4\,(f'f g''^2h'^2 + g''g h'^2f'^2 + h''h f'^2g''^2),$$
$$\Theta'\Theta_1 - \Delta''\Phi = 4\,(ff'g''^2h''^2 + gg'h''^2f''^2 + hh'f''^2g''^2);$$

also
$$\frac{\Theta_2}{\Delta} = \frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}, \quad \frac{\Theta_2'}{\Delta} = \frac{g'h'}{gh} + \frac{h'f'}{hf} + \frac{f'g'}{fg}, \quad \frac{\Delta'}{\Delta} = \frac{f'g'h'}{fgh},$$
$$\frac{\Theta_1'}{\Delta} = \frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h}, \quad \frac{\Theta_1}{\Delta} = \frac{g''h''}{gh} + \frac{h''f''}{hf} + \frac{f''g''}{fg}, \quad \frac{\Delta''}{\Delta} = \frac{f''g''h''}{fgh},$$
$$\frac{\Theta'}{\Delta''} = \frac{f'}{f''} + \frac{g'}{g''} + \frac{h'}{h''}, \quad \frac{\Theta}{\Delta''} = \frac{g'h'}{g''h''} + \frac{h'f'}{h''f''} + \frac{f'g'}{f''g''}, \quad \frac{\Delta'}{\Delta''} = \frac{f'g'h'}{f''g''h''},$$
$$\frac{\Theta_1'\Theta_2 - \Delta\Phi}{\Delta^2} = \frac{f'f''}{f'^2} + \frac{g'g''}{g'^2} + \frac{h'h''}{h'^2}, \quad \frac{\Theta\Theta_2' - \Delta'\Phi}{\Delta'^2} = \frac{f''f}{f'^2} + \frac{g''g}{g'^2} + \frac{h''h}{h'^2},$$
$$\frac{\Theta'\Theta_1 - \Delta''\Phi}{\Delta''^2} = \frac{ff'}{f''^2} + \frac{gg'}{g''^2} + \frac{hh'}{h''^2}.$$

8. It follows that when the three conics have three points in common, of the three equations,
$$\Delta\, x^3 - \Theta_2 x^2 + \Theta_2' x - \Delta' = 0,$$
$$\Delta\, x^3 - \Theta_1 x^2 - \Theta_1 x - \Delta'' = 0,$$
$$\Delta'' x^3 - \Theta' x^2 + \Theta\, x - \Delta = 0,$$
the roots of any one are the quotients of those of another divided each by one of those of the third. This gives three relations among the coefficients of the three equations; viz., that the coefficients of the last should be proportional to those of the cubic, p. 410.

It appears also that if the equation be formed which has for its roots the sums of the products of those of the first of the three cubics above, each multiplied by one of those of the second, the expression $\frac{\Theta_1'\Theta_2 - \Delta\Phi}{\Delta^2}$ must satisfy that equation. In the "Quarterly Journal of Mathematics," Vol. VI., pp. 331–333, I have investigated the form* of this derivative (of the sixth degree) of two cubics, and have shown that, assuming
$$\mathrm{K} = \Theta_2^2\Theta_1 + \Theta_1'^2\Theta_2' - 3\Delta\Theta_1\Theta_2',$$
$$2\mathrm{L} = 2\,(\Delta''\Theta_2^3 + \Delta'\Theta_1'^3 - \Theta_1\Theta_2\Theta_1'\Theta_2') + 3\,(\Theta_2\Theta_2' - 3\Delta\Delta')\,(\Theta_1\Theta_1' - 3\Delta\Delta''),$$
$$\mathrm{M} = 27\Delta^2\Delta'^2 + 4\Delta\Theta_2'^3 + 4\Delta'\Theta_2^3 - \Theta_2^2\Theta_2'^2 - 18\Delta\Delta'\Theta_2\Theta_2',$$
$$\mathrm{M}' = 27\Delta^2\Delta''^2 + 4\Delta\Theta_1^3 + 4\Delta''\Theta_1'^3 - \Theta_1^2\Theta_1'^2 - 18\Delta\Delta''\Theta_1\Theta_1',$$
it is
$$4\,(\Delta^4\phi^3 - \Delta^2\Theta_2\Theta_1'\phi^2 + \Delta\mathrm{K}\phi - \mathrm{L})^2 - \mathrm{M}\mathrm{M}' = 0.$$

* Which may be verified *à posteriori* by a much shorter process, by showing that the left side vanishes when two roots of either cubic are equal.

Substituting for ϕ, $\frac{\Theta_1'\Theta_2 - \Delta\Phi}{\Delta^2}$, the condition then is

$$4\{\Delta^2\Phi^3 - 2\Delta\Theta_1'\Theta_2\Phi^2 + (\Theta_1'^2\Theta_2^2 + \Delta K)\Phi - \Theta_1'\Theta_2 K + L^2\} - \Delta^2 MM' = 0.$$

It will be remarked that, though Φ, Θ, Θ', considered separately as functions of the roots of

$$\Delta x^3 - \Theta_1' x^2 + \Theta_1 x - \Delta' = 0,$$
$$\Delta x^3 - \Theta_2 x^2 + \Theta_2' x - \Delta'' = 0,$$

have each six values, they are so related that, one being determined, the other two are thereby determined. Consequently, to obtain *sufficient* as well as necessary conditions, relations must be found between Θ and Θ' in the first degree with Φ. These may be obtained as follows.

Consider the identity

$$(\beta-\gamma)(\beta\gamma'+\beta'\gamma) + (\gamma-\alpha)(\gamma\alpha'+\gamma'\alpha) + (\alpha-\beta)(\alpha\beta'+\alpha'\beta)$$
$$-2\{\beta\gamma(\beta'-\gamma') + \gamma\alpha(\gamma'-\alpha') + \alpha\beta(\alpha'-\beta')\}$$
$$+(\alpha+\beta+\gamma)\{(\beta-\gamma)\alpha' + (\gamma-\alpha)\beta' + (\alpha-\beta)\gamma'\} = 0,$$

or, what is the same thing,

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma'+\beta'\gamma & \gamma\alpha'+\gamma'\alpha & \alpha\beta'+\alpha'\beta \end{vmatrix} - 2\begin{vmatrix} 1 & 1 & 1 \\ \alpha' & \beta' & \gamma' \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} + (\alpha+\beta+\gamma)\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} = 0,$$

which may be transformed into

$$\begin{vmatrix} 1 & 1 & 1 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \\ \beta\gamma+\beta'\gamma & \gamma\alpha'+\gamma'\alpha & \alpha\beta'+\alpha'\beta \end{vmatrix} - 2\begin{vmatrix} 1 & 1 & 1 \\ \beta'+\gamma' & \gamma'+\alpha' & \alpha'+\beta' \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix}$$
$$= (\alpha+\beta+\gamma)\begin{vmatrix} 1 & 1 & 1 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \\ \beta'+\gamma' & \gamma'+\alpha' & \alpha'+\beta' \end{vmatrix}.$$

Multiplied by $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}$, this becomes

$$\begin{vmatrix} 3 & 2(\alpha+\beta+\gamma) & \alpha(\beta'+\gamma')+\ldots \\ \alpha+\beta+\gamma & 2(\beta\gamma+\gamma\alpha+\alpha\beta) & 2(\alpha'\beta\gamma+\beta'\gamma\alpha+\gamma'\alpha\beta) \\ \alpha'+\beta'+\gamma' & \alpha'(\beta+\gamma)+\ldots & 2(\alpha\beta'\gamma'+\beta\gamma'\alpha'+\gamma\alpha'\beta') \end{vmatrix}$$
$$-2\begin{vmatrix} 3 & 2(\alpha'+\beta'+\gamma') & \beta\gamma+\gamma\alpha+\alpha\beta \\ \alpha+\beta+\gamma & \alpha(\beta'+\gamma')+\ldots & 3\alpha\beta\gamma \\ \alpha'+\beta'+\gamma' & 2(\beta'\gamma'+\gamma'\alpha'+\alpha'\beta') & \alpha'\beta\gamma+\beta'\gamma\alpha+\gamma'\alpha\beta \end{vmatrix}$$
$$= (\alpha+\beta+\gamma)\begin{vmatrix} 3 & \alpha+\beta+\gamma & \alpha'+\beta'+\gamma' \\ \alpha+\beta+\gamma & 2(\beta\gamma+\ldots) & \alpha(\beta'+\gamma')+\ldots \\ \alpha'+\beta'+\gamma' & \alpha'(\beta+\gamma)+\ldots & 2(\beta'\gamma'+\ldots) \end{vmatrix}.$$

Now let $\alpha = \frac{f'}{f}$, $\beta = \frac{g'}{g}$, $\gamma = \frac{h'}{h}$, $\alpha' = \frac{f''}{f}$, $\beta' = \frac{g''}{g}$, $\gamma' = \frac{h''}{h}$; then the above identity becomes

$$\Delta \begin{vmatrix} 3\Delta & 2\Theta_1' & \Phi \\ \Theta_1' & 2\Theta_1 & 2\Theta \\ \Theta_2 & \Phi & 2\Theta' \end{vmatrix} - 2\Delta \begin{vmatrix} 3\Delta & 2\Theta_2 & \Theta_1 \\ \Theta_1' & \Phi & 3\Delta' \\ \Theta_2 & 2\Theta_2' & \Theta \end{vmatrix} = \Theta_1' \begin{vmatrix} 3\Delta & 2\Theta_1' & 2\Theta_2 \\ \Theta_1' & 2\Theta_1 & \Phi \\ \Theta_2 & \Phi & 2\Theta_2' \end{vmatrix}.$$

From this equation a similar one may plainly be deduced by interchanging Δ' and Δ'', Θ_1' and Θ_2, Θ_1 and Θ_2', Θ and Θ'; viz.,

$$\Delta \begin{vmatrix} 3\Delta & 2\Theta_2 & \Phi \\ \Theta_2 & 2\Theta_2' & 2\Theta' \\ \Theta_1' & \Phi & 2\Theta \end{vmatrix} - 2\Delta \begin{vmatrix} 3\Delta & 2\Theta_1' & \Theta_2' \\ \Theta_2 & \Phi & 3\Delta'' \\ \Theta_1' & 2\Theta_1 & \Theta' \end{vmatrix} = \Theta_2 \begin{vmatrix} 3\Delta & 2\Theta_2 & 2\Theta_1' \\ \Theta_2 & 2\Theta_2' & \Phi \\ \Theta_1' & \Phi & 2\Theta_1 \end{vmatrix}.$$

Three analogous relations might be obtained from these by interchanging Δ and Δ', Θ and Θ_1', Θ' and Θ_1; and three others by interchanging Δ and Δ'', Θ_2 and Θ', Θ_2' and Θ.

8. The covariant

$$\Delta x^3 + \Delta' y^3 + \Delta'' z^3 + \Theta y^2 z + \Theta' z^2 y + \Theta_1 z^2 x + \Theta_1' x^2 z + \Theta_2 x^2 y + \Theta_2' x y^2 + \Phi x y z$$

evidently breaks up into three linear factors when the three conics have three points in common; for, referring them to their common inscribed triangle, the discriminant of $\lambda u + \mu v + \nu w$ is

$$(\lambda f + \mu f' + \nu f'')\,(\lambda g + \mu g' + \nu g'')\,(\lambda h + \mu h' + \nu h'');$$

but this is $\Theta\lambda^3 + \ldots + \Phi\lambda\mu\nu$. The Jacobian of u, v, w in the same case breaks up into three right lines; viz., those joining the common points. But this takes place also, as Dr. Salmon has pointed out, when the three conics have a common self-conjugate triangle. In fact, as may easily be verified, it takes place even though only one vertex and the opposite side of the common self-conjugate triangle of two of the conics are pole and polar with respect to the third.

9. The condition that the three conics,

$$\begin{aligned} 2f\,yz + 2g\,zx + 2h\,xy &= 0, \\ 2f'\,yz + 2g'\,zx + 2h'\,xy &= 0, \\ 2f''\,yz + 2g''\,zx + 2h''\,xy &= 0, \end{aligned}$$

should have a fourth point common, is found by eliminating the products of the variables linearly from the above three equations, and is therefore

$$\begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = 0,$$

or
$$fgh\begin{vmatrix} 1 & 1 & 1 \\ \frac{f'}{f} & \frac{g'}{g} & \frac{h'}{h} \\ \frac{f''}{f} & \frac{g''}{g} & \frac{h''}{h} \end{vmatrix} = 0,$$

or
$$\Delta\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} = 0;$$

squaring which and rejecting $\Delta = 0$,
$$\begin{vmatrix} 3\Delta & 2\Theta_1' & 2\Theta_2 \\ \Theta_1' & 2\Theta_1 & \Phi \\ \Theta_2 & \Phi & 2\Theta_2' \end{vmatrix} = 0.$$

In fact, since generally, if the three conics should have three points in common, it has been shown that each of the four determinants,

$$\begin{vmatrix} 3\Delta & 2\Theta_1' & \Phi \\ \Theta_1' & 2\Theta_1 & 2\Theta \\ \Theta_2 & \Phi & 2\Theta' \end{vmatrix} \qquad \begin{vmatrix} 3\Delta & 2\Theta_2 & \Theta_1 \\ \Theta_1' & \Phi & 3\Delta' \\ \Theta_2 & 2\Theta_2' & \Theta \end{vmatrix}$$

$$\begin{vmatrix} 3\Delta & 2\Theta_2 & \Phi \\ \Theta_2 & 2\Theta_2' & 2\Theta' \\ \Theta_1' & \Phi & 2\Theta \end{vmatrix} \qquad \begin{vmatrix} 3\Delta & 2\Theta_1' & \Theta_2 \\ \Theta_2 & \Phi & 3\Delta'' \\ \Theta_1' & 2\Theta_1 & \Theta' \end{vmatrix}$$

has the determinant $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}$ as a factor, as well as the determinant which has been shown to be the square of the preceding, it follows that when the conics have a fourth point also common, each of the above four determinants will also vanish.

Four conditions which are necessary, and sufficient, in order that the three conics should have four points in common are, therefore,

$$4\,(\Delta^2\Phi^3 - 2\Delta\Theta_2\Theta_1'\Phi^2 + \mathrm{K}'\Phi - \mathrm{L}')^2 - \Delta^2\mathrm{M}\mathrm{M}' = 0,$$
$$\begin{vmatrix} 3\Delta & 2\Theta_1' & 2\Theta_2 \\ \Theta_1' & 2\Theta_1 & \Phi \\ \Theta_2 & \Phi & 2\Theta_2' \end{vmatrix} = 0,$$

together with *any two* of following four equations:

$$\begin{vmatrix} 3\Delta & 2\Theta_2 & \Theta_1 \\ \Theta_1' & \Phi & 3\Delta' \\ \Theta_2 & 2\Theta_2' & \Theta \end{vmatrix} = 0, \qquad \begin{vmatrix} 3\Delta & 2\Theta_1' & \Theta_2' \\ \Theta_2 & \Phi & 3\Delta'' \\ \Theta_1' & 2\Theta_1 & \Theta' \end{vmatrix} = 0,$$

$$\begin{vmatrix} 3\Delta & 2\Theta_1' & \Phi \\ \Theta_1' & 2\Theta_1 & 2\Theta \\ \Theta_2 & \Phi & 2\Theta' \end{vmatrix} = 0, \qquad \begin{vmatrix} 3\Delta & 2\Theta_2 & \Phi \\ \Theta_2 & 2\Theta_2' & 2\Theta' \\ \Theta_1' & \Phi & 2\Theta \end{vmatrix} = 0.$$

But the existence of any three of the last four together with the first, though necessary, would not be *sufficient*, in general, because each of the last four determinants has another factor besides $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}$; nor would the existence of the second and three of the last four in general be sufficient, because the second contains Φ in the second degree, and is therefore satisfied by a value of Φ which does not necessarily also satisfy the first.

It is hardly necessary to remark that in the above enumeration of conditions the same interchange of symbols may be made as before.

10. Since, when the three conics have four points in common, $\begin{vmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{vmatrix} = 0$, it at once appears that in this case the three lines whose equation is $\Delta x^3 + \Delta' y^3 + \Delta'' z^3 + \Theta y^2 z + \ldots\ldots + \Phi xyz = 0$, have a common point of intersection. The Jacobian of the system becomes indeterminate—every one of its coefficients vanishing—as must be the case, since it should represent the right lines joining two and two any three of the four points common to the conics; and at the same time their common self-conjugate triangle.

APPENDIX.

THE following is the Abstract furnished by Mr. W. Barrett Davis of his communication "On Methods used in the construction of Tables of Divisors," (p. 10):—

"The following three distinct numerical calculations were originally undertaken to obtain large series of prime numbers, and to exhibit the status of large numbers with regard to the number of divisors. I do not know that any algebraical means exist of ascertaining how many numbers are composed of two or three prime divisors, say, or binary and ternary &c. composites, between given limits.

"The first Table exhibits the divisors of numbers in the neighbourhood of 10^{xi}. I have calculated the place of all the prime numbers as far as 80,021, and after that as far as the square root, viz., $\sqrt{312000}$ by estimation, those that would enter within my limits. The second is a similar Table for 10^8. An extract from this has been published in

Liouville's 'Journal de Mathématiques' for 1866. The third computation is an attempt to find the factors of the differences of powers, and was suggested by a paper by M. Lamé, in the *Comptes Rendus*, and the facility of the process in the case of cubes and quintics led me to go on. I have succeeded in obtaining the primes and composites of large numbers, that would be quite unapproachable otherwise. These factors are all of one form for each power; thus for cubes they are $6m+1$, for quintics $10m+1$; and this reminds me to ask the members of the Society for an algebraical form that will approximately solve the following problem: 'Required the number of primes of the form $6m+1$ between given limits A and B.'

"In conclusion I beg to say that though the matter is simple multiplication, some mechanical contrivances have been used to shorten the labour, which could not have been carried to the same extent without the use of Col. Oakes' reciprocals and others. Dase, Burckhardt, and Chernac have been constantly used; and I have carefully looked through the 'Commentationes Arithmeticæ' of Euler to find a theorem which I could apply in a few cases only. I should have been glad to use Gauss's Table and Method for detecting a prime number, but I had not his book long enough in my own private use."

Prof. Cayley communicated an article to the "London, Edinburgh, and Dublin Philosophical Magazine" for May, 1872, "On a Bicyclic Chuck." See p. 104.

The remarkable echo at Bedgebury Park (p. 185) became the subject of a correspondence, initiated by Lord Rayleigh in "Nature," under the heading, "Harmonic Echoes." See "Nature," No. 199 (vol. viii., pp. 319, 320, August 21, 1873), Nos. 200, 208, and 210 (vol. ix. p. 6).

With reference to Mr. Macleod's paper (p. 236) it may be noted, as was pointed out to the Secretaries, that the very oldest books on artillery (as Santbech, 1561) appear to use the principle of the hodograph, though in a bungling fashion. It may be of interest to notice that it was thought that it would be desirable to introduce a similar treatment of the question into elementary works on the subject of projectiles.

There is an analysis given in the June (1873) number of the "London, Edinburgh, and Dublin Philosophical Magazine" (pp. 450—454), which bears upon Mr. Hayward's communication (p. 289) in little more than name only. The title of the article in question is, "The first extension of the term *Area* to the case of an autotomic plane circuit; by Thomas Muir, MA., Assistant-Professor of Mathematics in Glasgow University." The writer quotes De Morgan's words, "No such extension of the word has been made that I ever met with;" and then he directs attention to a paper by Alb. Ludov. Fried. Meister in the Göttingen Commentaries for 1769-1770, entitled, "De genesi figurarum planarum et inde pendentibus earum affectionibus," which, he says, contains almost all that even yet can be said on the subject.

A Member of the Council supplies the following note upon Prof. Wolstenholme's paper (p. 321): "Eckardt, in Schlömilch's Zeitschrift for 1870, has shown that if P and Q move in the same circle with uniform velocities, a point dividing the chord PQ in a constant ratio describes an epitrochoid."

Lord Rayleigh communicated to the November (1873) number of the London, Edinburgh, and Dublin Philosophical Magazine a "Note on Vibrations of approximately Simple Systems," which bears upon his paper on page 357 *suprà*.

Prof. Wolstenholme's Note on p. 330 does away with the necessity of inserting *in extenso* his paper read before the Society at the June (1873) meeting (p. 356), but a few of its results may be given here. Designating by a', b', c' the quantities given in his Note (p. 330):

If we take $\alpha=\frac{3}{2}\pi$, $a=b$, we get the well-known case of the cardioid $a'=0$, $b'=0$, $c'=3a$.

If we take $\alpha=\frac{1}{2}\pi$, $a=b$, or $\alpha=\frac{5}{2}\pi$, we get in each case the same locus, as was shown in the Author's former paper.

Again, if we take $a=-3b$, $\alpha=\frac{1}{2}\pi$, we have again $a'=b'=0$, $c'=b$; or the locus of the intersection of perpendicular tangents to a tricusped hypocycloid is the inscribed circle.

The locus will in general be a circle, if either $\frac{a+b}{a+2b}\alpha=r\pi$, or if $\frac{b}{a+2b}\alpha=r\pi$, r being any integer. Thus if we take $2a=-3b$ and $\alpha=\frac{1}{2}\pi$, we again get the tricusped hypocycloid under its alternative method of generation, and obtain the circle by making $c'=0$.

If $a=-4b$, and $\alpha=\frac{2}{3}\pi$ or $\frac{4}{3}\pi$, we have $a'=b'=0$, $c'=2b$, giving the inscribed circle of the four-cusped hypocycloid; and it has been shown by the Author, in his paper on epicycloids, that three of the tangents from any point of this circle are so inclined.

In general, if we make $a'=b'=0$, we get

$$c'=(a+2b)\sin\frac{b}{a+b}\cdot r\pi\div\sin\frac{a+2b}{a+b}r\pi=(-1)^r(a+2b),$$

giving the circle through the vertices of the epicycloid (or hypocycloid).

Also, if we make $c'=0$, we have

$$\frac{a'}{a}=\frac{b'}{b}=(-1)^r\frac{a+2b}{a+b},$$

therefore

$$a'+b'=(-1)^r(a+2b),$$

giving again the circle through the vertices.

It appears, then, that the only cases in which the locus is a circle are those discussed in the paper on epicycloids, p. 321.

The locus can become an epicycloid only by taking $c'=b'$ on to $-b'$, which gives the equation

$$b\sin\frac{a+b}{a+2b}\alpha=\pm(a+b)\sin\frac{b}{a+2b}\alpha,$$

or with the notation of the paper referred to,

$$m \sin n\theta = \pm n \sin m\theta.$$

The form of the equation shows that if C be the centre of the fixed circle, P, P′ the points of contact of two moving circles when the tangents are inclined at the given angle, Q the corresponding point of contact of the moving circle in the epitrochoid; then CQ bisects the angle PCP′.

In the February number (1873, p. 98) of the "Philosophical Magazine" appears a "Note on the History of certain Formulæ in Sphèrical Trigonometry" (see the Appendix to Vol. iii. of the Proceedings of the London Mathematical Society, pp. 320, 321). In this Mr. Todhunter combats the view taken in the Appendix. He remarks that "the writer (Klügel) states correctly the positions of Gauss and Mollweide; and then he adds that Delambre published the formulæ in the *Connaissance des Tems* for 1808, and so French writers usually call them after him. But these few words relating to Delambre seem to me to fall below the usual standard of German accuracy. For, in the first place, the erroneous date (1808) must have been borrowed without verification, although there is nothing to warn us of this. And in the next place, the writer apparently puts the claims of Mollweide and Delambre as equal, by ascribing to both the date 1808, overlooking the fact that the *Connaissance des Tems* for an assigned year is published in advance of that year. Thus, finally, although Mollweide has priority over Gauss, yet he comes about a year and a half after Delambre; and therefore, until any other person can be shown to have published the formulæ before April, 1807, they must be justly ascribed to Delambre."

Further on, Mr. Todhunter points out that Delambre's second mode of proof is substantially the same as that independently discovered and printed in the "Proceedings," vol. iii., p. 13. One step, however, in Prof. Crofton's proof he states is simpler than the corresponding step in Delambre's, viz., the proof of the equality of the angles MVA and CVP.

A Mathematical Society of Paris has been founded on the plan of the similar Societies of London, Moscow, and Berlin, having for its object to encourage mathematical studies, and increase mathematical knowledge, and to form a bond of union of those interested in the mathematical sciences. The Society publishes a bulletin of its Proceedings (*Nature*, August 1, 1872; and for a more detailed account see *Nouvelles Annales*, August, 1872).

Want of leisure has precluded Prof. Henrici from drawing up a full obituary notice of the late Dr. Clebsch. We shall content ourselves with the following brief sketch he has outlined.

Rudolf Friedrich Alfred Clebsch was born 19th January, 1833, at Königsberg in Prussia, where his father was a medical officer in the army. He was a pupil at the Gymnasium, and at the early age of

seventeen he became a student at the University of Königsberg. Here he attended the lectures of Neumann, Richelot, and Hesse on Mathematical Physics and Mathematics. The influence of each of these three men is perceptible in his writings. He took his degree in 1854, and went to Berlin, where he was for several years a teacher of mathematics at different schools, developing, in a short time, his extraordinary talents in this capacity. In his teaching we are told that it was not his aim to commence with a series of "difficult abstract definitions, but to start from concrete presentation, and rouse the pupil's interest by intuition. This was subsequently the characteristic of his university lectures; the subject-matter of the lecture grows up organically before his audience." We learn further that "it was, in the highest sense of the word, an æsthetic pleasure to listen to his exposition." Clebsch accepted, in 1858, the appointment of Professor of Theoretical Mechanics at the Polytechnic School at Carlsruhe; this he resigned for the chair of Pure Mathematics at the University of Giessen. Here he found, for the first time, the proper sphere for the full and simultaneous development of the gifts which he possessed, both for teaching and for original research; and consequently here it is that his scientific work rapidly increases. In the autumn of 1868 he left Giessen for Göttingen, where he died four years afterwards, on the 7th November, 1872.

His writings are too numerous to be specially considered in the brief space at our command; they may, however, be divided, according to contents and succession, into six groups.

First, he wrote on Mathematical Physics, Hydrostatics, Elasticity, and Optics; next on the Calculus of Variations, and on Partial Differential Equations of the first order. There follow his first geometrical papers on general Theory of Curves and Surfaces. In connection with these comes a period characterised by his study of Abelian Functions and their use in Geometry. In his last years he was occupied at the same time by Plane-Representation of Surfaces and Theory of Invariants.*

A full report of his works is given by a number of his pupils in the Mathem. Ann., vol. vii., p. 1.

Dr. Clebsch was elected an Honorary Foreign Member of the Society on December 14th, 1871. R. T.

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* This account has been chiefly drawn up from the account in the "Mathematische Annalen," vol. vi., p. 1, which is itself a reprint from the "Nachrichten der Königl. Gesellschaft der Wissenschaften in Göttingen." Translations are given in the "Nouvelles Annales," and also in the "Giornale di Matematiche" (Gennaio e Febbraio, 1874). In this latter journal also appeared, in Jan.-Feb. 1873, "Commemorazione di R. F. A. Clebsch indirizzata all' Accademia di Göttingen dal Prof. Carlo Neumann," from the "Göttingen Nachrichten."

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